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COPULA STRUCTURAL SHIFT IDENTIFICATION

The paper aims at presenting the research results of revealing structural shift in the copula-models of multivariate time-series. Non-parametric method of structural shift identification and estimation is used. The asymptotical characteristics (probabilities of the I-type and II-type errors, probability of estimation error) of the method are analysed. The simulation method verification results for Clayton and Gumbel copulas are presented and discussed. The empirical part of the paper is devoted to structural shift identification for multivariate time series of interest rates for Euro-, US Dollar- and Ruble-zones. The empirical application provides strong evidence of the efficiency for the proposed method of structural shift identification.

Key words: Copula, structural shift, Kolmogorov-Smirnov statistics, interest rates

1. Introduction

Let us take a continuous random vector $\mathbf{X} = \{X_1, \ldots, X_d\}$ with the joint cumulative distribution function (d.f.) marked as V and marginal distribution functions of its components $F_1, \ldots, F_d$. ***Copula*** for the joint d.f V than can be written as follows:

$$V(x_1, \ldots, x_d) = G(F_1(x_1), \ldots, F_d(x_d)),$$

where G - the only continuous cumulative d.f. which has univariate marginals equally distributed on $[0,1]$.

We are thinking of a copula when G function is unknown, but is of the following class:

$$\mathfrak{A} = \{G_\theta : \theta \in \Theta\},$$

where Θ - is an open set in R^p space.

The two mostly known books containing the detailed description of parametric copula families are those of Harry Joe [Joe (1997)] and Roger Nelsen [Nelsen (2006)]. Copulas are of often use in empirical applications in the modern actuarial calculations, econometrics and hydrology (e.g. [Frees and Valdez (1998)], [Cui and Sun (2004)], [Genest and Favre (2007))]. Never-the-less, they are more and more often applied to solving financial and risk-management tasks (e.g. [Cherubini et al. (2004)], [McNeil et al. (2005)]).

The article presented aims at discussing the problem of structural shift identification within copulas. Non-stationary copulas with discrete time are taken given their structural parameters may rapidly change in the unknown moments in time. The problem is of vital importance as the majority of real financial time series are unstable and subject to structural shifts (the notoriously known one is the world financial crisis of 2007 – 2009 that revealed most of the models used to be inadequate). Thus we argue that structural shift identification within copula models is of prior importance for further empirical research of joint financial data distributions.

The recent scientific findings made in the field of copula application can be generally classified into two principal groups:

(1) Papers devoted to parametric copula models estimation and goodness-of-fit testing (e.g. for Gaussian copula [(Malevergne and Sornette (2003))], for Clayton one [(Shih, (1998)], [Glidden, (1999)]; [Cui and Sun, (2004)]); and

(2) Articles on non-parametric methods of testing goodness-of-fit of the copula-models, including blanket tests (e.g. [Genest et al. (2006)], [Breymann et al. (2003)], [Dobric and Schmid (2005)], [Junker and May (2005)]).

The article proposes a non-parametric way of copula structural shift moment estimation. The exact problem statement is being formulated below.

2. Problem Statement and Proposed Solution

Shall start from the selection $\{\mathbf{X}_1,\ldots,\mathbf{X}_N\}$ of independent R^d -dimensional vectors with the cumulative d.f. $V_1,...,V_N$.

Suppose there exist two alternatives. The null hypothesis H_0 (cf. (2) below) is that the copula rests the same, i.e. $G_1 = G_2$. The alternative is that the copula changes after some moment $m = [\theta N]$ in time. Here we do suppose all the marginal d.f. $F_1,..., F_d$ are unchanged. To summarize the joint d.f. $V_i(x_1,..., x_d)$ in each moment i might be represented by the following system (1).

$$V_i(x_1,...,x_d) = \begin{cases} G_1(F_1(x_1),...,F_d(x_d)), & 1 \le i \le m, \\ G_2(F_1(x_1),...,F_d(x_d)), & m < i \le n. \end{cases} \tag{1}$$

Thus we want to test the following null hypothesis (2):

$$H_0 : G_1 = G_2 \tag{2}$$

Given H_0 is rejected we are interested in estimating the moment m of copula structural shift.

In other words, we are testing whether there exists a structural shift in a pattern of comovement of observed vector components. The goal is to propose a method having (1) Ist ("false emergency") and IInd ("false calmness") type estimation errors relatively small (tend to zero given the increase in the observation set N); and (2) the moment parameter estimation $\hat{\theta}_N$ to be consistent, i.e. tend to the true value of θ given the increase in the observation set.

The proposed method is based on non-parametric approach. Let us take empirical copula-processes so that for every $l = 1,\dots,N-1$ we have the following (3):

$$D_l(u) = \frac{1}{l}\sum_{i=1}^{l} I(U_{i,l} \le u) = \frac{1}{l}\sum_{i=1}^{l}\prod_{j=1}^{d} I(U_{ij,l} \le u_j)$$
$$D_{N-l}(u) = \frac{1}{N-l}\sum_{i=l+1}^{N} I(U_{i,N-l} \le u) = \frac{1}{N-l}\sum_{i=l+1}^{N}\prod_{j=1}^{d} I(U_{ij,N-l} \le u_j), \qquad (3)$$

where $U_{i,l} = (U_{i1,l},\dots,U_{id,l})$ and for every $j = [1,\dots,d]$

$$U_{ij,l} = \frac{l}{l+1}F_{j,l}(X_{ij}) = rank(X_{ij})/(l+1), \quad 1 \le i \le l,$$
$$U_{ij,N-l} = rank(X_{ij})/(N-l+1), \quad l+1 \le i \le N. \qquad (4)$$

Aiming at step-by-step structural shift identification we do fix the constant and use the following statistics as a modification of Kolmogorov-Smirnov one (5):

$$\Psi_{l,N-l}(u) = (D_l(u) - D_{N-l}(u))\sqrt{l(N-l)}/N \qquad (5)$$

and

$$T_N = \max_{[\beta N] \le l \le [(1-\beta)N]} \sup_u |\Psi_{l,N-l}(u)|. \qquad (6)$$

Therefore we may arrive at the structural shift point estimation (7):

$$\hat{m}_N \in \operatorname*{arg\,max}_{[\beta N] \le l \le [(1-\beta)N]} \left(\sup_u |\Psi_{l,N-l}(u)| \right), \qquad (7)$$

Then the structural shift parameter estimation will be as follows $\hat{\theta}_N = \hat{m}_N / N$.

To verify whether the shift-point found is a proper one we are using three measures of method efficiency listed herebelow:

1) Ist type error probability («to condemn an innocent»):

$$\alpha_N = P_0\{T_N > C\}, \qquad (8)$$

where $C > 0$ is the significance level that we do accept in order to test the null hypothesis of structural shift absence;

2) IInd type error probability:

$$\delta_N = P_m\{T_N \le C\}. \qquad (9)$$

3) The probability of estimation error: for all epsilon in $0 < \varepsilon < 1/2$ we use the following measure:

$$\gamma_N = P_m\{|\hat{\theta}_N - \theta| > \varepsilon\}. \qquad (10)$$

3. Major Findings

To remind the major assumption $X_1,\dots,X_n$ being independent random d-dimensional vectors with continuous univariate marginal d.f. Then it follows that random variables $U_{i,l}$ being defined above in (3) – (4) are independent in different moments in time $i=1,\dots,l$. Besides, their distributions are the same under the null hypothesis meaning the absence of structural shift and verify fon Kramer condition $E_0\exp(tU_{i,l})<\infty$ given $|t|<T$ for some $T>0$, as we may rewrite the assumption as follows $U_{i,l}=(U_{i1,l},\dots,U_{id,l})\in[0,1]^d$.

Next theorem 1 provides an upper exponential estimation for the Ist type error probability w.r.t. the method proposed.

Theorem 1.

$$\alpha_N \le L_1\exp(-L_2C^2N), \qquad (11)$$

Where L_1, L_2 are positive constants independent of N.

The proof of the theorem 1 comes from the logic described below. Given marginal distributions being continuous, it is true (cf. [Tsukahara (2005)]) that under the null $H_0: G_1=G_2$ and given $[\beta N]\le l\le[(1-\beta)N]$ we have the following:

$$\begin{aligned}&\sqrt{l}(D_l(u)-G_1(u))\to W_1(u),\\&\sqrt{N-l}(D_{N-l}(u)-G_1(u))\to W_2(u),\end{aligned}$$

where $W_1(\cdot),W_2(\cdot)$ - independent Winer processes on $[0,1]^d$, and the symbol "$\to$" is used to signify weak convergence in $D[0,1]^d$ space given $N\to\infty$. Therefore it is true that:

$$(D_l(u)-D_{N-l}(u))\frac{\sqrt{l(N-l)}}{N}\to\frac{1}{\sqrt{N}}((1-\frac{l}{N})^{1/2}W_1(u)-(\frac{l}{N})^{1/2}W_2(u))$$

Based on the result above and using the exponential estimations for the probability of Winer process intersecting the horizontal border we arrive at the theorem 1 result.

Equivalently the IInd type error and estimation error probabilities might be deducted. To be more precise the following theorem 2 is true.

Theorem 2.

Denote $\eta=\sup_u|G_1(u)-G_2(u)|$ and assume that $0<C<\eta/4$. Let $d=\eta/4-C$. Then the following is true:

$$\begin{aligned}&\delta_N\le L_1\exp(-L_2\min(d,d^2)N)\\&\gamma_N\le C_1\exp(-C_2\min(\varepsilon,\varepsilon^2)N)\end{aligned}, \qquad (12)$$

where L_1,L_2,C_1,C_2 are positive constants independent of N.

Proof.

Below the main idea of the proof of theorem 2 is presented.

Start with the case of $[\beta N] \le l \le m$. As $D_l(u) = \frac{1}{l}\sum_{i=1}^{l} I(U_{i,l} \le u)$, then

$$ED_l(u) = G_1(u).$$

Then it is possible to write the following:

$$D_{N-l}(u) = \frac{1}{N-l}(\sum_{i=l+1}^{m} I(U_{i,N-l} \le u) + \sum_{i=m+1}^{N} I(U_{i,N-l} \le u))$$

And therefore we have

$$ED_{N-l}(u) = \frac{1}{N-l}((m-l)G_1(u) + (N-m)G_2(u)).$$

After the transformation the following is true

$$E(D_l(u) - D_{N-l}(u)) = G_1(u)(1 - \frac{m-l}{N-l}) - G_2(u)\frac{N-m}{N-l} = \frac{N-m}{N-l}(G_1(u) - G_2(u)).$$

Thus we may conclude the following

$$\max_{l \le m} \sup_u E\Psi_{l,N-l}(u) = \frac{\sqrt{m(N-m)}}{N} \sup_u | G_1(u) - G_2(u) |.$$

In the same manner we deal with another case of $m < l \le [(1-\beta)N]$. To remark

$$\max_m \sqrt{m(N-m)} / N \le 1/4 .$$

We have received an upper estimation of the expectation for the T_N statistics. With regard to the stochastic additive component of the statistics (like in theorem 1), upper exponential estimations for the error estimation probability (12) come from upper exponential estimations for the sums of independent, identically distributed and centered random variates that satisfy fon Kramer condition (cf. [Petrov (1987)]).

Now simulation method verification results will be discussed.

4. **Simulation Method Verification**

The proposed method was tested using bidimensional vectors whose joint d.f. was characterised by (1) Clayton copula; and (2) Gumbel copula.

Clayton copula: for any $u, v \in (0,1)$ and $\kappa > 0$:

$$C_\kappa(u,v) = (u^{-\kappa} + v^{-\kappa} - 1)^{-1/\kappa},$$

Gumbel copula: for any $u, v \in (0,1)$ and $\kappa > 0$:

$$C_\kappa(u,v) = \exp[-\{(-\log u)^{1/\kappa} + (-\log v)^{1/\kappa}\}^\kappa]$$

We assume copula function does not change throughout the period, whereas its parameter κ might change in some moment $m = [\theta N]$.

Firstly we examine the critical bounds for the method proposed w.r.t. different observation sets and copula types. Initially we deal with a homogeneous set, i.e. without structural shift. For each observation set N the experiment was independently simulated 500 times. 95th and 99th quantiles for the T_N statistics maximum were estimated. 95th quantile values were then used as critical bounds for the rejection of the null hypothesis given the existence of structural shift. Simulation results are presented in tables 1-2.

Table 1

Critical Bounds, Clayton Copula, Homogeneous Set $\kappa = 0,3$

N	50	100	200	300	500	700	1000	1500	2000
95%	0,1156	0,0850	0,0615	0,0492	0,0372	0,0314	0,0278	0,0213	0,0197
99%	0,1343	0,0945	0,0674	0,0550	0,0426	0,0348	0,0323	0,0232	0,0214

Table 2.

Critical Bounds, Gumbel Copula, Homogeneous Set $\kappa = 0,3$

N	50	100	200	300	500	700	1000	1500	2000
95%	0,1033	0,0749	0,0508	0,0402	0,0313	0,0243	0,0206	0,0158	0,0146
99%	0,1187	0,0836	0,0585	0,0461	0,0343	0,0292	0,0233	0,0168	0,0154

As the tables 1-2 show the critical bounds are not very sensitive to the concrete copula underlying the observations. It permits us to undertake robust parameter calibration procedure for the purpose of structural shift identification and estimation. The respective results are provided in tables 3-4 below.

Table 3

Structural Shift Identification and Estimation, Clayton Copula, Parameter Values Before and After the Structural Shift $\kappa_1 = 0,3;\ \kappa_2 = 1,0$; Structural Shift Parameter $\theta = 0,3$, C – Critical Bound; w_2 - IInd Type Error.

$\theta = 0,3$	$\kappa_1 = 0,3;\ \kappa_2 = 1,0$			
N	500	700	1000	1500
C	0,037	0,031	0,027	0,020
w_2	0,56	0,43	0,15	0,02
θ_N	0,337	0,335	0,303	0,30

Table 4

Structural Shift Identification and Estimation, Gumbel Copula, Parameter Values Before and After the Structural Shift $\kappa_1 = 0,3;\ \kappa_2 = 1,0$; Structural Shift Parameter $\theta = 0,3$, C – Critical Bound; w_2 - II[nd] Type Error.

$\theta = 0,3$	$\kappa_1 = 0,3;\ \kappa_2 = 0,7$						
N	100	200	300	500	700	1000	1500
C	0,07	0,05	0,04	0,03	0,02	0,017	0,015
w_2	0,69	0,60	0,44	0,33	0,04	0,01	0
θ_N	0,45	0,40	0,35	0,33	0,31	0,305	0,30

Based on simulation results above we may summarize the major findings:

1) The proposed method enables us to properly identify the structural shifts in copula-models and to arrive at their parameter estimates. To remark, we do understand any unpredicted (rapid) change in the multivariate copula reflecting certain type of dependence in-between univariate components;
2) The critical bounds estimated do not depend neither on the copula type (e.g. Clayton, Gumbel or other), nor on the copula parameters under the null hypothesis. It makes them of great value when carrying out non-parametric tests for the structural shift identification in copula-models.

5. Application to Real World Data

The proposed method was applied to multivariate financial time series of interest rates to test the existence of structural shift. The basic data set contained 21 time series for 7 maturity buckets and 3 currencies (i.e. interest rates for borrowings in certain currency). The maturities were taken (1) overnight, (2) 1 month, (3) 3 months, (4) 6 months, (5) 1 year, (6) 3 years, (7) 5 years. For short-term maturities (less than one year) interbank rates (EURIBOR, USD LIBOR, MosPrime) were taken. For long-term maturities - the interest-rate swap contract quotes for the respective interbank rates. The initial set contained daily data announced by the respective organisations (European Banking Federation for EURIBOR, British Bankers Association for USD LIBOR, National Currency Association (NVA) for MosPrime) from August 6, 2007 to May 21, 2009. Bloomberg was used as a data source. Time series graphical representation is provided in Annex 1.

Methodology Used:

Copulas for joint distributions were estimated by semiparametric[1] method in order to avoid marginal d.f. misspecification. Thus empirical marginals of daily log-returns were taken, and copula was estimated parametrically. Six major copulas were regarded: Archimedean ones (Clayton, Gumbel), extreme value one (Cochi, or Student's t with 1 degree of freedom (d.o.f.)) and elliptical ones (Gaussian and Student's T with 5 and 10 d.o.f.). Copula parameter estimates[2] for both methods used (IFM and ITAU) [3] can be found in Annexes 2.1 and 2.2.

Econometric Findings Interpretation.

1. The structural shift in ruble-zone interest rates copula was estimated equal to December 3, 2008. Before the shift the joint comovement of interest rates was best characterized by Cochi copula, whereas afterwards – by Gaussian one. To remind Cochi copula has the strongest tail dependence compared to other elliptical copulas. Tail dependence indexes (both upper and lower as the copula is symmetric) equaled to 85% for the IFM method and 96.2% for the rank-transformed data (ITAU method). On the contrary Gaussian (Normal) copula has got zero tail dependence, i.e. the conditional probability of simultaneous rise or fall (in the highest and lowest quantiles) of copula components is nil. Evidently, the pooled estimation provided biased results by indicating Student's t with 8 d.o.f. to fit the data best, i.e. some average in-between the Gaussian and the Cochi copulas.

   To comment on the sources of the interest rates joint behavior it is necessary to trace the principal interest rate determinant – the refinance rate. Though in Russia it is not as much linked to interbank lending rates as in case of The European Central Bank or The US Federal Reserve, it still provides a government indication of changes in economic environment, particularly in the amount of accessible liquid funds.

   Before December 1, 2008 the Central Bank of Russian Federation (CB RF) was constantly shifting up the refinance rate up to 13 % p.a. Thus the regulator was affirming that it was necessary to limit the lending activity in order to limit the increase in the monetary base and to prevent future inflation escalation. Instead the market was in need of

[1] Authors [Kim G., Silvapulle M., Silvapulle P. (2007)] argue that semiparametric approach should be preferred enabling to arrive at consistent and robust estimates compared to parametric approaches in cases when the marginal d.f. is unknown.

[2] R software was used to undertake the estimation described. The codes and data are readily available from the authors upon request.

[3] ITAU method enables researches to estimate copula parameters based not on the probability space, but on the transformed-to-ranks probability space. Would like to remark that the parameter estimate did not depend on the d.o.f. number when using ITAU method. The only thing that was influenced by the number of d.o.f. was the value of tail dependence index. Therefore we would recommend using inference-for-margins (IFM) method in R when carrying out the estimation procedure in R.

facilitating lending (starting from the interbank one) that could be possible by the means of decreasing refinance rate.

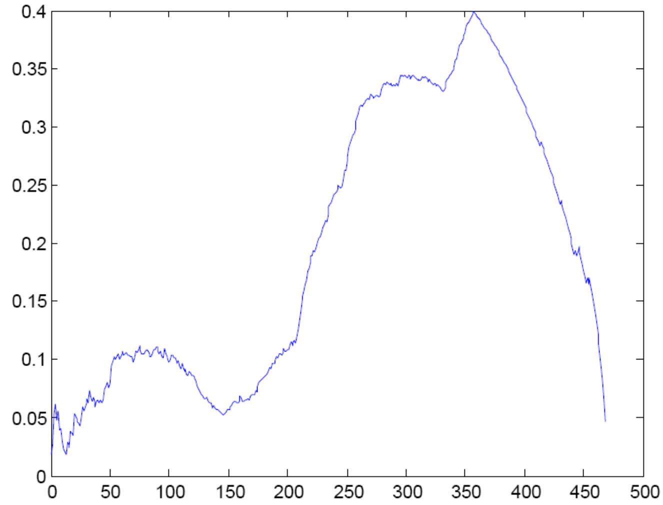

Figure 1. Key Statistics $\Psi_{l,N-l}(u)$ Dynamics Revealing the Structural Shift Point in Ruble-Zone Interest Rates Copula.
The observation number can be found on horizontal axis, the statistics value – on the vertical one.

We would also like to point out the shift-even points on the figure 1 above indicating the search of structural shift moment. It can be seen that though the global maximum belongs to observation No. 348 (i.e. December 3, 2008), the local maximum exists in the nearby of November 12, 2008. That is on November 11 and 12 that CB RF initiated two consecutive up-shifts in refinance rate by 11% and 12%, respectively (cf. Figure 2). Probably, if the economic environment was not characterized by the shortage of liquidity, the interest rates co-movement could have been satisfying the normal copula assumption after that. Never-the-less, it was exactly the period of October-November 2008 that was called a ‘banking liquidity crisis’ that the interest rate up and down co-movement was extremely strong proving the Cochi copula to fit the data best.

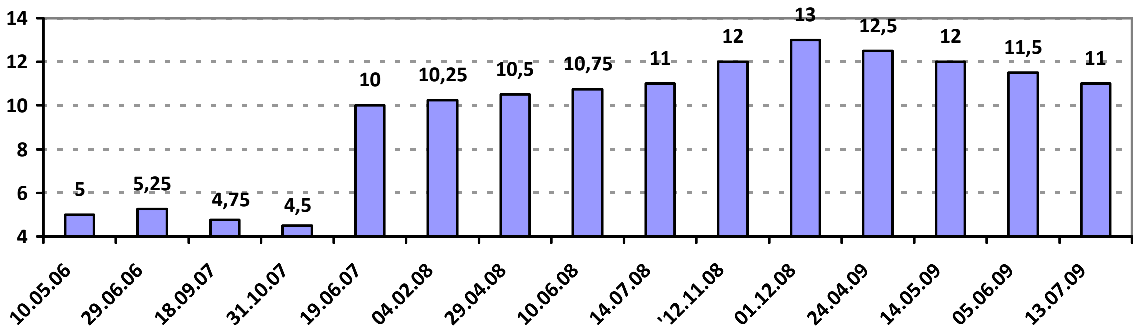

Figure 2. Key Dates of Refinance Rate Changes by the Central Bank of Russia.
Horizontal axis marks the time schedule and the vertical one – the refinance rate in percentage points.

Source: http://cbr.ru/print.asp?file=/statistics/credit_statistics/refinancing_rates.htm

2. As for US Dollar-linked interest rates, the structural shift was estimated to be on July 17, 2008 (cf. Figure 3). For the first part of the data set (before the shift-date) the Clayton copula was found to fit best, whereas afterwards it was also Gaussian copula. To remind Clayton copula has positive lower tail dependence (equal to 89.5% for the IFM method and 99% for the rank-transformed one in our case) of the components and zero for upper tail ones.

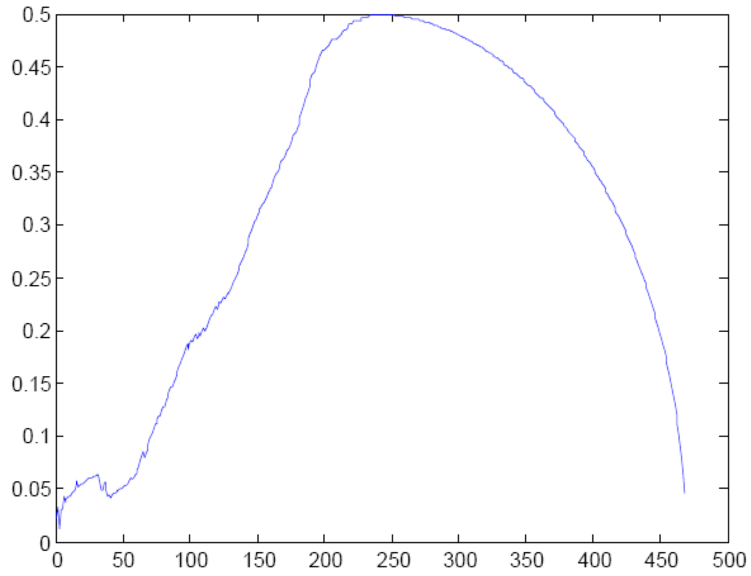

Figure. 3. Key Statistics $\Psi_{l,N-l}(u)$ Dynamics Revealing the Structural Shift Point in US Dollar-Zone Interest Rates Copula.

The observation number can be found on horizontal axis, the statistics value – on the vertical one.

Thus the comovement of US Dollar-zone interest rates tended to simultaneously fall, than to rise. It is closely related to the dynamics of the US Dollar interest rates' principal determinant – the Fed funds rate (cf. Figure 4).

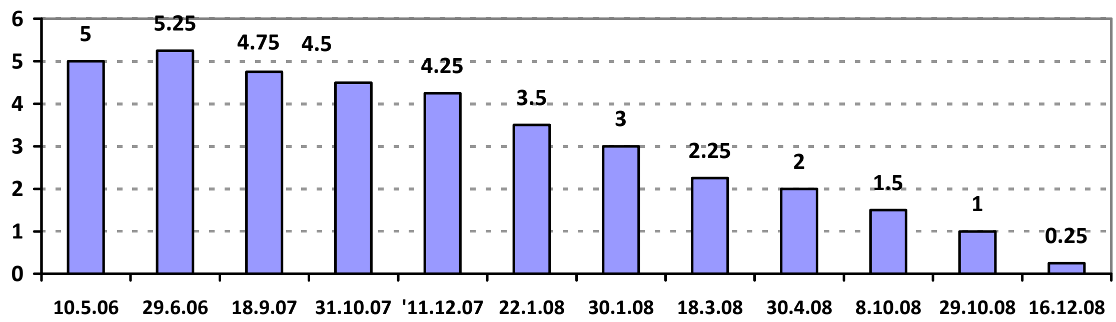

Figure 4. Key Dates of Refinance Rate Changes by the US Federal Reserve.

Horizontal axis marks the time schedule and the vertical one – the refinance rate in percentage points.

Source: www.cbonds.info/index/index_detail/type_id/160/

From the very rise of credit crunch the US Federal Reserve was decreasing the refinance rate aiming at stimulating the market participants. Never-the-less, the two periods (before and after the shift) do differ. The prior period was marked by a comparably greater decrease in refinance rate (from 4.75% to 2%, i.e. by .75%). The consequent – by lesser decrease (from 2% to 0.25%, i.e. by 1.75%).

Therefore we tend to interpret the results obtained as follows. During the first period before the structural shift in July 17, 2008 the market participants were expecting and were in need of fed funds rate decrease. By contrast in the after-the-shift period further decrease was not as desirable and vital as before. That is why Clayton copula was identified to fit the data best for the first period and the Gaussian – for the second.

3. The analysis of Euro-zone interest rate co-dynamics was not as evident, as those with the ruble-zone and USD-zone ones. Never-the-less, the period before and after the estimated structural shift date (September 24, 2008; cf. Figure 5) can be differentiated based on the copula parameters' estimates. For the first 'before-the-shift' period Clayton copula seems to be most relevant to describing the interest rates comovement pattern (based on the maximum likelihood function value). For the second 'after-the-shift' period there is no strictly preferable copula found as IFM method was being unable to complete the calculation because of the infinite value of all copulas' likelihood functions (it was obtained only for the Clayton one). Not to mention that the Clayton copula parameter estimated value decreased three times using ITAU method from 75.27 to 22.16. It can be interpreted as the decrease in the tightness of interest rates comovement. Never-the-less, the parameter significance has grown three times from 1.6 to 4.81 (on the opposite, for the Student's t copula its parameter's significance decreased four times using ITAU method). Thus we may conclude that disregarding the decrease in the measure of associativity of euro-zone interest rates, Clayton copula has revealed itself to be more adequate to describing the joint rates dynamics.

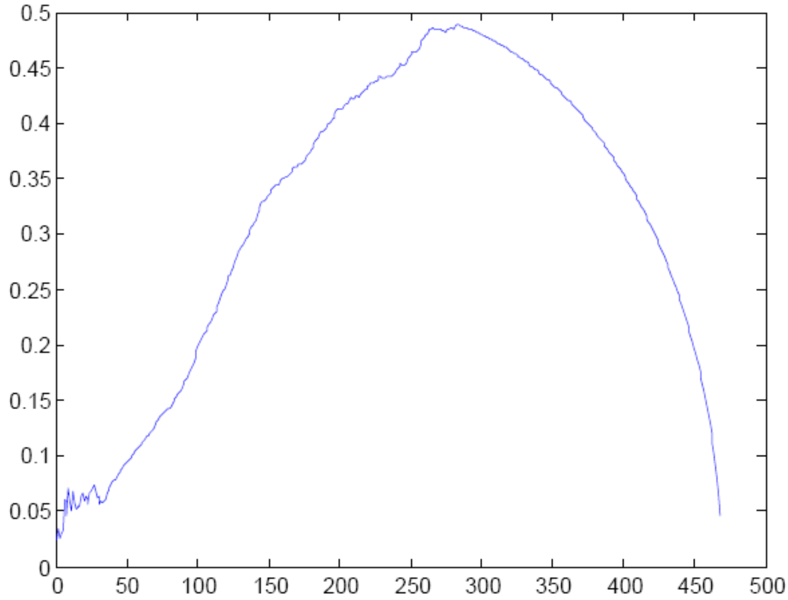

Figure. 5. Key Statistics $\Psi_{l,N-l}(u)$ Dynamics Revealing the Structural Shift Point in Euro-Zone Interest Rates Copula.

The observation number can be found on horizontal axis, the statistics value – on the vertical one.

To add economic interpretation of the structural shift moment estimate we have to trace the European Central Bank's (ECB) policy towards the refinance rate (cf. Figure 6).

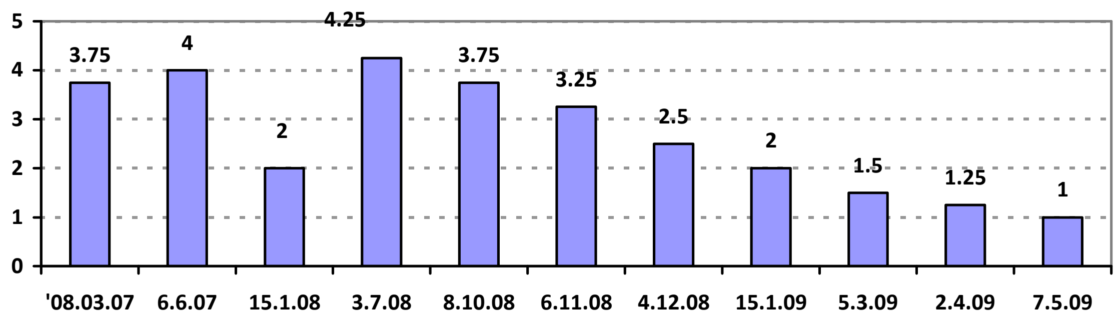

Figure 6. Key Dates of Refinance Rate Changes by the European Central Bank.

Horizontal axis marks the time schedule and the vertical one – the refinance rate in percentage points.

Source: www.cbonds.info/index/index_detail/type_id/161/

During all the 'before-the-shift' period (before September 24, 2008) the ECB was perpetually increasing the refinance rate in order to tighten inflationary pressures. But it was on October 8, 2008 when the ECB first had cut the rate by 0.5%. Here-and-after the ECB continued rate-cutting process to arrive at 1% on May 5, 2009, i.e. overall decrease by 3.25%. We argue that it was this downward movement that might be best described by the Clayton copula, than that of the Gaussian.

To conclude we would like to comment on the findings presented above and those to be found in the previous research. For example the paper [Penikas, Simakova, Titova (2009)] stated that Gumbel copula best fits the interest rate joint distribution. Current research has revealed Gaussian and Clayton copulas to be the best candidates. Aiming at understanding the differences three major issues must be accounted for .

Primarily, the earlier research did not consider the structural shift thus providing biased parameter estimates.

Secondly, as was presented above, the refinance rate is an important determinant of interest rates comovement pattern. The previous research dealt with the period January 17, 2007 to November 17, 2008 when the CB RF was constantly up-shifting the rate. To remind Gumbel copula is characterised by the positive upper tail dependence, i.e. significant probability of simultanious realisations of high quantiles of all the copula components.

Thirdly, the previous research data set contained five time series of Ruble-zone interest rates, the actual one – seven for each of the three currencies chosen. Not to mention Archimedean copulas (incl. Gumbel and Clayton) has the drawback of decreasing the parameter estimate significance given the rise in the copula dimension. Another problem is that the comovement pattern is characterised by the sole parameter (elliptical copulas additionally take in account the covariance matrix). To solve the multidimensionality problem hierarchical copulas might be constructed (as proposed in e.g. [Savu, Trede (2006)]).

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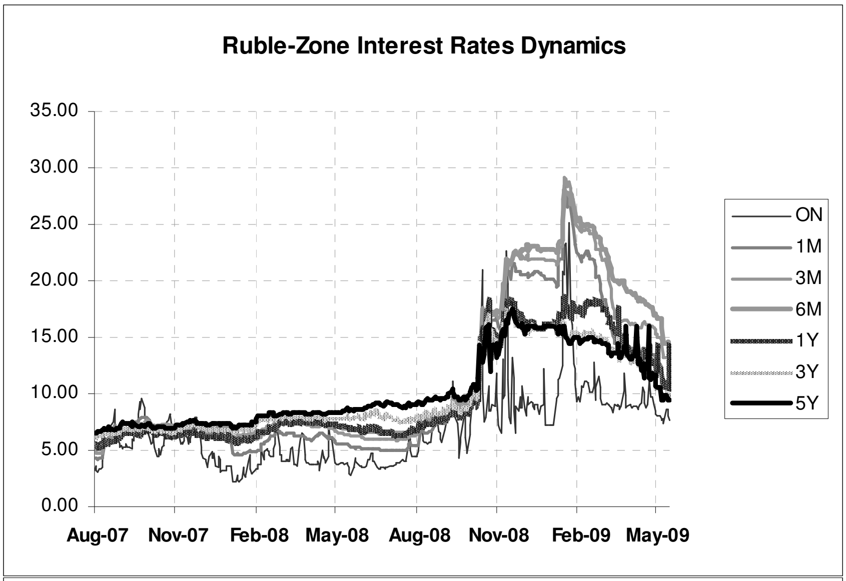
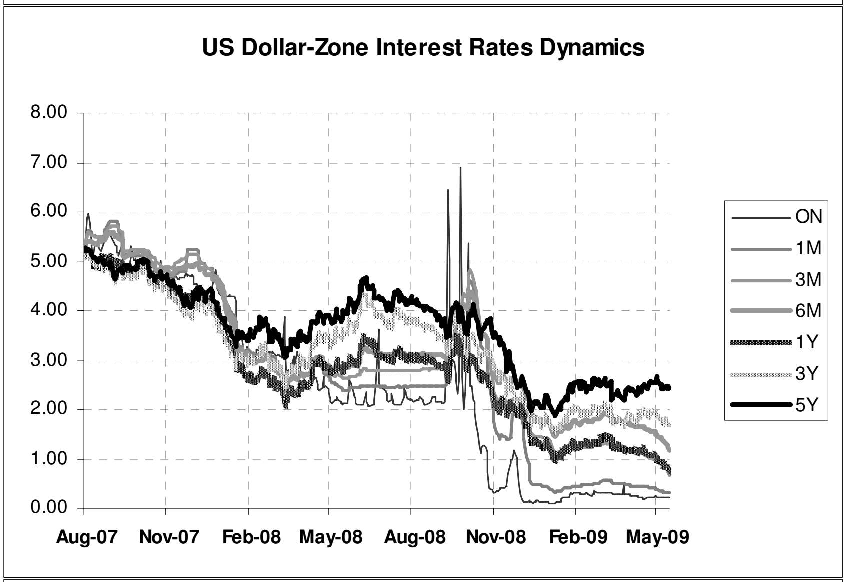
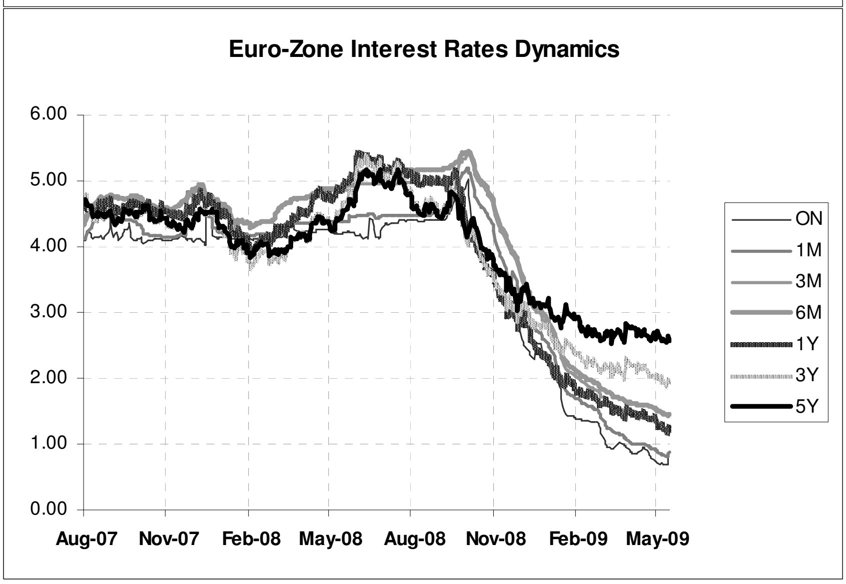
Annex 1. Interest Rate Time Series Dynamics' Graphical Representation.







Notes: The maturities presented are as follows: ON - overnight, 1M - 1 month, 3M - 3 months, 6M - 6 months, 1Y - 1 year, 3Y - 3 years, 5Y - 5 years.

Annex 2.1. Copula Parameter Estimates Using Inference-For-Margins (IFM) Method for The Daily Logreturns of Interest Rates

Copula		Ruble-zone Rates Period			US Dollar-zone Rates Period			Euro-zone Rates Period		
		Before-the-shift	After-the-shift	Pooled Data	Before-the-shift	After-the-shift	Pooled Data	Before-the-shift	After-the-shift	Pooled Data
Gumbel	Parameter	4,72	2,42	4,09	2,20	2,79	2,52	1,65		
	Z-statistics	61,09	37,53	72,01	48,84	48,01	67,96	60,34		
	ML	2 460,88	598,12	3 100,43	911,95	1 019,92	1 959,64	1 093,70	infinite	infinite
	UTDI	84,2%	66,8%	81,5%	62,9%	71,8%	68,3%	47,9%		
	LTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
Clayton	Parameter	11,23	6,08	10,12	6,26	6,46	6,42	5,21	2,71	4,31
	Z-statistics	58,12	23,26	65,23	31,60	34,09	46,47	28,77	17,03	31,66
	ML	2 524,54	584,28	3 095,12	1 093,70	991,31	2 082,37	1 420,04	763,85	2 140,48
	UTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
	LTDI	94,0%	89,2%	93,4%	89,5%	89,8%	89,8%	87,5%	77,4%	85,1%
Student's t 1 d.o.f	Parameter	0,96	0,72	0,93	0,51	0,71		0,56		
	Z-statistics	325,61	29,87	236,42	21,80	37,82		27,05		
	ML	2 880,78	538,74	3 488,68	426,46	880,44	infinite	925,83	infinite	infinite
	UTDI	85,1%	62,5%	81,2%	50,6%	62,1%		53,2%		
	LTDI	85,1%	62,5%	81,2%	50,6%	62,1%		53,2%		
Student's t 5 d.o.f	Parameter	0,98	0,88	0,97	0,78	0,88		0,75		
	Z-statistics	943,80	98,44	751,65	70,05	138,54		66,66		
	ML	2 869,21	625,04	3 527,30	829,41	1 003,76	infinite	1 244,64	infinite	infinite
	UTDI	80,6%	55,8%	76,5%	42,6%	56,5%		39,4%		
	LTDI	80,6%	55,8%	76,5%	42,6%	56,5%		39,4%		
Student's t 10 d.o.f	Parameter	0,98	0,90	0,97 **	0,82	0,90		0,78		
	Z-statistics	1 116,75	130,33	875,49 **	96,05	187,49		84,43		
	ML	2 854,39	636,73	3529,88 **	871,51	1 030,76	infinite	1 262,99	infinite	infinite
	UTDI	73,9%	45,2%	71,9%	31,3%	46,5%		26,5%		
	LTDI	73,9%	45,2%	71,9%	31,3%	46,5%		26,5%		
Guassian	Parameter	0,98	0,91	0,97	0,84 *	0,92		0,79 *		
	Z-statistics	1 374,01	187,98	1 199,99	136,30 *	279,67		112,8 *		
	ML	2 803,58	652,71	3 507,54	908,0962 *	1 074,85	infinite	1282,31 *	infinite	infinite

Notes: ML - the value of the maximum likelihood function; UTDI and LTDI stand for the vaule of the upper and lower tail dependence indexes; * - the estimate for the Student's t copula with 100 d.o.f. was taken as the first proxy for the Gaussian copula estimate; ** - the maximum ML value presented belongs to the 8 d.o.f. Student's t copula case; The yellow colour marks the best copula chosen based on the maximum ML value.

Annex 2.2. Copula Parameter Estimates Using ITAU Method for The Rank-Transformed Probabilities of Daily Logreturns of Interest Rates.

Copula		Ruble-zone Rates Period			US Dollar-zone Rates Period			Euro-zone Rates Period		
		Before-the-shift	After-the-shift	Pooled Data	Before-the-shift	After-the-shift	Pooled Data	Before-the-shift	After-the-shift	Pooled Data
Gumbel	Parameter	49,92	14,16	52,93	36,07	11,03	42,53	38,63	12,08	45,09
	Z-statistics	3,32	5,68	5,65	1,83	4,98	2,05	1,64	5,25	1,83
	UTDI	98,6%	95,0%	98,7%	98,1%	93,5%	98,4%	98,2%	94,1%	98,5%
	LTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
Clayton	Parameter	97,85	26,33	103,86	70,14	20,07	83,07	75,27	22,16	88,19
	Z-statistics	3,25	5,28	5,55	1,78	4,53	2,00	1,60	4,81	1,78
	UTDI	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%	0,0%
	LTDI	99,3%	97,4%	99,3%	99,0%	96,6%	99,2%	99,1%	96,9%	99,2%
Student's t 1 d.o.f	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352
	UTDI	96,2%	93,0%	97,0%	97,6%	90,8%	98,0%	97,6%	90,8%	98,1%
	LTDI	96,2%	93,0%	97,0%	97,6%	90,8%	98,0%	97,6%	90,8%	98,1%
Student's t 5 d.o.f	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352
	UTDI	92,9%	86,9%	94,3%	95,6%	82,9%	96,3%	95,4%	82,8%	96,4%
	LTDI	92,9%	86,9%	94,3%	95,6%	82,9%	96,3%	95,4%	82,8%	96,4%
Student's t 10 d.o.f	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352
	UTDI	90,2%	82,0%	92,2%	93,9%	76,6%	94,9%	93,7%	76,4%	95,0%
	LTDI	90,2%	82,0%	92,2%	93,9%	76,6%	94,9%	93,7%	76,4%	95,0%
Guassian	Parameter	0,9971	0,9901	0,9982	0,9989	0,9832	0,9992	0,9988	0,9830	0,9993
	Z-statistics	1 389	362	3 148	884	188	1 403	832	216	1 352

Notes: UTDI and LTDI stand for the vaule of the upper and lower tail dependence indexes;