

**LIMITS OF GAUDIN ALGEBRAS, QUANTIZATION OF BENDING FLOWS,  
JUCYS–MURPHY ELEMENTS AND GELFAND–TSETLIN BASES**

A. CHERVOV, G. FALQUI, AND L. RYBNIKOV

ABSTRACT. Gaudin algebras form a family of maximal commutative subalgebras in the tensor product of  $n$  copies of the universal enveloping algebra  $U(\mathfrak{g})$  of a semisimple Lie algebra  $\mathfrak{g}$ . This family is parameterized by collections of pairwise distinct complex numbers  $z_1, \dots, z_n$ . We obtain some new commutative subalgebras in  $U(\mathfrak{g})^{\otimes n}$  as limit cases of Gaudin subalgebras. These commutative subalgebras turn to be related to the hamiltonians of bending flows and to the Gelfand–Tsetlin bases. We use this to prove the simplicity of spectrum in the Gaudin model for some new cases.

KEY WORDS. Gaudin model, Bethe ansatz, bending flows, Gelfand–Tsetlin bases.

CONTENTS

1.	Introduction	1
2.	Preliminaries	3
2.1.	Construction of Gaudin subalgebras.	3
2.2.	Quantum Mishchenko-Fomenko "shift of argument" subalgebras.	4
2.3.	Talalaev's formula.	4
2.4.	The generators of $\mathcal{A}(z_1, \dots, z_n)$ .	4
3.	Limits of Gaudin algebras	6
4.	Quantum hamiltonians of bending flows	7
5.	Schur–Weyl duality and Jucys–Murphy elements	8
6.	$(\mathfrak{gl}_N, \mathfrak{gl}_M)$ duality and Gelfand–Tsetlin algebra	8
7.	Simplicity of the joint spectrum of the Gaudin algebras in the $\mathfrak{gl}_N$ case	9
	References	10

1. INTRODUCTION

Gaudin model was introduced in [G76] as a spin model related to the Lie algebra  $\mathfrak{sl}_2$ , and generalized to the case of arbitrary semisimple Lie algebras in [G83], 13.2.2. The generalized Gaudin model has the following algebraic interpretation. Let  $V_\lambda$  be an irreducible representation of  $\mathfrak{g}$  with the highest weight  $\lambda$ . For any collection of integral dominant weights  $(\lambda) = \lambda_1, \dots, \lambda_n$ , let  $V_{(\lambda)} = V_{\lambda_1} \otimes \dots \otimes V_{\lambda_n}$ . For any  $x \in \mathfrak{g}$ , consider the operator  $x^{(i)} = 1 \otimes \dots \otimes 1 \otimes x \otimes 1 \otimes \dots \otimes 1$  ( $x$  stands at the  $i$ th place), acting on the space  $V_{(\lambda)}$ . Let  $\{x_a\}$ ,  $a = 1, \dots, \dim \mathfrak{g}$ , be an orthonormal basis of  $\mathfrak{g}$  with respect to Killing form, and let  $z_1, \dots, z_n$  be pairwise distinct complex numbers. The hamiltonians of Gaudin model are the following commuting operators acting

in the space  $V_{(\lambda)}$ :

$$(1) \quad H_i = \sum_{k \neq i} \sum_{a=1}^{\dim \mathfrak{g}} \frac{x_a^{(i)} x_a^{(k)}}{z_i - z_k}.$$

We can treat the  $H_i$  as elements of the universal enveloping algebra  $U(\mathfrak{g})^{\otimes n}$ . In [FFR], an existence of a large commutative subalgebra  $\mathcal{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$  containing  $H_i$  was proved. For  $\mathfrak{g} = sl_2$ , the algebra  $\mathcal{A}(z_1, \dots, z_n)$  is generated by  $H_i$  and the central elements of  $U(\mathfrak{g})^{\otimes n}$ . In other cases, the algebra  $\mathcal{A}(z_1, \dots, z_n)$  has also some new generators known as higher Gaudin hamiltonians. Their explicit construction for  $\mathfrak{g} = gl_r$  was obtained in [T04], see also [CT04],[CT06]. (For  $\mathfrak{g} = gl_3$  see [CRT]). The construction of  $\mathcal{A}(z_1, \dots, z_n)$  uses the quite nontrivial fact [FF] that the completed universal enveloping algebra of the affine Kac–Moody algebra  $\hat{\mathfrak{g}}$  at the critical level has a large center  $Z(\hat{\mathfrak{g}})$ . There is a natural homomorphism from the center  $Z(\hat{\mathfrak{g}})$  to the enveloping algebra  $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}])$ . To any collection  $z_1, \dots, z_n$  of pairwise distinct complex numbers, one can naturally assign the evaluation homomorphism  $U(\mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]) \rightarrow U(\mathfrak{g})^{\otimes n}$ . The image of the center under the composition of the above homomorphisms is  $\mathcal{A}(z_1, \dots, z_n)$ . We will call  $\mathcal{A}(z_1, \dots, z_n)$  the *Gaudin algebra*.

The main problem in Gaudin model is the problem of simultaneous diagonalization of (higher) Gaudin hamiltonians. The bibliography on this problem is enormous (cf. [BF, Fr95, Fr02, FFR, FFTL, MV, MTV05, MTV06-1]). It follows from the [FFR] construction that all elements of  $\mathcal{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$  are invariant with respect to the diagonal action of  $\mathfrak{g}$ , and therefore it is sufficient to diagonalize the algebra  $\mathcal{A}(z_1, \dots, z_n)$  in the subspace  $V_{(\lambda)}^{sing} \subset V_{(\lambda)}$  of singular vectors with respect to  $\text{diag}_n(\mathfrak{g})$  (i.e., with respect to the diagonal action of  $\mathfrak{g}$ ). The standard conjecture says that, for generic  $z_i$ , the algebra  $\mathcal{A}(z_1, \dots, z_n)$  has simple spectrum in  $V_{(\lambda)}^{sing}$ . This conjecture is proved in [MV] for  $\mathfrak{g} = sl_r$  and  $\lambda_i$  equal to  $\omega_1$  or  $\omega_{r-1}$  (i.e., for the case when every  $V_{\lambda_i}$  is the standard representation of  $sl_r$  or its dual) and in [SV] for  $\mathfrak{g} = sl_2$  and arbitrary  $\lambda_i$ .

In the present paper we consider some limits of the Gaudin algebras when some of the points  $z_1, \dots, z_n$  glue together. We obtain some new commutative subalgebras this way. We consider the "most degenerate" subalgebra of this type. In the case of  $\mathfrak{g} = gl_N$ , this subalgebra gives a quantization of the (higher) "bending flows hamiltonians", introduced in [FM03-1]. Original bending flows were introduced in [KM96]. They are related to  $SU(2)$ . See [BR98, FIM01, FM03-2, FM04, FM06] for further developments. Further, we establish a connection of this subalgebra to the Gelfand–Tsetlin bases via the results of Mukhin, Tarasov and Varchenko on  $(gl_N, gl_M)$  duality (cf [MTV05, MTV06-1]). This result was obtained first in [FIM01] by different methods. We use this to prove the simple spectrum conjecture for  $\mathfrak{g} = gl_N$  and  $\lambda_i = m_i \omega_1$ ,  $m_i \in \mathbb{Z}_+$ .

The paper is organized as follows. In section 2 we collect some well-known facts on Gaudin algebras. In section 3 we describe some limits of Gaudin algebras. In section 4 we obtain quantum hamiltonians of bending flows as limits of higher Gaudin hamiltonians. In sections 5 and 6 we establish a connection between quantum bending flows hamiltonians and Gelfand–Tsetlin theory. Finally, in section 7 we apply our results and prove the simple spectrum conjecture.

**Acknowledgments** G.F. acknowledges support from the ESF programme MISGAM, and the Marie Curie RTN ENIGMA. The work of A.C. and L.R. has been partially supported by the RFBR grant 04-01-00702 and by the Federal agency for atomic energy of Russia. A.C. has been partially supported by the Russian President Grant MK-5056.2007.1, grant of Support for the Scientific Schools 8004.2006.2, and by the INTAS grant YSF-04-83-3396, by the ANR grant GIMP (Geometry and Integrability in Mathematics and Physics), the part of work was done during the visits to SISSA (under the INTAS project), and to the University of Angers (under ANR grant GIMP). A.C. is deeply indebted to SISSA and especially to B. Dubrovin, as well as to the University of Angers and especially to V. Rubtsov, for providing warm hospitality, excellent working conditions and stimulating discussions. The work of L.R. was partially supported by RFBR grant 05 01 00988-a and RFBR grant 05-01-02805-CNRS-a. L.R. gratefully acknowledges the support from Deligne 2004 Balzan

prize in mathematics. The work was finished during L.R.'s stay at the Institute for Advanced Study supported by the NSF grant DMS-0635607. The authors are indebted to A. Vershik for the interest in their work, to A. Mironov for useful discussion.

## 2. PRELIMINARIES

**2.1. Construction of Gaudin subalgebras.** Consider the infinite-dimensional pro-nilpotent Lie algebra  $\mathfrak{g}_- := \mathfrak{g} \otimes t^{-1}\mathbb{C}[t^{-1}]$  – it is a "half" of the corresponding affine Kac–Moody algebra  $\hat{\mathfrak{g}}$ . The universal enveloping algebra  $U(\mathfrak{g}_-)$  bears a natural filtration by the degree with respect to the generators. The associated graded algebra is the symmetric algebra  $S(\mathfrak{g}_-)$  by the Poincaré–Birkhoff–Witt theorem. The commutator operation on  $U(\mathfrak{g}_-)$  defines the Poisson–Lie bracket  $\{\cdot, \cdot\}$  on  $S(\mathfrak{g}_-)$ : for the generators  $x, y \in \mathfrak{g}_-$  we have  $\{x, y\} = [x, y]$ . For any  $g \in \mathfrak{g}$ , we denote the element  $g \otimes t^m \in \mathfrak{g}_-$  by  $g[m]$ .

The Poisson algebra  $S(\mathfrak{g}_-)$  contains a large Poisson-commutative subalgebra  $A \subset S(\mathfrak{g}_-)$ . This subalgebra can be constructed as follows.

Consider the following derivations of the Lie algebra  $\mathfrak{g}_-$ :

$$(2) \quad \partial_t(g[m]) = mg \otimes t^{m-1} \quad \forall g \in \mathfrak{g}, m = -1, -2, \dots$$

$$(3) \quad t\partial_t(g[m]) = mg \otimes t^m \quad \forall g \in \mathfrak{g}, m = -1, -2, \dots$$

The derivations (2), (3) extend to the derivations of the associative algebras  $S(\mathfrak{g}_-)$  and  $U(\mathfrak{g}_-)$ . The derivation (3) induce a grading of these algebras.

Let  $i_{-1} : S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$  be the embedding, which maps  $g \in \mathfrak{g}$  to  $g[-1]$ . Let  $\Phi_l$ ,  $l = 1, \dots, \text{rk } \mathfrak{g}$  be the generators of the algebra of invariants  $S(\mathfrak{g})^{\mathfrak{g}}$ .

**Fact 1.** (1) [BD, FFR, Fr02] *The subalgebra  $A \subset S(\mathfrak{g}_-)$  generated by the elements  $\partial_t^n \overline{S}_l$ ,  $l = 1, \dots, \text{rk } \mathfrak{g}$ ,  $n = 0, 1, 2, \dots$ , where  $\overline{S}_l = i_{-1}(\Phi_l)$ , is Poisson-commutative.*

(2) *There exist the homogeneous with respect to  $t\partial_t$  elements  $S_l \in A$  such that  $\text{gr } S_l = \overline{S}_l$ .*

(3)  *$A$  is a free commutative algebra generated by  $\partial_t^n S_l$ ,  $k = 1, \dots, \text{rk } \mathfrak{g}$ ,  $n = 0, 1, 2, \dots$ .*

*Remark.* The generators of the subalgebra  $A \subset S(\mathfrak{g}_-)$  can be described in the following equivalent way. Let  $i(z) : S(\mathfrak{g}) \hookrightarrow S(\mathfrak{g}_-)$  be the embedding depending on the formal parameter  $z$ , which maps  $g \in \mathfrak{g}$  to  $\sum_{k=1}^{\infty} z^{k-1} g[-k]$ . Then the coefficients of the power series  $\overline{S}_l(z) = i(z)(\Phi_l)$  in  $z$  freely generate the subalgebra  $A \subset S(\mathfrak{g}_-)$ .

*Remark.* The subalgebra  $A \subset U(\mathfrak{g}_-)$  comes from the center of  $U(\hat{\mathfrak{g}})$  at the critical level by the AKS-scheme (see [FFR, ER, CT06] for details).

The Gaudin subalgebra  $\mathcal{A}(z_1, \dots, z_n) \subset U(\mathfrak{g})^{\otimes n}$  is the image of the subalgebra  $A \subset U(\mathfrak{g}_-)$  under the homomorphism  $U(\mathfrak{g}_-) \rightarrow U(\mathfrak{g})^{\otimes n}$  of specialization at the points  $z_1, \dots, z_n$  (see [FFR, ER]). Namely, let  $U(\mathfrak{g})^{\otimes n}$  be the tensor product of  $n$  copies of  $U(\mathfrak{g})$ . We denote the subspace  $1 \otimes \dots \otimes 1 \otimes \mathfrak{g} \otimes 1 \otimes \dots \otimes 1 \subset U(\mathfrak{g})^{\otimes n}$ , where  $\mathfrak{g}$  stands at the  $i$ th place, by  $\mathfrak{g}^{(i)}$ . Respectively, for any  $f \in U(\mathfrak{g})$  we set

$$(4) \quad f^{(i)} = 1 \otimes \dots \otimes 1 \otimes f \otimes 1 \otimes \dots \otimes 1 \in U(\mathfrak{g})^{\otimes n}.$$

Let  $\text{diag}_n : U(\mathfrak{g}_-) \hookrightarrow U(\mathfrak{g}_-)^{\otimes n}$  be the diagonal embedding. For any collection of pairwise distinct complex numbers  $z_i, i = 1, \dots, n$ , we have the following homomorphism:

$$(5) \quad \varphi_{z_1, \dots, z_n} = (\varphi_{z_1} \otimes \dots \otimes \varphi_{z_n}) \circ \text{diag}_n : U(\mathfrak{g}_-) \rightarrow U(\mathfrak{g})^{\otimes n}.$$

More explicitly, we have

$$\varphi_{z_1, \dots, z_n}(g \otimes t^m) = \sum_{i=1}^n z_i^m g^{(i)}.$$

Set

$$\mathcal{A}(z_1, \dots, z_n) = \varphi_{z_1, \dots, z_n}(\mathcal{A}) \subset U(\mathfrak{g})^{\otimes n}$$

**2.2. Quantum Mishchenko-Fomenko "shift of argument" subalgebras.** Consider the subalgebra  $\mathcal{A}(z_1, z_2) \subset U(\mathfrak{g}) \otimes U(\mathfrak{g})$ . The associated graded quotient of  $\mathcal{A}(z_1, z_2)$  with respect to the second tensor factor is a commutative subalgebra  $\overline{\mathcal{A}(z_1, z_2)} \subset U(\mathfrak{g}) \otimes S(\mathfrak{g})$ . Any element  $\mu \in \mathfrak{g}^* = \text{Spec } S(\mathfrak{g})$  gives the evaluation homomorphism  $\text{id} \otimes \mu : U(\mathfrak{g})^{\otimes(n-1)} \otimes S(\mathfrak{g})$ . The image of  $\overline{\mathcal{A}(z_1, z_2)}$  under this homomorphism is a commutative subalgebra  $A_\mu \subset U(\mathfrak{g})$ , which does not depend on  $z_1, z_2$ . This subalgebra is a quantum version of the Mishchenko-Fomenko "shift of argument" subalgebra  $A_\mu \subset S(\mathfrak{g})$  (see [CT06, FF06, MTV06-2, Ryb]). The latter is generated by the derivatives (of any order) along  $\mu$  of the generators of the Poisson center  $S(\mathfrak{g})^{\mathfrak{g}}$ , (or, equivalently, generated by central elements of  $S(\mathfrak{g}) = \mathbb{C}[\mathfrak{g}^*]$  shifted by  $t\mu$  for all  $t \in \mathbb{C}$ ) [MF]. In [FF06, Ryb] it is shown, that  $\text{gr } A_\mu = A_\mu$ . The algebra  $A_\mu$  is a free commutative algebra with  $\frac{1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g})$  generators, and hence has the maximal possible transcendence degree (see [MF]).

**2.3. Talalaev's formula.** In [T04] D. Talalaev constructed explicitly some elements of  $U(\mathfrak{g})^{\otimes n}$  commuting with quadratic Gaudin hamiltonians for the case  $\mathfrak{g} = \mathfrak{gl}_r$ . The formulas of [T04] are universal, i.e. actually they describe a commutative subalgebra of  $U(\mathfrak{g}_-)$  which gives a commutative subalgebra of  $U(\mathfrak{g})^{\otimes n}$  as the image of the specialization homomorphism at the points  $z_1, \dots, z_n$  (see [CT06]).

Namely, set

$$L(z) = \sum_{1 \leq i, j \leq r} \sum_{n=1}^{\infty} z^{n-1} e_{ij}[-n] \otimes e_{ji} \in U(\mathfrak{g}_-) \otimes \text{End } \mathbb{C}^r,$$

where  $z$  is a formal parameter, and consider the following differential operator in  $z$  with the coefficients from  $U(\mathfrak{g}_-)$ :

$$D = \text{Tr } A_r \prod_{i=1}^r (L(z)^{(i)} - \partial_z) = \partial_z^r + \sum_{k=1}^r \sum_{n=1}^{\infty} Q_{n,k} z^{n-1} \partial_z^{r-k}.$$

Here we denote by  $L(z)^{(i)} \in U(\mathfrak{g}_-) \otimes (\text{End } \mathbb{C}^r)^{\otimes r}$  the element obtained by putting  $L(z)$  to the  $i$ -th tensor factor, and  $A_r$  denotes the projector onto  $U(\mathfrak{g}_-) \otimes \text{End}(\Lambda^r \mathbb{C}^r) \subset U(\mathfrak{g}_-) \otimes (\text{End } \mathbb{C}^r)^{\otimes r}$ . It follows from [CT06] that the elements  $Q_{n,k} \in U(\mathfrak{g}_-)$  pairwise commute.

In [Ryb06], it is shown that  $\sum_{n=1}^{\infty} Q_{n,k} z^{n-1} = S_l(z)$ , and hence the elements  $Q_{n,k} \in U(\mathfrak{g}_-)$  generate the same commutative subalgebra  $\mathcal{A} \subset U(\mathfrak{g}_-)$ .

**2.4. The generators of  $\mathcal{A}(z_1, \dots, z_n)$ .** For our purposes, we need some specific set of generators of  $\mathcal{A}(z_1, \dots, z_n)$ . Let us describe it.

Consider the following  $U(\mathfrak{g})^{\otimes n}$ -valued functions in the variable  $w$

$$S_l(w; z_1, \dots, z_n) := \varphi_{w-z_1, \dots, w-z_n}(S_l)$$

Let  $S_l^{i,m}(z_1, \dots, z_n)$  be the coefficients of the principal parts of the Laurent series of  $S_l(w; z_1, \dots, z_n)$  at the points  $z_1, \dots, z_n$ . I.e. we set

$$S_l(w; z_1, \dots, z_n) = \sum_{m=1}^{m=\deg \Phi_l} S_l^{i,m}(z_1, \dots, z_n) (z - z_i)^{-m} + O(1) \text{ as } z \rightarrow z_i.$$

The following assertion is standard.

---

<sup>1</sup>It was first observed in [MTV0512], that one can use simply the column determinant to define  $D$ ,  $D = \det^{col}(L(z) - \partial_z)$ , where  $\det^{col} M = \sum_{\sigma} (-1)^{\text{sigma}} \prod M_{\sigma(i), i}$ . This becomes natural due to an observation of [CF07], that  $L(z) - \partial_z$  is a "Manin's matrix", so use of column-determinant is dictated by Manin's general theory.

- Proposition 1.** (1) The subalgebra  $\mathcal{A}(z_1, \dots, z_n)$  is a free commutative algebra generated by the elements  $S_l^{i,m}(z_1, \dots, z_n) \in U(\mathfrak{g})^{\otimes n}$ , where  $i = 1, \dots, n-1$ ,  $l = 1, \dots, \text{rk } \mathfrak{g}$ ,  $m = 1, \dots, \text{deg } \Phi_l$ , and  $S_l^{n, \text{deg } \Phi_l}(z_1, \dots, z_n) = \Phi_l^{(n)} \in U(\mathfrak{g})^{\otimes n}$ , where  $l = 1, \dots, \text{rk } \mathfrak{g}$ .
- (2) The elements  $S_l^{i,m}(z_1, \dots, z_n) \in U(\mathfrak{g})^{\otimes n}$  are stable under simultaneous affine transformations of the parameters  $z_i \mapsto az_i + b$ .
- (3) All the elements of  $\mathcal{A}(z_1, \dots, z_n)$  are invariant with respect to the diagonal action of  $\mathfrak{g}$ .
- (4) The center of the diagonal  $\text{diag}_n(U(\mathfrak{g})) \subset U(\mathfrak{g})^{\otimes n}$  is contained in  $\mathcal{A}(z_1, \dots, z_n)$ .

*Proof.* (1) The algebra  $\mathcal{A}(z_1, \dots, z_n)$  is generated by the elements  $\varphi_{z_1, \dots, z_n}(\partial_t^n S_l)$ . The latter are Taylor coefficients of  $S_l(w; z_1, \dots, z_n) = \varphi_{w-z_1, \dots, w-z_n}(S_l)$  about  $w = 0$ . The function  $S_l(w; z_1, \dots, z_n)$  is meromorphic in  $w$  having a zero of order  $\text{deg } \Phi_l$  at  $\infty$ . Since the poles of  $S_l(w)$  are exactly  $z_1, \dots, z_n$ , the Taylor coefficients of  $S_l(w)$  about  $w = 0$  are linear expressions in the coefficients of the principal part of the Laurent series for the same function about  $z_1, \dots, z_n$ . Since the function  $S_l(w)$  has zero of order  $\text{deg } \Phi_l$  at  $\infty$ , for each  $m = 1, \dots, \text{deg } \Phi_l - 1$  the Laurent coefficient  $S_l^{m,m}(z_1, \dots, z_n) = \text{Res}_{w=z_n}(w-z_n)^{m-1} S_l(w; z_1, \dots, z_n)$  is a linear combination of the Laurent coefficients at  $z_1, \dots, z_{n-1}$ .

Now, it remains to check that the generators  $S_l^{i,m}$  are algebraically independent. Equivalently, we need to prove that the transcendence degree of  $\mathcal{A}(z_1, \dots, z_n)$  is  $\frac{n-1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g}) + \text{rk } \mathfrak{g}$ . We shall deduce this from the maximality of quantum Mishchenko-Fomenko subalgebras. Namely, it is sufficient to prove that the associated graded quotient of  $\mathcal{A}(z_1, \dots, z_n)$  with respect to the  $n$ -th tensor factor has the transcendence degree  $\frac{n-1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g}) + \text{rk } \mathfrak{g}$ . This associated graded quotient is a commutative subalgebra  $\overline{\mathcal{A}(z_1, \dots, z_n)} \subset U(\mathfrak{g})^{\otimes(n-1)} \otimes S(\mathfrak{g})$ . Any element  $\mu \in \mathfrak{g}^* = \text{Spec } S(\mathfrak{g})$  gives the evaluation homomorphism  $\text{id}^{\otimes(n-1)} \otimes \mu : U(\mathfrak{g})^{\otimes(n-1)} \otimes S(\mathfrak{g})$ . This homomorphism sends the central generators  $\overline{S_l^{n, \text{deg } \Phi_l}(z_1, \dots, z_n)} = \Phi_l^{(n)} \in \overline{\mathcal{A}(z_1, \dots, z_n)} \subset U(\mathfrak{g})^{\otimes(n-1)} \otimes S(\mathfrak{g})$  to constants. By Theorems 2 and 3 of [Ryb], the algebra  $(\text{id}^{\otimes(n-1)} \otimes \mu)(\overline{\mathcal{A}(z_1, \dots, z_n)}) \subset U(\mathfrak{g})^{\otimes(n-1)}$  is the tensor product of quantum Mishchenko-Fomenko subalgebras,  $\mathcal{A}_\mu^{(1)} \otimes \dots \otimes \mathcal{A}_\mu^{(n-1)}$  which is known to be of the transcendence degree  $\frac{n-1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g})$  (see [MF]). Thus, the algebra  $\overline{\mathcal{A}(z_1, \dots, z_n)}$  has the transcendence degree  $\frac{n-1}{2}(\dim \mathfrak{g} + \text{rk } \mathfrak{g})$  over  $\mathbb{C}[\Phi_1^{(n)}, \dots, \Phi_{\text{rk } \mathfrak{g}}^{(n)}] = S(\mathfrak{g})^{\mathfrak{g}}$ . Hence the assertion.

(2) The Laurent coefficients of the functions  $S_l(w; z_1 + b, \dots, z_n + b)$  about  $z_i + b$  are equal to those of  $S_l(w - b; z_1, \dots, z_n)$  about  $z_i$ , hence our generators are stable under simultaneous transformations of the parameters  $z_i \mapsto z_i + b$ .

Now consider simultaneous transformations  $z_i \mapsto az_i$ . The Laurent coefficients of the functions  $S_l(w; az_1, \dots, az_n)$  about  $az_i$  are proportional to those of  $S_l(aw; az_1, \dots, az_n)$  about  $z_i$ . Since the elements  $S_l \in U(\mathfrak{g}_-)$  are homogeneous with respect to  $t\partial_t$ , we see that  $\exp((\log a)t\partial_t)S_l$  is proportional to  $S_l$ . Note that  $S_l(aw; az_1, \dots, az_n) = \varphi_{w-az_1, \dots, w-az_n}(\exp((\log a)t\partial_t)S_l)$ . This means that the Laurent coefficients of the functions  $S_l(aw; az_1, \dots, az_n)$  about  $z_i$  are proportional to those of  $S_l(w; z_1, \dots, z_n)$  about  $z_i$ .

(3) The elements  $S_l \in U(\mathfrak{g}_-)$  are  $\mathfrak{g}$ -invariant, and the homomorphisms  $\varphi_{w-z_1, \dots, w-z_n}$  are  $\mathfrak{g}$ -equivariant for any  $w$ , hence the images of  $S_l$  under the homomorphisms  $\varphi_{w-z_1, \dots, w-z_n}$  are  $\mathfrak{g}$ -invariant as well.

(4) The  $\text{deg } \Phi_l$ -th Taylor coefficient of  $S_l(w; z_1, \dots, z_n)$  at  $\infty$  is the  $l$ -th generator of the center of  $\text{diag}_n(U(\mathfrak{g})) \subset U(\mathfrak{g})^{\otimes n}$ .  $\square$

### 3. LIMITS OF GAUDIN ALGEBRAS

The algebra  $U(\mathfrak{g})^{\otimes n}$  has an increasing filtration by finite-dimensional spaces,  $U(\mathfrak{g})^{\otimes n} = \bigcup_{k=0}^{\infty} (U(\mathfrak{g})^{\otimes n})_{(k)}$  (by degree with respect to the generators). We define the limit  $\lim_{s \rightarrow \infty} B(s)$  for any one-parameter family of subalgebras  $B(s) \subset U(\mathfrak{g})^{\otimes n}$  as  $\bigcup_{k=0}^{\infty} \lim_{s \rightarrow \infty} B(s) \cap (U(\mathfrak{g})^{\otimes n})_{(k)}$ . It is clear that the limit of a family of *commutative* subalgebras is a commutative subalgebra. It is also clear that passage to the limit commutes with homomorphisms of filtered algebras (in particular, with the projection onto any factor and with finite-dimensional representations).

We shall consider the limits of Gaudin subalgebras when some of the points  $z_1, \dots, z_n$  glue together. More precisely, let  $z_1, \dots, z_k$  be independent on  $s$ , and  $z_{k+i} = z + su_i$ ,  $i = 1, \dots, n-k$ , where  $z_1, \dots, z_k, z \in \mathbb{C}$  are pairwise distinct and  $u_1, \dots, u_{n-k} \in \mathbb{C}$  are pairwise distinct. Let us describe the limit subalgebra  $\lim_{s \rightarrow 0} \mathcal{A}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}) \subset U(\mathfrak{g})^{\otimes n}$ .

Consider the following homomorphisms

$$D_{k,n} := \text{id}^{\otimes k} \otimes \text{diag}_{\mathfrak{S}_{n-k}} : U(\mathfrak{g})^{\otimes(k+1)} \hookrightarrow U(\mathfrak{g})^{\otimes n},$$

and

$$I_{k,n} := 1^{\otimes k} \otimes \text{id}^{\otimes(n-k)} : U(\mathfrak{g})^{\otimes(n-k)} \hookrightarrow U(\mathfrak{g})^{\otimes n},$$

where  $\text{id} : U(\mathfrak{g}) \rightarrow U(\mathfrak{g})$  is the identity map,  $\text{diag}_{\mathfrak{S}_{n-k}} : U(\mathfrak{g}) \hookrightarrow U(\mathfrak{g})^{\otimes(n-k)}$  is the diagonal embedding. Clearly, the image of  $[U(\mathfrak{g})^{\otimes(n-k)}]^{\mathfrak{g}}$  under the homomorphism  $I_{k,n}$  commutes with the image of the homomorphism  $D_{k,n}$ .

Let  $z_1, \dots, z_k, z$  and  $u_1, \dots, u_{n-k}$  be two collections of pairwise distinct complex numbers. We assign to these data a commutative subalgebra

$$\mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})} := D_{k,n}(\mathcal{A}(z_1, \dots, z_k, z)) \cdot I_{k,n}(\mathcal{A}(u_1, \dots, u_{n-k})) \subset U(\mathfrak{g})^{\otimes n}.$$

**Proposition 2.** *The subalgebra  $\mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})}$  is a free commutative algebra generated by the elements  $D_{k,n}(S_l^{i,m}(z_1, \dots, z_k, z))$ , with  $i = 1, \dots, k$ ,  $l = 1, \dots, \text{rk } \mathfrak{g}$ ,  $m = 1, \dots, \deg \Phi_l$ ,  $I_{k,n}(S_l^{i,m}(u_1, \dots, u_{n-k}))$ , with  $i = 1, \dots, n-k-1$ ,  $l = 1, \dots, \text{rk } \mathfrak{g}$ ,  $m = 1, \dots, \deg \Phi_l$  and  $S_l^{m, \deg \Phi_l}(z_1, \dots, z_n) = \Phi_l^{(n)} \in U(\mathfrak{g})^{\otimes n}$ , where  $l = 1, \dots, \text{rk } \mathfrak{g}$ .*

*Proof.* Note that by the assertion (4) of Proposition 1 the center of  $\text{id}^{\otimes k} \otimes \text{diag}_{\mathfrak{S}_{n-k}}(U(\mathfrak{g}))$  is contained in  $I_{k,n}(\mathcal{A}(u_1, \dots, u_{n-k}))$ . Hence, by (1) of Proposition 1, the elements defined above generate the algebra  $\mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})}$ . Thus it remains to show that these elements are algebraically independent. But this is so due to the same argument as (1) of Proposition 1.  $\square$

**Theorem 1.**  $\lim_{s \rightarrow 0} \mathcal{A}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}) = \mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})} \subset U(\mathfrak{g})^{\otimes n}$ .

*Proof.*

**Lemma 1.**  $\lim_{z \rightarrow \infty} \varphi_z = \varepsilon$ , where  $\varepsilon : U(\mathfrak{g}_-) \rightarrow \mathbb{C} \cdot 1 \subset U(\mathfrak{g})$  is the co-unit.

*Proof.* It is sufficient to check this on the generators. We have

$$\lim_{z \rightarrow \infty} \varphi_z(g[m]) = \lim_{z \rightarrow \infty} z^m g = 0 \quad \forall g \in \mathfrak{g}, m = -1, -2, \dots$$

$\square$

Let us choose the generators of  $\mathcal{A}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k})$  as in Proposition 1. The coefficients of the Laurent expansion of  $S_l(w; z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k})$  at  $z + su_j$  are proportional to the Laurent coefficients of  $S_l(w; \frac{z_1-z}{s}, \dots, \frac{z_k-z}{s}, u_1, \dots, u_{n-k})$  at the point  $u_j$ . On the other hand, by Lemma 1 we have

$$\begin{aligned} \lim_{s \rightarrow 0} S_l(w; \frac{z_1-z}{s}, \dots, \frac{z_k-z}{s}, u_1, \dots, u_{n-k}) &= \\ &= \lim_{s \rightarrow 0} \varphi_{w-\frac{z_1-z}{s}, \dots, w-\frac{z_k-z}{s}, w-u_1, \dots, w-u_{n-k}}(S_l) = \\ &= (\varepsilon \otimes \dots \otimes \varepsilon \otimes \varphi_{w-u_1} \otimes \dots \otimes \varphi_{w-u_{n-k}}) \circ \text{diag}_n(S_l) = I_{n,k} S_l(w; u_1, \dots, u_{n-k}). \end{aligned}$$

Therefore,  $\lim_{s \rightarrow 0} S_l^{i+k,m}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}) = I_{n,k}(S_l^{i,m}(u_1, \dots, u_{n-k}))$  for  $i = 1, \dots, n-k$ .

Now let us compute the limits of the coefficients of the Laurent expansion of  $S_l(w; z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k})$  at  $z_i$ .

**Lemma 2.**  $\lim_{s \rightarrow 0} \varphi_{z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}} = D_{n,k} \circ \varphi_{z_1, \dots, z_k, z}$ .

*Proof.* It is sufficient to check this on the generators. We have

$$\begin{aligned} \lim_{s \rightarrow 0} \varphi_{z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}}(g[m]) &= \\ &= \lim_{s \rightarrow 0} \left( \sum_{i=1}^k z_i^m g^{(i)} + \sum_{i=k+1}^n (z + su_{i-k})^m g^{(i)} \right) = \\ &= D_{n,k} \circ \varphi_{z_1, \dots, z_k, z}(g[m]). \end{aligned} \quad \square$$

By Lemma 2 we have  $\lim_{s \rightarrow 0} S_l^{i,m}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}) = D_{n,k}(S_l^{i,m}(z_1, \dots, z_k, z))$  for  $i = 1, \dots, k$ .

Thus we have  $\lim_{s \rightarrow 0} \mathcal{A}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}) \supset \mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})}$ . On the other hand, from propositions 1 and 2, it follows that the algebras  $\mathcal{A}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k})$  and of  $\mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})}$  have the same Poincare series. Hence  $\lim_{s \rightarrow 0} \mathcal{A}(z_1, \dots, z_k, z + su_1, \dots, z + su_{n-k}) = \mathcal{A}_{(z_1, \dots, z_k, z), (u_1, \dots, u_{n-k})}$   $\square$

#### 4. QUANTUM HAMILTONIANS OF BENDING FLOWS

In [FM03-1], a complete system of Poisson-commuting elements in the Poisson algebra  $S(gl_N)^{\otimes n}$  was constructed. Namely, the following functions on  $gl_n \oplus \dots \oplus gl_N$  commute with respect to the Poisson bracket.

$$\overline{H}_{l,k}^{(\alpha)}(X_1, \dots, X_n) := \text{Res}_{s=0} \frac{1}{z^{\alpha+1}} \text{Tr}(X_k + z(\sum_{i=k+1}^n X_i))^l,$$

where  $X_1, \dots, X_n \in gl_N$ . Now, we construct a system of commuting elements  $H_{l,k}^{(\alpha)} \in U(gl_N)^{\otimes n}$  such that  $\text{gr } H_{l,k}^{(\alpha)} = \overline{H}_{l,k}^{(\alpha)}$ .

One can iterate the limiting procedure described in the previous section to obtain some new commutative subalgebras in  $U(\mathfrak{g})^{\otimes n}$ . In particular, we can obtain the following subalgebra  $\mathcal{A}_{(z_1, z_2), \dots, (z_1, z_2)} \subset U(\mathfrak{g})^{\otimes n}$ , which is generated by

$$D_{n,2}(\mathcal{A}_{z_1, z_2}), 1 \otimes D_{n-1,2}(\mathcal{A}_{z_1, z_2}), \dots, 1^{\otimes(n-2)} \otimes \mathcal{A}_{z_1, z_2}.$$

By Proposition 1 assertion (2), this subalgebra is independent on  $z_1, z_2$ . We denote it by  $\mathcal{A}_{\text{lim}}$ . The following is checked by an easy computation.

**Proposition 3.** For  $\Phi_l(X) = \text{Tr } X^l$  ( $X \in gl_N$ ),  $S_l(z) = i_z(\Phi_l)$ , we have  $\text{gr } D_{n-k+1,2}(S_l^{1,\alpha}) = \overline{H}_{l,k}^{(\alpha)}$ .

This means that, for  $\mathfrak{g} = \mathfrak{gl}_N$ , the algebra  $\mathcal{A}_{\text{lim}}$  gives a quantization of the "bending flows" system from [FM03-1].

## 5. SCHUR–WEYL DUALITY AND JUCYS–MURPHY ELEMENTS

Let  $\mathfrak{g} = \mathfrak{gl}_N$ , and  $V_{(\lambda)} = \mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N$ . By Schur–Weyl duality, the centralizer of the diagonal  $\mathfrak{g}$  in  $\text{End}(V_{(\lambda)})$  is the image of the group algebra  $\mathbb{C}[S_n]$ . Equivalently, the space  $V_{(\lambda)}^{\text{sing}} = [\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N]^{\text{sing}}$  decomposes into the sum of irreducible  $S_n$ -modules with multiplicities 0 or 1. Since the elements of  $\mathcal{A}(z_1, \dots, z_n)$  commute with the diagonal action of  $\mathfrak{g}$ , we can treat them as commuting elements of  $\mathbb{C}[S_n]$ . In particular, one can rewrite quadratic Gaudin hamiltonians 1 as follows:

$$H_i = \sum_{j \neq i} \frac{(i, j)}{z_i - z_j}.$$

The quadratic elements of  $\mathcal{A}_{\text{lim}}$  can be rewritten as follows

$$H_i = \sum_{j < i} (i, j).$$

The latter are known as Jucys–Murphy elements. By [OV], these elements generate the Gelfand–Tsetlin subalgebra in  $\mathbb{C}[S_n]$  (in other words, this subalgebra is generated by the centers of the group subalgebra  $\mathbb{C}[S_{n-1}] \subset \mathbb{C}[S_n]$ ,  $\mathbb{C}[S_{n-2}] \subset \mathbb{C}[S_{n-1}] \subset \mathbb{C}[S_n]$  and so on). This algebra has a simple spectrum in any irreducible representation of  $S_n$ . We can obtain from this the following result of Mukhin and Varchenko.

**Proposition 4.** [MV] *Suppose  $\mathfrak{g} = \mathfrak{gl}_N$  and  $(\lambda) = (\lambda_1, \dots, \lambda_n)$ , where  $\lambda_i = \omega_1$ . The Gaudin algebra  $\mathcal{A}(z_1, \dots, z_n)$  (and, moreover, its quadratic part) has simple joint spectrum in  $V_{(\lambda)}^{\text{sing}}$  for generic values of the parameters  $z_1, \dots, z_n$ .*

*Proof.* The Gelfand–Tsetlin subalgebra in  $\mathbb{C}[S_n]$  has a simple spectrum in any irreducible representation of  $S_n$ . This means that the algebra  $\mathcal{A}_{\text{lim}}$  has a simple spectrum in  $V_{(\lambda)}^{\text{sing}} = [\mathbb{C}^N \otimes \cdots \otimes \mathbb{C}^N]^{\text{sing}}$  (since the latter is multiplicity-free as an  $S_n$ -module). Since the subalgebra  $\mathcal{A}_{\text{lim}}$  belongs to the closure of the family of Gaudin subalgebras  $\mathcal{A}(z_1, \dots, z_n)$ , for generic values of  $z_i$  the algebra  $\mathcal{A}(z_1, \dots, z_n)$  has simple spectrum in  $V_{(\lambda)}^{\text{sing}}$  as well.  $\square$

## 6. $(\mathfrak{gl}_N, \mathfrak{gl}_M)$ DUALITY AND GELFAND–TSETLIN ALGEBRA

Consider the space  $W := \mathbb{C}^N \otimes \mathbb{C}^M$  with the natural action of the Lie algebra  $\mathfrak{gl}_N \oplus \mathfrak{gl}_M$ . The universal enveloping algebra  $U(\mathfrak{gl}_N \oplus \mathfrak{gl}_M) = U(\mathfrak{gl}_N) \otimes U(\mathfrak{gl}_M)$  acts on the symmetric algebra  $S(W) = \mathbb{C}[W^*]$  by differential operators. Let  $D(W)$  be the algebra of differential operators on  $W^*$ . We shall use the following classical result.

**Fact 2.** *The image of  $U(\mathfrak{gl}_M)$  is the centralizer of the image of  $U(\mathfrak{gl}_N)$  in  $D(W)$ . Equivalently, the space  $S(W)^{\text{sing}}$  is multiplicity-free as  $\mathfrak{gl}_M$ -module.*

We can treat the space  $W$  as the direct sum of  $M$  copies of  $\mathbb{C}^N$ , and hence we have the action of  $U(\mathfrak{gl}_N \oplus \cdots \oplus \mathfrak{gl}_N) = U(\mathfrak{gl}_N)^{\otimes M}$  on  $S(W)$ . The elements of the Gaudin subalgebra  $\mathcal{A}(z_1, \dots, z_M) \subset U(\mathfrak{gl}_N)^{\otimes M}$  commute with the diagonal  $\mathfrak{gl}_N$ , and hence they can be rewritten as the elements of  $U(\mathfrak{gl}_M)$ . Thus, we obtain a commutative subalgebra in  $U(\mathfrak{gl}_M)$ . Let us describe this subalgebra explicitly. Consider the diagonal  $M \times M$ -matrix  $Z$  with the diagonal entries equal to  $z_1, \dots, z_M$ . To any diagonal matrix  $Z$  one can naturally assign a

quantum Mishchenko-Fomenko subalgebra  $\mathcal{A}_Z \subset U(\mathfrak{gl}_M)$ . For generic  $Z$ , the subalgebra  $\mathcal{A}_Z \subset U(\mathfrak{gl}_M)$  is the centralizer of the following family of commuting quadratic elements [Ryb05]:

$$(6) \quad Q_Z := \left\{ \sum_{\alpha \in \Delta_+} \frac{\langle \alpha, h \rangle}{\langle \alpha, Z \rangle} e_\alpha e_{-\alpha} \mid h \in \mathfrak{h} \right\},$$

where  $\mathfrak{h} \subset \mathfrak{gl}_M$  is the subalgebra of diagonal matrices,  $\Delta$  is the root system of  $\mathfrak{gl}_M$ ,  $\Delta_+$  is the set of positive roots, and  $e_\alpha$  are certain nonzero elements of the root spaces  $\mathfrak{g}_\alpha$ ,  $\alpha \in \Delta$ . The following assertion is due to Mukhin, Tarasov and Varchenko.

**Fact 3.** [MTV06-1] *The image of the Gaudin subalgebra  $\mathcal{A}(z_1, \dots, z_M) \subset U(\mathfrak{gl}_N)^{\otimes M}$  in  $\text{End } S(W)$  coincides with the image of  $\mathcal{A}_Z \subset U(\mathfrak{gl}_M)$ . The space of quadratic Gaudin hamiltonians coincides with the image of  $Q_Z \subset U(\mathfrak{gl}_M)$ .*

**Remark.** The quasi-classical version of this fact goes back to [AHH90] (see also [GGM97] section 5.4 and especially formula 5.27 page 23). But the relation with the  $(\mathfrak{gl}_N, \mathfrak{gl}_M)$  duality was not understood and the full picture was not developed. These facts go back to the well-known in integrability theory fact that the Toda chain has two Lax representations: one is by  $2 \times 2$  matrices another by  $n \times n$  matrices.

**Remark.** The  $(\mathfrak{gl}_N, \mathfrak{gl}_M)$  duality for quadratic hamiltonians was first observed by Toledano Laredo in [TL].

In [Sh] Shuvalov described the closure of the family of subalgebras  $\mathcal{A}_Z \subset S(\mathfrak{g})$  under the condition  $Z \in \mathfrak{h}^{reg}$  (i.e., for regular  $Z$  in the fixed Cartan subalgebra). In particular, the following assertion is proved in [Sh].

**Fact 4.** *Suppose that  $Z(t) = Z_0 + tZ_1 + t^2Z_2 + \dots \in \mathfrak{h}^{reg}$  for generic  $t$ . Set  $\mathfrak{z}_k = \bigcap_{i=0}^k \mathfrak{z}_{\mathfrak{g}}(\mu_i)$  (where  $\mathfrak{z}_{\mathfrak{g}}(Z_i)$  is the centralizer of  $Z_i$  in  $\mathfrak{g}$ ),  $\mathfrak{z}_{-1} = \mathfrak{g}$ . Then we have*

- (1) *the subalgebra  $\lim_{t \rightarrow 0} \mathcal{A}_{Z(t)} \subset S(\mathfrak{g})$  is generated by all elements of  $S(\mathfrak{z}_k)^{\mathfrak{z}_k}$  and their derivatives (of any order) along  $Z_{k+1}$  for all  $k$ .*
- (2)  *$\lim_{t \rightarrow 0} \mathcal{A}_{Z(t)}$  is a free commutative algebra.*

This means, in particular, that the  $\lim_{z_1 \rightarrow z_M} (\dots (\lim_{z_{M-2} \rightarrow z_M} (\lim_{z_{M-1} \rightarrow z_M} \mathcal{A}_Z)) \dots)$  for  $\mathfrak{g} = \mathfrak{gl}_M$  is generated by the Poisson centers of  $S(\mathfrak{gl}_{M-1}) \subset S(\mathfrak{gl}_M)$ , of  $S(\mathfrak{gl}_{M-2}) \subset S(\mathfrak{gl}_{M-1}) \subset S(\mathfrak{gl}_M)$  etc. (this is a maximal commutative subalgebra known as the Gelfand–Tsetlin algebra). This was observed earlier by Vinberg in [Vin], 6.1–6.4.

Let  $\mathcal{A}_{G-T_s} \subset U(\mathfrak{gl}_M)$  be the Gelfand–Tsetlin commutative subalgebra generated by the center of  $U(\mathfrak{gl}_{M-1}) \subset U(\mathfrak{gl}_M)$ , the center of  $U(\mathfrak{gl}_{M-2}) \subset U(\mathfrak{gl}_{M-1}) \subset U(\mathfrak{gl}_M)$  etc.

**Theorem 2.** *The image of  $\mathcal{A}_{\text{lim}} \subset U(\mathfrak{gl}_N)^{\otimes M}$  in  $\text{End } S(W)$  coincides with  $\mathcal{A}_{G-T_s} \subset U(\mathfrak{gl}_M)$ .*

*Proof.* It is sufficient to prove, that the corresponding limit of  $\mathcal{A}_Z \subset U(\mathfrak{gl}_M)$  is the Gelfand–Tsetlin subalgebra. On the level of Poisson algebras this is proved in [Vin] and [Sh]. On the other hand, in [Tar], it is proved that there is a unique lifting of such limit subalgebra to  $U(\mathfrak{gl}_M)$ . Thus, the limit of  $\mathcal{A}_Z \subset U(\mathfrak{gl}_M)$  is  $\mathcal{A}_{G-T_s} \subset U(\mathfrak{gl}_M)$ .  $\square$

## 7. SIMPLICITY OF THE JOINT SPECTRUM OF THE GAUDIN ALGEBRAS IN THE $\mathfrak{gl}_N$ CASE

Let  $V$  be the tautological representation of  $\mathfrak{g} = \mathfrak{gl}_N$  (of the highest weight  $\omega_1$ ) and  $V(\lambda) = \bigotimes_{i=1}^n S^{m_i} V$  (or, equivalently,  $(\lambda) = (\lambda_1, \dots, \lambda_n)$ , where  $\lambda_i = m_i \omega_1$ ,  $m_i \in \mathbb{Z}_+$ ). We shall show in this section, that the joint spectrum of the Gaudin algebra  $\mathcal{A}(z_1, \dots, z_n)$  in the space  $V(\lambda)^{sing}$  is simple for generic values of the parameters  $z_1, \dots, z_n$ .

**Theorem 3.** Suppose  $(\lambda) = (\lambda_1, \dots, \lambda_n)$ , where  $\lambda_i = m_i \omega_1$ ,  $m_i \in \mathbb{Z}_+$ . The Gaudin algebra  $\mathcal{A}(z_1, \dots, z_n)$  has simple spectrum in  $V_{(\lambda)}^{sing}$  for generic values of the parameters  $z_1, \dots, z_n$ .

*Proof.* The Gelfand–Tsetlin subalgebra in  $U(\mathfrak{gl}_M)$  has simple spectrum in any irreducible representation of  $\mathfrak{gl}_M$ . Since the image of  $\mathcal{A}_{G-T_s} \subset U(\mathfrak{gl}_M)$  coincides with  $\mathcal{A}_{\text{lim}}$ , and  $S(W)^{sing}$  is multiplicity-free as  $\mathfrak{gl}_M$ -module, the algebra  $\mathcal{A}_{\text{lim}}$  has a simple spectrum in  $S(W)^{sing}$ .

Note that the representation  $V_{(\lambda)}$  occurs in  $S(W)$ . Thus the algebra  $\mathcal{A}_{\text{lim}}$  has simple spectrum in  $V_{(\lambda)}^{sing}$ .

Since the subalgebra  $\mathcal{A}_{\text{lim}}$  belongs to the closure of the family of Gaudin subalgebras  $\mathcal{A}(z_1, \dots, z_n)$ , for generic values of  $z_i$  the algebra  $\mathcal{A}(z_1, \dots, z_n)$  has simple spectrum in  $V_{(\lambda)}$  as well.  $\square$

## REFERENCES

- [AHH90] MR Adams, J Harnad, J Hurtubise *Dual moment maps into loop algebras*, Lett.Math.Phys. 20,(1990),299
- [BR98] A. Ballesteros, O. Ragnisco, *A systematic construction of completely integrable Hamiltonians from coalgebras*, J. Phys. A **31** (1998), 3791–3813
- [BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin’s integrable system and Hecke eigen-sheaves*, Preprint, available at [www.ma.utexas.edu/~benzvi/BD](http://www.ma.utexas.edu/~benzvi/BD).
- [BF] H.M. Babujian and R. Flume, *Off-shell Bethe Ansatz equation for Gaudin magnets and solutions of Knizhnik-Zamolodchikov equations*, Mod. Phys. Lett. **A 9** (1994) 2029–2039.
- [CF07] A. Chervov, G. Falqui, *Manin’s matrices and Talalaev’s formula*, arXiv:0711.2236
- [CT04] Chervov, A. and Talalaev, D. *Universal G-oper and Gaudin eigenproblem*, hep-th/0409007
- [CT06] Chervov, A. and Talalaev, D. *Quantum spectral curves, quantum integrable systems and the geometric Langlands correspondence*, hep-th/0604128
- [CRT] Chervov, A., Rybnikov L., and Talalaev, D. *Rational Lax operators and their quantization*, hep-th/0404106
- [ER] B.Enriquez, V.Rubtsov, *Hitchin systems, higher Gaudin hamiltonians and r-matrices*. Math. Res. Lett. 3 (1996), no. 3, 343–357. alg-geom/9503010
- [FIM01] H. Flaschka, J. Millson, *The moduli space of weighted configurations on projective space*, math.SG/0108191
- [FM03-1] Gregorio Falqui, Fabio Musso, *Gaudin Models and Bending Flows: a Geometrical Point of View*, J. Phys. A 36 (2003), no. 46,11655–11676. nlin.SI/0306005
- [FM03-2] Gregorio Falqui, Fabio Musso, *Bi-hamiltonian Geometry and Separation of Variables for Gaudin Models: a case study*, nlin.SI/0306008
- [FM04] Gregorio Falqui, Fabio Musso, *On Separation of Variables for Homogeneous  $SL(r)$  Gaudin Systems*, nlin.SI/0402026
- [FM06] Gregorio Falqui, Fabio Musso, *Quantisation of bending flows*, nlin.SI/0610003
- [GGM97] A.Gorsky, S.Gukov, A.Mironov, *Multiscale  $N=2$  SUSY field theories, integrable systems and their stringy/brane origin – I* hep-th/9707120
- [FF] B. Feigin and E. Frenkel, *Affine Kac–Moody algebras at the critical level and Gelfand–Dikiĭ algebras*, Int. Jour. Mod. Phys. **A7**, Supplement 1A (1992) 197–215.
- [FFR] B.Feigin, E.Frenkel, N.Reshetikhin, *Gaudin model, Bethe Ansatz and critical level*. Comm. Math. Phys., 166 (1994), pp. 27–62.
- [FFTL] B.Feigin, E.Frenkel, V. Toledano Laredo, *Gaudin model with irregular singularities*. math.QA/0612798.
- [Fr95] E.Frenkel, *Affine Algebras, Langlands Duality and Bethe Ansatz*, XIth International Congress of Mathematical Physics (Paris, 1994), 606–642, Internat. Press, Cambridge, MA, 1995. q-alg/9506003
- [Fr02] E.Frenkel, *Lectures on Wakimoto modules, opers and the center at the critical level*, math.QA/0210029.
- [G76] M.Gaudin, *Diagonalisation d’une classe d’hamiltoniens de spin*, J. de Physique, t.37, N 10, p. 1087–1098, 1976.
- [G83] Gaudin, Michel. *La fonction d’onde de Bethe*. (French) [The Bethe wave function] Collection du Commissariat a’ l’Energie Atomique: Se’rie Scientifique. [Collection of the Atomic Energy Commission: Science Series] Masson, Paris, 1983. xvi+331 pp.
- [KM96] M. Kapovich, J. Millson, *The symplectic geometry of polygons in Euclidean space*, J. Differ. Geom. **44**, 479–513 (1996)
- [MF] Mishchenko, A. S. and Fomenko, A. T. *Integrability of Euler’s equations on semisimple Lie algebras*. (Russian) Trudy Sem. Vektor. Tenzor. Anal. No. 19 (1979), 3–94.
- [MTV05] E. Mukhin, V. Tarasov and A. Varchenko, *Bispectral and  $(\mathfrak{gl}_N, \mathfrak{gl}_M)$  dualities*, math.QA/0510364.
- [MTV0512] E. Mukhin, V. Tarasov, A. Varchenko, *The B. and M. Shapiro conjecture in real algebraic geometry and the Bethe ansatz*, arXiv:math/0512299
- [MTV06-1] E. Mukhin, V. Tarasov and A. Varchenko, *A generalization of the Capelli identity*, math.QA/0610799.
- [MTV06-2] E. Mukhin, V. Tarasov and A. Varchenko, *Bethe eigenvectors of higher transfer matrices*, math.QA/0605015.

- [MV] E. Mukhin and A. Varchenko, *Norm of a Bethe vector and the Hessian of the master function*, Compos. Math. 141 (2005), no. 4, 1012–1028. math.QA/0402349.
- [OV] Okounkov, Andrei and Vershik, Anatoly. *A new approach to representation theory of symmetric groups*. Selecta Math. (N.S.) 2 (1996), no. 4, 581–605. math.RT/0503040.
- [Ryb] Rybnikov, L. G., *Argument shift method and Gaudin model*, Func. Anal. Appl., **40** (2006), No 3, translated from Funktsional'nyi Analiz i Ego Prilozheniya, vol. 40 (2006), No 3, pp. 30–43. math.RT/0606380.
- [Ryb05] L. G. Rybnikov, *Centralizers of certain quadratic elements in Poisson-Lie algebras and Argument Shift method. (Russian)* Uspekhi Mat. Nauk 60 (2005), no. 2(362), 173–174; math.QA/0608586.
- [Ryb06] L.G.Rybnikov, *Uniqueness of higher Gaudin hamiltonians*, math.QA/0608588.
- [Sh] Shuvalov, V. V. *On the limits of Mishchenko-Fomenko subalgebras in Poisson algebras of semisimple Lie algebras*. Russian) Funktsional. Anal. i Prilozhen. 36 (2002), no. 4, 55–64; translation in Funct. Anal. Appl. 36 (2002), no. 4, 298–305
- [SV] Scherbak, I.; Varchenko, A. *Critical points of functions,  $sl_2$  representations, and Fuchsian differential equations with only univalued solutions*. Dedicated to Vladimir I. Arnold on the occasion of his 65th birthday. Mosc. Math. J. 3 (2003), no. 2, 621–645, 745.
- [T04] Talalaev, D. *Quantization of the Gaudin system*, Functional Analysis and Its application Vol. **40** No. 1 pp.86-91 (2006) hep-th/0404153
- [Tar] Tarasov, A. A. *On the uniqueness of the lifting of maximal commutative subalgebras of the Poisson-Lie algebra to the enveloping algebra*. (Russian) Mat. Sb. 194 (2003), no. 7, 155–160; translation in Sb. Math. 194 (2003), no. 7-8, 1105–1111.
- [TL] Toledano Laredo, Valerio. *A Kohno-Drinfeld theorem for quantum Weyl groups*. Duke Math. J. 112 (2002), no. 3, 421–451, math.QA/0009181.
- [Vin] E. Vinberg, *On some commutative subalgebras in universal enveloping algebra*, Izv. AN USSR, Ser. Mat., 1990, vol. 54 N 1 pp 3-25

INSTITUTE FOR THEORETICAL AND EXPERIMENTAL PHYSICS, MOSCOW  
*E-mail address:* [chervov@itep.ru](mailto:chervov@itep.ru)

UNIVERSITA DEGLI STUDI DI MILANO - BICOCCA  
*E-mail address:* [E-mail:gregorio.falqui@unimib.it](mailto:gregorio.falqui@unimib.it)

PONCELET LABORATORY (INDEPENDENT UNIVERSITY OF MOSCOW AND CNRS) AND INSTITUTE FOR THEORETICAL AND EXPERIMENTAL PHYSICS, MOSCOW  
*E-mail address:* [leo.rybnikov@gmail.com](mailto:leo.rybnikov@gmail.com)