Introduction to Social Choice Theory I: Axiomatic Results for Two Alternatives

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A long tradition

The aim of social choice theory is to provide a mathematical analysis of collective decision mechanism (e.g. voting rules), in order to unveil their normative properties.

There is a long tradition of comparing the voting rules between them. At the end of 18^{th} century, Condorcet and Borda were the first, in the French Royal Academy of Sciences, to propose mathematical and logical arguments

The modern theory is due to Kenneth Arrow, who published his book *Social Choice and Individual Values* in 1952. His main result states that there is no "perfect" voting rule; He proved that no voting mechanism can satisfy simultaneously four appealing normative conditions, as soon as there are more then two voters and three candidates.

Program of the day

Today, I will not present his result, but introduce Social Choice Theory in the simple case where there is only two candidates.

Just after the publication of Arrow's book, Kenneth May presented the first characterization of majority voting.

Then, I will discuss some extension, like Mac Garvey's theorem, the existence of cycles and Black's single peaked condition.

If time permits, I will put my own research on indirect voting into perspective.

The simplest voting model

$X = \{a, b\}$, the (finite) set of candidates for an election.

$N = \{1, 2, \ldots, i, \ldots n\}$ is the finite set of voters.

The preference of voter i on X will be depicted by a binary relation R_i on X. $x\,R_i\,y$ means that *x is at least as good as alternative y for voter i*.
Indifference:

$$x\,R_i\,y \textit{ and } y\,R_i\,x \;\Rightarrow\; x\,I_i\,y$$

Strict Preference:

$$x\,R_i\,y \textit{ and } \neg(y\,R_i\,x) \;\Rightarrow\; x\,P_i\,y$$

We will not consider here the case of incomparibility.

Voting Rules

Let $\mathcal{B}$, the set of all binary relations on X, the set of alternatives. Let $\pi = (R_1, R_2, \ldots, R_i, \ldots, R_n)$ be an n-tuple of binary relations, one for each individual. $\pi \in \mathcal{B}^n$. We call π a preference profile.

$$\pi = (a\, P_1\, b,\ a\, I_2\, b,\ b\, P_3\, a,\ b\, P_4\, a,\ a\, I_5\, b,\ )$$

Aggregation Procedure: An aggregation procedure for the society N and a set of alternatives X is a mapping from $\mathcal{B}^\setminus$ into $\mathcal{B}$.

$$\begin{aligned} f\ :\ \mathcal{B}^n &\ \rightarrow\ \mathcal{B} \\ \pi = (R_1, \ldots R_n) &\ \rightarrow\ R\ =\ f(\pi) \end{aligned}$$

Many rules can be proposed, even for two alternatives:

- Dictatorship, veto rules,
- Majority rule, absolute or relative, Super majority rules
- Voting into jurisdictions
- inventing complex cascades of committees, etc.

The majority rule

Majority rule. The majority rule is the aggregation function f such that, for all profile $\pi \in \mathcal{B}^n$,

$$x\, R(\pi)\, y \;\Leftrightarrow \#\{i \in N :\; x\, R_i\, y\} > \#\{i \in N :\; y\, R_i\, x\}$$

In our example:

$$\#\{i \in N :\; a\, R_i\, b\} = 3$$

$$\#\{i \in N :\; b\, R_i\, a\} = 4$$

Hence, collectively, $b\, R(\pi)\, a$ and $\neg(a\, R(\pi)\, b)$, thus, we get $a\, P(\pi)\, b$.

Anonymity

Let σ be a permutation on N, and Σ the set of all possible permutations on N. For any profile $\pi \in \mathcal{B}^n$, any permutation $\sigma \in \Sigma$,

we can define the permuted profile $\pi' = \sigma(\pi)$. It is the profile where the new preference of voter i is the previous preference of voter $\sigma(i)$; the voters have "exchanged" their preferences.

$$\sigma(1) = 3,\ \sigma(2) = 5,\ \sigma(3) = 1,\ \sigma(4) = 4,\ \sigma(5) = 2$$

Hence, we get:

$$\sigma(\pi) = \pi' = \left(b\, P'_1\, a,\ a\, I'_5\, b,\ a\, P'_3\, b,\ b\, P'_4\, a,\ a\, I'_5\, b,\right)$$

Anonymity

Anonymity. A aggregation function f is anonymous if, for all profile $\pi \in \mathcal{B}^n$, and all permutation $\sigma \in \Sigma$,

$$f(\sigma(\pi)) = f(\pi)$$

Anonymity means that no voter plays a specific role in the decision process; all the voters are equal. The majority rule is anonymous.

Sometimes, anonymity is not desirable: the chairman may have a tie breaking right, some countries in international organization may have more weight than others, etc.

Neutrality

Let γ be a permutation on X, the set of alternatives, and Γ, the set of all such permutations on X. For any relation $R \in \mathcal{B}$, we can define $\gamma(R)$ in the following way:

$$\gamma(x)\,\gamma(R)\,\gamma(y) \;\Leftrightarrow\; x\,R\,y$$

For two alternatives there is just one permutation in Γ: $\gamma(a) = b$ and $\gamma(b) = a$.

$$\gamma(\pi)\;\pi' \;=\; \big(b\,P_1'\,a,\; a\,I_2'\,b,\; a\,P_3'\,b,\; a\,P_4'\,b,\; a\,I_5'\,b,\;\big)$$

Neutrality

Neutrality. A aggregation function f is neutral if, for all profile $\pi \in \mathcal{B}^n$, and all permutation $\gamma \in \Gamma$,

$$f(\gamma(\pi)) = \gamma(f(\pi))$$

Neutrality means that, when the names of the candidates are permuted, so does the social ordering; As a consequence, no candidate is favored, and neutrality is a democratic requirement. The majority rule is neutral.

Monotonicity

Monotonicity Let $i \in N$ be a voter, and π and π' two profiles in $\mathcal{B}^n$ such that:

- $\forall i \neq j,\ R_j = R'_j$
- $(x\, I_i\, y \text{ and } x\, P'_i\, y)$ or $(y\, P_i\, x \text{ and } x\, R'_i\, y)$

Then, f is monotonic if:

$$\big(x\, I\, y \;\Rightarrow\; x\, P'\, y\big) \text{ and } \big(x\, P\, y \;\Rightarrow\; x\, P'\, y\big)$$

An improvement of the position of x in the preference should not harm it, and should even favor it in case of a tie. The majority rule is responsive to improvements of the positions of the candidate.

May's Theorem

Theorem: Let $f : \mathcal{B}^n \to \mathcal{B}$, an aggregation function. f is the majority rule if and only iff f is neutral, anonymous and monotonic.

It is easy to prove that the majority rule satisfies the three conditions. They are all necessary:

- Giving extra votes to a voters breaks the anonymity condition.
- Electing the elder in case of a tie breaks the neutrality condition.
- If we need $2/3$ of the votes to declare x strictly better than y, monotonicity is violated.

More alternatives

$X = \{a, b, c, \dots\}$ or $\{x_1, \dots x_m\}$, the (finite) set of candidates for an election.

$N = \{1, 2, \dots, i, \dots n\}$ is the finite set of voters.

The preference of voter i on X will be a weak ordering R_i on X, that is:

- R_i is complete. $\forall x, y \in X$, $x \neg y$, $x \, R_i \, y$ or $y \, R_i \, x$.
- R_i is reflexive. $\forall x \in X$, $x \, R_i \, x$.
- R_i is transitive. $\forall x, y, z \in X$, $x \, R_i \, y$ and $y \, R_i \, z$ imply $x \, R_i \, z$

To summarize, all the individuals are able to rank all the alternatives, from their best preferred to their least preferred, possibly with ties.

Using Majority Rule

$N = \{1, 2, 3, 4, 5\}$,
$X = \{x, y, z, w\}$

$i:$	1	2	3	4	5
	x	w	z	y	xz
$R_i:$	yz	x	y	w	yw
	w	z	x	x	
		y	w	z	

$$x \, P \, z \, P \, y \, P \, w$$

Cycles

$N = \{1, 2, 3, 4, 5, 6, 7\}$,
$X = \{x, y, z, w\}$

$i:$	1	2	3	4	5	6	7
	x	x	w	z	y	z	x
$R_i:$	y	y	x	w	z	w	z
	z	z	y	x	w	x	y
	w	w	z	y	x	y	w

$$x \, P \, y, \;\; y \, P \, z, \;\; z \, P \, w, \;\; w \, P \, x$$

Mac Garvey's theorem

Let T be a complete relation on X. Then, there exist a population of size n and profile of preferences $\pi \in \mathcal{R}^n$ that the results of majority rule for π is exactly T.

For m alternatives in X, we need only $m(m-1)$ to create any outcome.

Possible escapes

Super majority rules (or quota rules) can be used to avoid cycles, at the expense of having more and more indifference among alternatives. (Craven, Ferejohn et Grether)

Another route is of course to abandon the majority principle, and to use the family of *scoring rules*. Each voter gives a number of points to the alternatives according to their rank, and the candidate with more points wins. Typically, in many modern democracies, we use the plurality rule, which gives one point to your preferred candidate, and zero to the others.

The existence of cycles rely on the fact that we are free to choose any preference pattern we need. But in the reality, not all the preferences can be encountered. In 1948, Duncan Black noticed that when preferences are *single peaked*, the majority winner always exists.

Two tiers voting rules

At the first stage of the voting procedure, the individuals elect one candidate in each constituency. At the second stage, another aggregating procedure collects the names of the candidates elected in each constituency in order to designate the final winner of the election.

Such electoral systems are used worldwide in international bodies (the United Nations, the United Nation Framework for Control of Climate Change, ...), in federal unions (the election of the president of the United States, the council of minister of the European Union) or in local councils (communities of cities in France).

So far, Social Choice Theory has not produced many papers on the issue "What is the best two step procedure?" until recently.

The 2000 US presidential election

Gerrymandering

Gerrymandering is a term that describes the deliberate rearrangement of the boundaries of electoral districts to influence the outcome of elections. The original gerrymander was created in 1812 by Massachusetts governor Elbridge **Gerry**, who crafted a district for political purposes that looked like a sala-**mander**. The purpose of gerrymandering is to either concentrate opposition votes into a few districts to gain more seats for the majority in surrounding districts (called packing), or to diffuse minority strength across many districts (called dilution).

The UK case

UK is divided into 647 constituencies for the election of the Members of Parliament (MP). During the 2005 campaign in United Kingdom , various websites and forums have been created invoking *tactical vote*.

On votedorset.net, we could find the following advertisement:

> *Four Lib Dems are seeking suitable Labour pairs, and our target is Oliver Letwin's seat (a Conservative). We would like to swap all four our tactical votes for Labour(East Staffordshire) with four equivalent tactical votes for the Lib Dems in Dorset West*

The objective was to manipulate the outcome by virtually changing the jurisdiction in which these voters could vote. In elections with two steps it seems possible to manipulate the result by changing jurisdiction. Are they some rules that are "movement-proof" ?

The model

$A = \{a, b\}$: set of candidates.

$N = \{1, \ldots, i, \ldots, n\}$: set of voters ($n \geq 3$).

$J = \{J_1, \ldots, J_j, \ldots, J_m\}$: $m \geq 2$ jurisdictions.

Partition function Let σ be a function from N to $\{1, 2, \ldots m\}$. $\forall i \in N$, $\sigma(i) = j \Leftrightarrow i \in J_j$.

Σ is the set of partitions σ such that, $\forall j \in \{1, \ldots, m\}$, $\sigma^{-1}(j) \neq \emptyset$. There is at least one voter per jurisdiction.

$\pi \in A^n$: *choice profile* or *vote profile*.

$$\pi = (a, b, a, a, b, a, a, b, b)$$

$\pi|_i$ denotes individual i's vote.

The federal constitution

In jurisdiction j, we chose the local winner with f_j:

$$\begin{array}{rcl} f_j : \Sigma \times A^n & \rightarrow & A \\ (\sigma, \pi) & \rightarrow & z \in A \end{array}$$

At the federal level, the federal winner is selected from the profile of local winners with the SCF g:

$$\begin{array}{rrcl} g: & A^m & \rightarrow & A \\ & \Pi = (z_1, \ldots, z_m) & \rightarrow & z \in A \end{array}$$

Federal constitution: $C = (g, f_1, \ldots, f_m)$
Federal winner:

$$g(f(\sigma, \pi)) = g(f_1(\sigma, \pi), \ldots, f_m(\sigma, \pi))$$

Properties

Local Unanimity: If $\forall z \in A$, $\pi|_i = \{z\}$ $\forall i \in J_j$ then $f_j(\sigma, \pi) = z$

Individual Stability $\forall \pi \in A^n, \forall i \in N, g(f(\sigma, \pi)) = g(f(\sigma', \pi))$, for all $\sigma, \sigma' \in \Sigma$ satisfying $\sigma(h) = \sigma'(h)$ $\forall h \neq i$ and $\sigma(i) \neq \sigma'(i)$

Stability: $\forall \pi \in A^n$, $g(f(\sigma, \pi)) = g(f(\sigma', \pi)) \forall \sigma$, $\sigma' \in \Sigma$

Stability $\Leftrightarrow$ Individual Stability
Stability $\Leftrightarrow$ Non Manipulability by Gerrymandering
Stability $\Rightarrow$ Vote Swapping Proofness.

Some stable rules

Constant Constitution A constitution $C = (g, f_1, \ldots, f_m)$ is called constant if there exists $z \in A$ such that $\forall \sigma \in \Sigma$, $\forall \pi \in A^n$, $g(f(\sigma, \pi)) = z$

Unanimity Rule Let $(x, z) \in A^2$. A g of f is a unanimous (x, z) voting rule, when z is elected if and only if all the voters vote for z, an $\bar{z}$ otherwise. The constitution C is a Unanimity Rule if all the f's are the same Unanimity Rule, and g is also a Unanimity Rule

The Constant Constitution and the Unanimous Constitution are anonymous, stable, satisfy local Pareto, but are not neutral.

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Theorem When $A = \{a, b\}$, a constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Pareto and Stability if and only if C is equivalent to the Constant Federal Rule or to one of the Unanimity Rules.

Proposition Let $A = \{a, b\}$. If a federal constitution $C = (g, f_1, \ldots, f_m)$ satisfies Local Pareto and Stability, then g is anonymous.

There is no way to reconciliate neutrality and indirect voting rules.

Open Question

We do not have a complete characterization for 3 alternatives or more, but at least, we know there will a be a “local” veto player.

Till now, all the proofs use jurisdiction with variable population sizes. Does the result hold with vote swapping only ? With additional conditions, yes.

All the rules are vulnerable to manipulation by gerrymandering. But it may be harder to manipulate some by redistricting, if the problem is NP-hard.

We can estimate the probability that direct an undirect voting rules leads to different results, to check which indirect mechanism minimizes the probability of conflicts for federal constitution. Tomorrow’s speech at the Franco-Russian conference.

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