MR2301007 (2008g:34085) 34C07 (37C27 37F75)
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The weakened infinitesimal Hilbert's 16th problem.5-02-034087-1
ISBN 5-02-034087-1 **(Russian. Russian summary)**
Tr. Mat. Inst. Steklova **254** (2006), *Nelineĭn. Anal. Differ. Uravn.*, 215–246; *translation in Proc. Steklov Inst. Math.* **2006,** *no. 3 (254),* 201–230.

The infinitesimal form of Hilbert's sixteenth problem requires one to place an upper bound on the number of limit cycles that can be created by a small polynomial perturbation of a planar polynomial Hamiltonian vector field. The answer should be given in terms of the degrees of the Hamiltonian $H \in \mathbb{R}[x, y]$ and the perturbation [cf. Y. S. Il′yashenko, Bull. Amer. Math. Soc. (N.S.) **39** (2002), no. 3, 301–354 (electronic); MR1898209 (2003c:34001)].

One may consider an opposite (in a sense) problem and ask how many cycles can always be generated from arbitrary level curves by a perturbation meeting the above constraints. The answer is equal to the dimension of the space of Abelian integrals of polynomial 1-forms over ovals (homological cycles) on the level curves of the Hamiltonian H. In such form the problem makes sense also for higher dimensions and complex settings, when integrals of polynomial $(k-1)$-forms are considered along homology $(k-1)$-cycles on the level curves of the algebraic hypersurface $\{H = c\}$, $H \in \mathbb{C}[x_1, \ldots, x_k]$.

The problem of computation of this dimension is completely solved by the author for a generic Hamiltonian of arbitrary degree. To provide an upper bound for this dimension over all such Hamiltonians is (perhaps somewhat confusingly) referred to as the weakened (or relaxed) infinitesimal Hilbert's sixteenth problem.

More precisely, in this beautiful paper (which is the exposition of the author's Ph.D. thesis) one can find the following results.

(1) Computation (exact) of the dimension of the space of Abelian integrals in the complex multidimensional settings.
(2) Solution of the analogous problem for integrals of Gel′fand-Leray residues of polynomial k-forms of any given degree.
(3) Sufficient conditions for the cycles to carry the space of integrals of maximal dimension.
(4) Real algebraic versions of the previous results (inequalities).
(5) Corollaries for the infinitesimal Hilbert's sixteenth problem (impossibility of arbitrary positioning of limit cycles in perturbations of constrained degree).

The proofs involve computation of the relative cohomology of algebraic hypersurfaces transversal to infinity and are based on the earlier results by the author [Funktsional. Anal. i Prilozhen. **31** (1997), no. 2, 34–44, 95; MR1475322 (98k:58183)].

The article is a pleasure to read. It contains complete and very transparent demonstrations of the above results, which were partially announced earlier in [I. A. Khovanskaya, Uspekhi Mat. Nauk **57** (2002), no. 5(347), 161–162; MR1992095 (2004e:34055)], and is likely to become a convenient source of references as well as food for thought for quite some time ahead.

Reviewed by *Sergei Yakovenko*

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Note: This list reflects references listed in the original paper as accurately as possible with no attempt to correct errors.