WEYL MODULES OVER MULTIVARIABLE CURRENTS

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Let $\mathfrak{g}$ be a reductive Lie algebra. By $\mathfrak{b}$ and $\mathfrak{h}$ denote its Borel and Cartan subalgebras, by R and R_+ denote the sets of roots and positive roots, by α_i, $i = 1 \dots \mathrm{rk}(\mathfrak{g})$, denote simple roots, by ω_i denote simple weights, by Q and Q^+ denote the root lattice and its positive cone, by P and P^+ denote the weight lattice and the dominant weight cone.

For $\mathfrak{g} = \mathfrak{g}l_r$ by ε_i, $i = 1 \dots r$, denote the standard basis in the weight space. Then partitions $\xi = (\xi_1 \geq \cdots \geq \xi_r)$ correspond to dominant weights $\sum \xi_i \varepsilon_i$ of $\mathfrak{g}l_r$. For each ξ by ξ^t denote the *transposed* partition $\xi^t_j = |\{i | \xi_i \geq j\}|$ corresponding to the reflected Young diagram.

For a representation U by U^μ denote the corresponding weight space, that is the common eigenspace of $\mathfrak{h}$ where $h \in \mathfrak{h}$ acts by the scalar $\mu(h)$.

In addition let A be a commutative finitely-generated algebra with a unit 1 and a co-unit (augmentation) $\epsilon : A \to \mathbb{C}$. Note that A can be treated as the algebra of functions on an affine scheme M, so ϵ is the evaluation at a certain closed point p of M.

By A_ϵ denote the augmentation ideal, that is, the kernel of ϵ. Note that the infinitesimal neighborhood spaces A/A_ϵ^n are always finite-dimensional.

Here we study finite-dimensional representations of $\mathfrak{g} \otimes A$, that is, the Lie algebra of $\mathfrak{g}$-valued functions on the scheme M.

Definition 1. *Let U be a representation of $\mathfrak{g} \otimes A$. We say that a vector $v_\lambda \in U$ is a* highest weight vector *of weight $\lambda \in \mathfrak{h}^*$ if*

$$(g \otimes P)v_\lambda = \lambda(g)\epsilon(P)v_\lambda \quad \textit{for } g \in \mathfrak{b},\ P \in A.$$

Theorem 1. [FL2]

(i) *There exists a universal finite-dimensional module $W^A_\epsilon(\lambda)$, such that any finite-dimensional module generated by v_λ is a quotient of $W^A_\epsilon(\lambda)$.*
(ii) *We have $W^A_\epsilon(\lambda) \neq 0$ if and only if $\lambda \in P^+$.*
(iii) *For any λ there exists N such that $\mathfrak{g} \otimes A_\epsilon^N$ acts on $W^A_\epsilon(\lambda)$ by zero.*
(iv) *We have $W^A_\epsilon(\lambda) \cong \oplus_\mu W^A_\epsilon(\lambda)^\mu$.*
(v) *Any finite-dimensional module, generated by a common eigenvector of $\mathfrak{b} \otimes A$, is a quotient of $W^A_{\epsilon_1}(\lambda^1) \otimes \cdots \otimes W^A_{\epsilon_k}(\lambda^k)$ for some λ^i, ϵ_i.*

Definition 2. [CP1][FL2] *The module $W^A_\epsilon(\lambda)$ is called* Weyl module.

For $A = \mathbb{C}[x^1, \dots, x^d]$ and $\epsilon(P) = P(0)$ let us denote it by $W^d(\lambda)$ as well.

Polynomiality Conjecture

Conjecture 1. *Let $\lambda = \sum \lambda_i \omega_i$. Fix $\mu = \sum \mu^i \alpha_i$.*

(i) *For $d > 0$ we have $\dim W^d(\lambda)^{\lambda-\mu}$ is a polynomial in $\lambda_1, \dots, \lambda_{\mathrm{rk}(\mathfrak{g})}$ of degree $\mu^i d$ in the variable λ_i, $i = 1 \dots \mathrm{rk}(\mathfrak{g})$.*

(ii) *Even for a singular point there exists ν, such that for $\lambda - \nu \in P^+$ we have $\dim W^A_\epsilon(\lambda)^{\lambda-\mu}$ is a polynomial in $\lambda_1, \dots, \lambda_{\mathrm{rk}(\mathfrak{g})}$ of degree $\mu^i d$ in the variable λ_i, $i = 1 \dots \mathrm{rk}(\mathfrak{g})$, where $d = \dim M$.*

Example 1. *Let $\mathfrak{g} = sl_2$, $\mu = \alpha$. Then $W^A_\epsilon(n\omega)^{n\omega-\alpha} \cong A/A^n_\epsilon$ (see [FL2]). And for a big enough n the integers $\dim A/A^n_\epsilon - \dim A/A^{n+1}_\epsilon = \dim A^n_\epsilon/A^{n+1}_\epsilon$ are values of the Hilbert polynomial for the completion of A at ϵ.*

Now let $\mathfrak{g} = sl_2$, $A = \mathbb{C}[x^1, \dots, x^d]$. The next table is organized as follows. First column indicates the number of variables. Second column contains the dimensions of Weyl modules together with the number of the sequence in the On-Line Encyclopedia of Integer Sequences [Sl]. In the third column there are the dimensions of weight spaces organized as polynomials. Last column provides a reference.

d	$\dim W_{n\omega}$	$\dim W^{n\omega-k\alpha}_{n\omega} = P(n)/P(k)$	reference
0	$n+1$	1 for $0 \le k \le n$	well known
1	2^n	**Binomial Coefficients** $\begin{bmatrix} n \\ k \end{bmatrix}$ $P(n) = \prod_{i=0}^{k-1} (n-i)$	[CP2]
2	**Catalan Number** $\begin{bmatrix} 2n+2 \\ n \end{bmatrix} \Big/ (n+1)$ A000108	**Narayana Numbers** $\begin{bmatrix} n+1 \\ k \end{bmatrix} \begin{bmatrix} n+1 \\ n-k \end{bmatrix} \Big/ (n+1)$ $P(n) = \prod_{i=-1}^{k-2} (n-i) \prod_{i=0}^{k-1} (n-i)$	[FL2] using [H2]
3	$\begin{bmatrix} 3n+3 \\ n \end{bmatrix} \Big/ \begin{bmatrix} n+2 \\ 2 \end{bmatrix}$ A000139	$\dfrac{\begin{bmatrix} n+k+2 \\ 2k+1 \end{bmatrix} \begin{bmatrix} 2n-k+1 \\ k \end{bmatrix}}{(k+1)(n+k+2)}$ $P(n) = \prod_{i=2}^{2k+1} (n+i-k) \prod_{i=-1}^{k-2} (2n-i-k)$	Conjec- ture
≥ 4	Huge prime factors	$P(n)$ is not factorizable $/\mathbb{Q}$	not known

Raw data for $\dim W_n$, $d = 4$: 1, 2, 7, 32, 172, 1030, 6664, 45694, 327812, 2438464, 18684534, 146751246, 1176973585, 9610391476, 79701687671, 670043134302, 5701013562013, 49027112699574, 425664727764880,

Relation to Diagonal Harmonics

First let us describe the classical *Schur-Weyl duality.* Let V be the r-dimensional vector representation of $\mathfrak{gl}_r$. For the classical settings note that $V^{\otimes n}$ inherits both the action of the Lie algebra $\mathfrak{gl}_r$ and the symmetric group Σ_n, and that these actions commute. Decomposition of this bi-module provides a reciprocity between the representations of these objects and it can be described as the following functor.

Definition 3. *Let π be a representation of Σ_n. By $\nabla_n^r(\pi)$ denote the representation $\left(V^{\otimes n}\otimes\pi\right)^{\Sigma_n}$ of $\mathfrak{gl}_r$.*

Remark 1. *Note that at the level of characters ∇_n^r is known as the* Frobenius characteristic map*.*

This functor can be generalized for our settings as follows.

Definition 4. *For an associative algebra A and an integer n by* wreath product *$\Sigma_n(A)$ denote the associative algebra, generated by $A^{\otimes n}$ and the elements of Σ_n under the relation*

$$(a_1\otimes\cdots\otimes a_n)\cdot\sigma=\sigma\cdot(a_{\sigma(1)}\otimes\cdots\otimes a_{\sigma(n)}) \tag{1}$$

and identification of the unit $e\in\Sigma_n$ with $1\otimes\cdots\otimes 1\in A^{\otimes n}$.

In other words, we have $\Sigma_n(A)\cong\mathbb{C}[\Sigma_n]\otimes A^{\otimes n}$ as a vector space with the multiplication given by the relation (1). Let us show that $\nabla_n^r(\pi)$ is indeed a functor from representations of $\Sigma_n(A)$ to the representations of the Lie algebra $\mathfrak{gl}_r\otimes A$. Note that for this observation A is not necessarily commutative, and $\mathfrak{gl}_r\otimes A$ can be defined as
the space of matrices with entries from A with the usual commutator.

Proposition 1. *Let π be a representation of $\Sigma_n(A)$. Then $\mathfrak{gl}_r\otimes A$ acts on $\nabla_n^r(\pi)$ as follows*

$$(g\otimes P)(u_1\otimes\cdots\otimes u_n\otimes v)=\sum_{k=1}^{n}u_1\otimes\cdots\otimes g(u_k)\otimes\cdots\otimes u_n\otimes\iota_k(P)v, \tag{2}$$

where ι_k is the inclusion of A into $A^{\otimes n}$ as the k-th factor $1\otimes\cdots\otimes A\otimes\cdots\otimes 1$.

Remark 2. *There are quantum analogs of this functor for a small number of variables. In the classical settings representations of the type A Hecke algebra corresponds to representations of $U_q(\mathfrak{gl}_r)$. In the one-variable settings this happens with the type A affine Hecke algebra and the affine quantum group. Finally for two variables there is a correspondence between double affine objects (see* [VV], [Gu]*). In these cases our correspondence is the classical limit of these constructions.*

Definition 5. *Let as above A be a commutative finitely-generated algebra with an augmentation ϵ. Note that ϵ can be extended as an augmentation of $S^n(A)$ in the natural way*

$$\epsilon(a_1 \circ \cdots \circ a_n) = \epsilon(a_1) \ldots \epsilon(a_n).$$

Introduce the space of diagonal harmonics

$$DH_n(A) = A^{\otimes n} \Big/ S^n(A)_\epsilon \cdot A^{\otimes n}$$

as a representation of $\Sigma_n(A)$.

This representation of Σ_n is known for $A = \mathbb{C}[x]$ due to the classical result of Chevalley, for $A = \mathbb{C}[x, y]$ due to [H2], for $A = \mathbb{C}[x, y]/xy$ due to [Ku]. Note that the table below proposes a description of this space for $A = \mathbb{C}[x, y, z]$.

Theorem 2. [FL2] *For $\mathfrak{g} = sl_r$ we have*

$$W_\epsilon^A(n\omega_1) \cong \nabla_n^r \left(DH_n(A)\right).$$

Note that by taking a big enough r we can reconstruct the action of Σ_n on $DH_n(A)$ from the action of $\mathfrak{gl}_r \otimes 1$ on $W_\epsilon^A(n\omega_1)$. And for $A = \mathbb{C}[x^1, \ldots, x^d]$, $d < 4$, the character of $W_\epsilon^A(n\omega_1)$ can be obtained from the following table.

d	1	2	3
$\dim W_{n\omega_1}$	$\begin{bmatrix} n+r \\ r \end{bmatrix}$	$\begin{bmatrix} r(n+1) \\ n \end{bmatrix} \Big/ (n+1)$	$r \begin{bmatrix} (2r-1)(n+1) \\ n-1 \end{bmatrix} \Big/ \begin{bmatrix} n+1 \\ 2 \end{bmatrix}$
$\dim W_{n\omega_1}^{k_1\varepsilon_1 + \cdots + k_r\varepsilon_r}$	$\frac{n!}{k_1! \ldots k_r!}$	$\begin{bmatrix} n+1 \\ k_1 \end{bmatrix} \begin{bmatrix} n+1 \\ k_2 \end{bmatrix} \ldots \begin{bmatrix} n+1 \\ k_r \end{bmatrix} \Big/ (n+1)$	$2^r (n+1)^{r-2} \prod_{i=1}^{r} \frac{\begin{bmatrix} 2(n+1)-k_i \\ k_i \end{bmatrix}}{2(n+1)-k_i}$
$\dim DH_n(A)$	$n!$	$(n+1)^{n-1}$	$2^n (n+1)^{n-2}$
$\chi_{k_1,\ldots,k_m}$	$n!\delta_{mn}$	$(n+1)^{m-1}$	$(n+1)^{m-2} \prod_{i=1}^{m} \begin{bmatrix} 2k_i \\ k_i \end{bmatrix}$

Here $\chi_{k_1,\ldots,k_m} = \mathrm{Sign}(\sigma)\mathrm{tr}(\sigma)$ on $DH_n(A)$, where $\sigma \in \Sigma_n$ is the element consisting of m cycles of length $k_1, \ldots, k_m$.

Binomial case $d = 1$

The following theorem first appeared as a conjecture and was proved for $\mathfrak{sl}_2$ in [CP2], then the proof was completed for $\mathfrak{sl}_r$ in [CL], for simply-laced Lie algebra in [FoLi] and recently H.Nakajima noticed that the general case can be obtained by combining the results of [BN], [K1], [K2], [N1], [N2].

Theorem 3. [CP2][CL][FoLi][N2] *We have*

$$W^1(\lambda) \cong \bigotimes_{i=1}^{\mathrm{rk}(\mathfrak{g})} W^1(\omega_i)^{\otimes \lambda_i}$$

as $\mathfrak{g} \otimes 1$-modules.

Let us present a construction for these modules based on the fusion product from [FL1]. Suppose that $V_1, \ldots, V_n$ — cyclic modules over $\mathfrak{g} \otimes \mathbb{C}[x]$ with cyclic vectors $v_1, \ldots, v_n$. Assume that there exists an integer N such that the ideal $\mathfrak{g} \otimes x^N \mathbb{C}[x]$ acts on these modules by zero.

For $z \in \mathbb{C}$ by $V_i(z)$ denote the representation $i_z^* V_i$, where i_z is the automorphism of $\mathfrak{g} \otimes \mathbb{C}[x]$ sending x to $x - z$.

Proposition 2. *Suppose that $z_i \neq z_j$ for $1 \leq i \neq j \leq n$. Then the vector $v = v_1 \otimes \cdots \otimes v_n$ is cyclic in $V_1(z_1) \otimes \cdots \otimes V_n(z_n)$.*

The grading on $\mathfrak{g}\otimes\mathbb{C}[x]$ by degree of x induces the natural grading on $U(\mathfrak{g}\otimes\mathbb{C}[x])$ and the filtration on our cyclic module as follows:

$$F^i \left(V_1(z_1) \otimes \cdots \otimes V_n(z_n)\right) = U(\mathfrak{g} \otimes \mathbb{C}[x])^{\leq i} v.$$

Definition 6. *By* fusion product $V_1 * \cdots * v_n$ *denote the cyclic representation* $\mathrm{gr}\, F^i \left(V_1(z_1) \otimes \cdots \otimes V_n(z_n)\right)$.

This is a big question in each case whether this module depends on the points z_i, and in which generality this product is associative. Nevertheless, in our generality both follows from Theorem 3.

Corollary 1. *We have*

$$W^1(\lambda) \cong W^1(\omega_1)^{*\lambda_1} * \cdots * W^1(\omega_r)^{*\lambda_r},$$

in particular, the right hand side is independent on the choice of points.

Corollary 2. *We have $W^1(\lambda) * W^1(\mu) \cong W^1(\lambda+\mu)$, in particular, fusion product of Weyl modules is associative.*

Catalan case $d = 2$

From here fix $\mathfrak{g} = \mathfrak{g}l_r$. Let us show that the character combinatorics of two-variable Weyl modules is pretty similar to the Catalan number combinatorics.

Consider the following elementary problem 'from a real life'. Imagine we have n cars and exactly n parking spaces along the road. Suppose that they are distributed into a sequence of N parking lots with capacities $m_1, \dots, m_N$. Now let each car has its own preferred lot, so we have a set-theoretical function $f : \{1, \dots, n\} \to \{1, \dots, N\}$. Now the spaces are distributed as follows. When i-th car appears on the road, it goes directly to $f(i)$-th lot, then parks there, if it is not full, or at the first available lot after $f(i)$ otherwise.

Proposition 3. *Each car find its space if and only if we have*

$$\left|f^{-1}(\{1, \dots, s\})\right| \geq m_1 + \dots + m_s, \qquad s = 1 \dots N.$$

Proof. Follows by induction on the number of cars. After the first car find the space at the k-th lot we consider all the other cars with the same preference function, and capacities $m_1, \dots, m_k - 1, \dots, m_N$. □

Note that the answer does not depend on the order of the cars. It motivates the following definition.

Definition 7. [PP][PS][Y] *For a vector* $\mathbf{m} = (m_1, \dots, m_N)$ *set* $|\mathbf{m}| = m_1 + \dots + m_N$ *and introduce the set of* generalized parking functions

$$PF(\mathbf{m}) = \{f : \{1, \dots, |\mathbf{m}|\} \to \mathbb{N} \mid \left|f^{-1}(\{1, \dots, s\})\right| \geq m_1 + \dots + m_s,\ 1 \leq s \leq N\}$$

with the action of $\Sigma_{|\mathbf{m}|}$ *by permutations of arguments.*

Definition 8. *Introduce the representation* $\mathbb{C}\mathrm{PF}(\mathbf{m})$ *of* $\Sigma_{|\mathbf{m}|}$ *as the space with basis* $PF(\mathbf{m})$ *and the corresponding action of the symmetric group.*

Remark 3. *It is shown in* [H2] *that we have* $DH_n(\mathbb{C}[x, y]) \cong \mathbb{C}\mathrm{PF}((n)^t) \otimes \mathrm{Sign}$ *as representations of* Σ_n*, where* Sign *is the one-dimensional sign representation of* Σ_n.

Note that the whole set of functions from $\{1, \dots, n\}$ to $\mathbb{N}$ enumerates the monomial basis of $\mathbb{C}[t_1, \dots, t_n] \cong \mathbb{C}[t]^{\otimes n}$, namely each f corresponds to $t_1^{f(1)-1} \dots t_n^{f(n)-1}$. So if A acts on $\mathbb{C}[t]$ not increasing the degree of polynomials then $\Sigma_{|\mathbf{m}|}(A)$ acts on $\mathbb{C}\mathrm{PF}(\mathbf{m})$. Note that for this end A is not necessarily commutative.

Let $\mathcal{A} = \mathbb{C}\langle X, Y\rangle$ be the associative algebra with the relation $XY - YX = X$. Then $\mathcal{A}$ acts on $\mathbb{C}[t]$ as follows: X acts as $\partial/\partial t$, Y acts as $t\partial/\partial t$. As this action does not increase the degree, the wreath product $\Sigma_{|\mathbf{m}|}(\mathcal{A})$ acts on $\mathbb{C}\mathrm{PF}(\mathbf{m})$. Note that $\mathcal{A}$ is a PBW-algebra, namely it is a flat deformation of the commutative algebra $\mathbb{C}[x, y]$, and we expect that the Weyl modules can be deformed to the parking function representations.

Theorem 4. [FL3] *For a partition ξ there is a filtration on $\mathbb{C}\langle X, Y\rangle$-module $\nabla_n^r(\mathbb{C}\mathrm{PF}(\xi^t)\otimes \mathrm{Sign})$ whose adjoint graded factor is a quotient of $W^2(\xi)$.*

Corollary 3. *There is an inclusion $\nabla_n^r(\mathbb{C}\mathrm{PF}(\xi^t)\otimes \mathrm{Sign})$ to $W^2(\xi)$ as representations of $\mathfrak{gl}_r\otimes 1$. Moreover, for $\xi=(n)$ we have $W^2(\xi)\cong \nabla_n^r(\mathbb{C}\mathrm{PF}(\xi^t)\otimes \mathrm{Sign})$.*

Remark 4. *Note that the main conjecture of* [FL3] *that these representations are indeed isomorphic.*

Remark 5. *Let $\mathbb{C}[x,y]_q$ be the associative algebra of q-polynomials, that is the algebra with relation $xy-qyx=0$. Then $\mathbb{C}[x,y]_q$ acts on $\mathbb{C}[t]$ not increasing the degree, namely x acts by $\partial/\partial t$ and y acts by $q^{t\partial/\partial t}$ (that is $y\cdot t^k=q^kt^k$). So $\Sigma_{|\mathbf{m}|}(\mathbb{C}[x,y]_q)$ acts on $\mathbb{C}\mathrm{PF}(\mathbf{m})$ and $\mathfrak{gl}_r\otimes\mathbb{C}[x,y]_q$ acts on $\nabla_n^r(\mathbb{C}\mathrm{PF}(\mathbf{m}))$.*

Remark 6. *For $\xi=(n)$ the space of parking function can be quantized to a representation of the double affine Hecke algebra (see* [Ch], [G]*), so Weyl modules can be quantized to a representation of the corresponding toroidal quantum group. Unfortunately, for other ξ there are no representations of these quantum algebras with suitable characters.*

Let us present a quadratic relation for characters of Weyl modules that generalizes the recurrence relation for Calatan numbers.

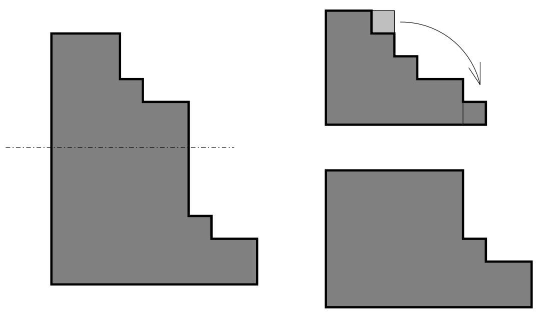
Definition 9. *Introduce a symmetric function $c_\xi=\mathrm{ch}\nabla_n^r(\mathbb{C}\mathrm{PF}(\xi^t)\otimes\mathrm{Sign})$, that is the Frobenius characteristic map of $\mathbb{C}\mathrm{PF}(\xi^t)\otimes\mathrm{Sign}$.*

Theorem 5. *We have*

$$c_\xi=\sum_{i=\xi_r}^{\xi_1-1} c_{(\xi_{>i})'}\cdot c_{\xi_{\leq i}},$$

where $(\xi_{>i})_j=\max(\xi_j-i,0)$, $(\xi_{\leq i})_j=\min(\xi_j,i)$,

$$(\xi')^t_j=\begin{cases}\xi^t_j+1 & 1=j<\xi_1\\ \xi^t_j-1 & 1<j=\xi_1\\ \xi^t_j & \textit{otherwise}\end{cases}$$



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