

A New Method of the Single Transferable Vote and its Axiomatic Justification

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We propose a new method of the Single Transferable Vote (STV) and give a unified way to describe classic procedures (Gregory Method, Inclusive Gregory Method and Weighted Inclusive Gregory Method) as an iterative procedure. A modification for quota definition is proposed which improves theoretical properties of the procedures. The method is justified by a new set of axioms. It is shown that this procedure extends the Weighted Inclusive Gregory Method with the modified definition of quota and random equiprobable selection of winning coalition on each iteration. The results are extended to the methods allowing fractional number of votes.

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1. Introduction

The Single Transferable Vote is a preferential voting system. Voters rank candidates according to their preferences, while some candidates can remain unranked. More than one candidate are elected. For the candidate to be elected the number of first place votes should exceed a quota. The quota is counted at the beginning of the procedure based on the number of voters and the number of seats. Vote counting process runs iteratively. On each iteration of the procedure the first place votes for each candidate are counted, then it is compared with the quota. Each candidate who achieves the quota is elected. If elected candidates have vote surplus, it is transferred to the other candidates. If all candidates have less than quota in first place votes and the number of candidates exceeds the number of seats which are not filled, then the candidate with a minimal number of votes is eliminated and her votes are transferred to other candidates. If the number of candidates equals the number of seats which are not filled then all candidates are declared elected.

There are several methods of implementing STV. The main difference is how a surplus is transferred. Gregory method was the first STV method. Other methods can be described as its modifications [2]. We give now short descriptions of different versions of Gregory method.

Gregory method (original version). Initially STV was implemented by random selection of transferred votes. The process of manual counting on STV election is described in [4]. According to the first place votes the ballots are sorted into bundles. Quota surplus votes are randomly removed from the bundle and transferred to the other candidate bundles. As soon as another candidate reaches the quota, only last parcel ballots are transferred to subsequent candidates.

Example 1. There are 4 candidates, 3 seats to be filled and the following voters' preferences

12 votes $a > b > c > d$,
12 votes a ,
10 votes $b > d$,
10 votes c ,
10 votes d .
Total 54 votes.

Quota is defined as

$$q = \left\lfloor \frac{\text{votes number}}{\text{seats number} + 1} \right\rfloor + 1, \quad (1)$$

where $\lfloor \cdot \rfloor$ indicates rounding down. In our example

$$q = \left\lfloor \frac{54}{3 + 1} \right\rfloor + 1 = 14.$$

It is the minimal number of votes which secure seat for the candidate who obtains the quota.

Candidate **a** receives 24 votes which exceeds the quota, so **a** is declared elected. 10 votes surplus is then distributed. Each vote with probability $10/24=41,67\%$ is transferred. Let us examine the most likely outcome. 5 votes are transferred to candidate **b**, 5 votes are moved to non-transferable votes, because these voters do not show subsequent preferences. On the next iteration we obtain

7 votes $a > b > c > d$,
7 votes a ,
5 votes $b > c > d$ (transfer to a),
10 votes $b > d$,
10 votes c ,
10 votes d ,

5 non-transferable votes.

Candidate **b** reaches the quota. According to Gregory method surplus votes are transferred from last parcel, i.e., from candidate **a** surplus. In other words, candidate **b** bundle goes up because of candidate **a** surplus transfer till it meets the quota. Residual ballots are transferred to subsequent candidates. In our example one vote is transferred to candidate **c**. Candidate **c** on the last iteration has more votes than candidate **d**, so she is declared elected. Finally, candidates **a**, **b**, **c** are the winners.

Gregory method (modern version). There is no random selection. Ballots are selected proportionally to the second place votes. Last parcel transfer principle remains unchanged. In our example on the first iteration $10/24=41,67\%$ of each group of votes is transferred. From 12 votes with the same second place votes $10/24*12=5$ votes are transferred. The second iteration surplus is homogenous (contains only $b > c > d$ vote). It is transferred to candidate **c**. Then candidates **a**, **b**, **c** are the winners.

Inclusive Gregory method. It differs from Gregory method in the way of transferring subsequent surpluses. The first iteration is the same (proportional version). On the second iteration in our example one vote from candidate **b** should be transferred. Inclusive Gregory method counts all votes for candidate **b** (22 votes):

$$\begin{aligned} 12 \text{ votes } a > b > c > d, \\ 10 \text{ votes } b > d. \end{aligned}$$

Despite of the fact that the majority from 12 votes are used for candidate **a** to win, the method takes into account initial preference profile. Thus $12/22=0,55$ votes should be transferred from first group of votes and $10/22=0,45$ from the second group. We consider method that transfers integer number of votes, i.e., one ballot goes to candidate **c**. Then candidates **a**, **b**, **c** are elected.

Weighted inclusive Gregory method. Facing with surplus transferring problem this method counts only such votes that were transferred to the candidate before current iteration. The first iteration is the same as in Gregory method. On the second iteration when one vote from the candidate **b** should be transferred, all 15 votes of all groups of voters are counted, i.e.

$$\begin{aligned} 5 (=10/24*12) \text{ votes } a > b > c > d, \\ 10 \text{ votes } b > d. \end{aligned}$$

The group of votes $b > d$ contains more voters, so the candidates **a**, **b**, **d** are elected.

In these methods only integer number of votes is transferred. This restriction comes from old manual counting practice, and later electoral laws fixed this feature. Modern computational methods allowing fractional votes cannot be implemented immediately. Because a desire to preserve a transparency of the procedure and the confidence to elections results lead to a some resistance to novice methods. Moreover, in real elections distortions arisen from the rounding are not significant because of the huge number of votes.

There are methods taking into account fractional number of votes. As an example, we can present weighted inclusive Gregory Method without corresponding rounding. We will consider a version of the proposed model with fractional votes in Section 4.

Essential differences in all methods are seen only when sequential surpluses are distributed and procedures become more and more complicated. Historical examples do not provide the best method. On the other hand, in real elections the number of iterations can approach several hundred which makes rather difficult interpretation of advantages of different methods. In addition, the methods used very often changed under some influence of different political forces when they believed that the results of elections obtained by using some method are unfair.

We use an axiomatic approach to study the Single Transferable Vote which allows comparing different methods on the basis of introduced axioms.

There are several publications showing that STV violates different rational choice axioms. In [1] it is shown that STV violates monotonicity, in [8] it is demonstrated that STV reveals No-Show Paradox (if some voters which prefer x participate in elections, x loses), violates Condorcet Principle, and does not satisfy Consistency Property (if a choice on two groups of ballots coincide, then the choice on the joined group should be the same).

N. Miller constructed an example [7] called ‘a butterfly effect’ which reveals some chaotic behavior of STV (see also discussion in [3]). Moreover, this example shows as well that STV is manipulable, i.e., a small change in the preference profile can lead to a substantial change of the result.

In [10] the axiomatics was provided for STV and an impossibility theorem was proved showing that there is no a procedure which satisfy all those axioms. The proof was made not only for STV but for aggregation procedures in general. This axiomatics does not allow a selection of some particular procedure.

2. A formalization of STV

To analyze the methods implementing STV, let us formalize the procedure and formulate the properties which these methods should satisfy to.

Let us denote

$V_0 = \{v_1, \dots, v_n\}$ - a set of voters indexed by k ,

$C_0 = \{c_1, \dots, c_m\}$ - a set of candidates indexed by j ,

s - the number of seats to be filled (assume that $s < m < n$),

i - index of iteration of counting of votes,

$V_i \subseteq V_0$ - the set of voters considered on the i -th iteration,

$C_i \subseteq C_0$ - the set of candidates on the i -th iteration,

E_i - the set of candidates elected before the i -th iteration,

\bar{P} - Preference profile of voters,

\bar{P}_i - a preference profile corresponding to the set of voters and the set of candidates on the i -th iteration,

$\sigma(c_j, V_i, C_i) \subseteq V_i$ - a coalition of voters for which a candidate c_j is ranked first in their preferences,

$\sigma^{\max}(c_j, V_i, C_i)$ - the maximal coalition of this type, i.e., all other voters are voted for other candidates,

$q(n, s)$ - the quota, i.e. the minimal number of votes necessary for a candidate to be elected. For an integer number of votes the quota is defined as

$$q(n, s) = \left\lfloor \frac{n}{s+1} \right\rfloor + 1, \quad (2)$$

If $|\sigma(c_j, V_i, C_i)| = q(n, s)$, then this coalition is a winning one since such coalition can provide the victory of a candidate c_j .

Let us define formally STV procedure which is performed iteratively until all seats are filled.

At the beginning of the procedure the quota is defined according to the formula (2). Moreover, on the initial iteration we put $i := 0$, $E_0 = \emptyset$.

The iteration $i \geq 0$.

a) if there exists a winning coalition $\sigma(c_j, V_i, C_i)$ for some candidate c_j , then the candidate c_j , supported by this coalition is declared elected. Then

$$E_{i+1} := E_i \cup \{c_j\}.$$

if $|E_{i+1}| < s$, then

$$\begin{aligned}
V_{i+1} &:= V_i \setminus \sigma(c_j, V_i, C_i), \\
C_{i+1} &:= C_i \setminus \{c_j\}, \\
i &:= i + 1, \\
&\text{and new iteration begins,}
\end{aligned}$$

otherwise, the procedure stops.

b) If there is no a winning coalition, then the algorithm proceeds as follows. If $s - |E_i| = |C_i|$, all candidates are elected, i.e., $E_{i+1} := E_i \cup C_i$ and the procedure stops. Otherwise, i.e., if $s - |E_i| < |C_i|$, we construct a partition of the set of voters to the coalitions $\sigma^{\max}(c_j, V_i, C_i)$ for all $c_j \in C_i$. The candidate $c_j \in C_i$ with the least cardinality of maximal coalition $|\sigma^{\max}(c_j, V_i, C_i)|$ loses, and

$$\begin{aligned}
E_{i+1} &:= E_i, \\
V_{i+1} &:= V_i, \\
C_{i+1} &:= C_i \setminus \{c_j\}, \\
i &:= i + 1,
\end{aligned}$$

then the new iteration begins.

Note that on some iteration a voter may not have a preference on the set of the candidates still not elected. This means that her ballot moves to the category of non-transferable votes, i.e., her votes are not included in the winning coalitions of those candidates for which this voter votes for, and these votes do not have any impact on the voting result after their candidates are excluded from the list. However, formally these voters remain in the list of voters until the end of the procedure.

Example 2. Consider the case with 3 candidates and 5 voters which illustrates how the sets of voters and candidates are decreasing. The initial preference profile is shown in Table 1.

Table 1. The preference profile

	Voters				
	1	2	3	4	5
First preferences	<u>a</u>	<u>a</u>	a	b	c
Second preferences	<u>c</u>	<u>b</u>	c	c	b
Third preferences	<u>b</u>	<u>c</u>	b	a	a

With $s=2$ to win it is necessary to get 2 votes, $q=2$. The candidate **a** has a winning coalition from the voters $\{1, 2\}$ (shown in Table 1 underlined), thus she is elected. The ballots of these voters and the candidate **a** are excluded from the profile. In Table 2 the changed preference profile is shown used by the procedure on the next iteration.

Table 2. The changed preference profile

	Voters		
	3	4	5
First preferences	<u>c</u>	b	<u>c</u>
Second preferences	<u>b</u>	c	<u>b</u>

The remaining set of voters is $\{3, 4, 5\}$, and the remaining set of candidates is $\{\mathbf{b}, \mathbf{c}\}$. Due to the exclusion of the candidate **a** the ballot 3 moves from the candidate **a** to the candidate **c**. There is a winning coalition for the candidate **c** (voters $\{3, 5\}$). The procedure is terminated, the candidates **a** and **c** are elected.

The rules implementing STV differ on the way how the winning coalition is chosen. This in turn defines which votes remain in favor of the candidate, and which are transferred. Thus, the rule how the winning coalition is defined determines all subsequent path of votes' transfers. That is why, the principle of how to choose the winning coalition on the each iteration should be determined in advance. Some natural constraints allow restricting a possible class of procedures.

3. Axioms and the representation theorem

1. *The independence of the previous iterations.* For any iteration i , if one begins the procedure as if it starts from the very beginning, but keeping unchanged the current distribution of votes, the current choice of the winning coalition should not be changed.
2. *The independence of consequent preferences.* Those preferences of voters which were not taken into account in the procedure, i.e., those but first preferences on the current iteration, should not influence the choice of the winning coalition.
3. *Anonymity.* The names of voters do not matter.
4. *Neutrality.* The names of the candidates do not matter.

The necessary condition for Axiom 1 to hold is the re-evaluation of the quota on each iteration according to the formula $q_i = \left\lfloor \frac{|V_i|}{s - |E_i| + 1} \right\rfloor + 1$. If the quota is not re-evaluated, then on some iteration the initial quota and the quota evaluated on the basis of votes and seats will not be equal, which naturally violates Axiom 1. The set of voters as well as the number of remaining seats are changed only at the moment of the election of the next candidate. On the iterations, when a candidate is not elected, the quota is not changed. The non-transferable votes are taken into account when the quota is re-evaluated. It turns out that the re-evaluation of the quota does not change the procedure crucially for the quota can decrease only on 1 over all iterations.

Lemma 1. The quota evaluated according to the formula $q_i = \left\lfloor \frac{|V_i|}{s - |E_i| + 1} \right\rfloor + 1$, does not increase on any iteration.

Proof. By definition of the quota, the number of the candidates equals to the number of the seats can get the quota, but larger number of the candidates cannot do it, i.e.,

$$s - |E_i| \leq \frac{|V_i|}{q_i} < s - |E_i| + 1. \quad (3)$$

Assume that after the election on some iteration of a current candidate the quota increases. This means that the old quota lead to the election of the number of candidates larger than $s - |E_i| - 1$, i.e.,

$$\begin{aligned} \frac{|V_i| - q_i}{q_i} &\geq s - |E_i|, \\ \frac{|V_i|}{q_i} &\geq s - |E_i| + 1, \end{aligned}$$

contradicting (3).

Thus, on each iteration the quota decreases or remains unchanged, i.e.,

$$q_i - q_0 \leq 0. \quad \blacksquare$$

Theorem 1. For the quota evaluated on the final iteration of the procedure $q_l = \left\lfloor \frac{|V_l|}{s - |E_l| + 1} \right\rfloor + 1$, the following inequality $-1 \leq q_l - q_0 \leq 0$ holds.

Proof. By Lemma 1 on each iteration the quota can decrease or does not change, i.e.,

$$q_i - q_0 \leq 0.$$

From the first i iterations the quota can change as

$$q_i - q_0 = \left\lfloor \frac{|V_i|}{s - |E_i| + 1} \right\rfloor - \left\lfloor \frac{|V_0|}{s + 1} \right\rfloor.$$

On each iteration when a candidate is elected the set of voters decreases. Then

$$q_i - q_0 = \left\lfloor \frac{|V_0| - \sum_{j \in J_i} q_j}{s - |E_i| + 1} \right\rfloor - \left\lfloor \frac{|V_0|}{s + 1} \right\rfloor,$$

where J_i is a set of iterations until i which succeeded by the election of a candidate, i.e., $|J_i| = |E_i|$.

Without rounding down this expression can increase to less than 1. Thus the integer part of the difference does not exceed the difference of integer parts, i.e.,

$$q_i - q_0 \geq \left\lfloor \frac{|V_0| - \sum_{j \in J_i} q_j}{s - |E_i| + 1} - \frac{|V_0|}{s + 1} \right\rfloor,$$

$$q_i - q_0 \geq \left\lfloor \frac{|E_i||V_0| - (s + 1)|E_i|q_0 - (s + 1)\sum_{j \in J_i}(q_j - q_0)}{(s - |E_i| + 1)(s + 1)} \right\rfloor.$$

Using the definition of q_0 , we obtain

$$q_i - q_0 \geq \left\lfloor \frac{\frac{|V_0||E_i|}{s+1} - |E_i|\left(\frac{|V_0|}{s+1} + 1 - \left\{\frac{|V_0|}{s+1}\right\}\right) - \sum_{j \in J_i}(q_j - q_0)}{(s - |E_i| + 1)} \right\rfloor,$$

where $\{x\} = x - \lfloor x \rfloor$,

$$q_i - q_0 \geq \left\lfloor \frac{\sum_{j \in J_i}\left(q_0 - q_j - 1 + \left\{\frac{|V_0|}{s+1}\right\}\right)}{(s - |E_i| + 1)} \right\rfloor.$$

Denote the last iteration when $q_i = q_0$ holds as iteration d , then beginning with $d+1$, the expression $\left\lfloor \frac{\sum_{j \in J_i}(q_0 - q_j - 1 + \left\{\frac{|V_0|}{s+1}\right\})}{(s - |E_i| + 1)} \right\rfloor$ does not decrease, and $q_i - q_0$ does not increase.

Consider the case when d is the initial iteration. Then $s - |E_d|$ seats are distributed and $q'_0 = q_d = q_0$. Since it continues the previous iteration, the quotas do not change, and the value $q'_{i-d} = q_i$ remains the same, $|V_d| = |V'_0|$. For this iteration it is true that

$$\left\lfloor \frac{\sum_{j \in J'_{i-d}}\left(q'_0 - q'_j - 1 + \left\{\frac{|V'_0|}{s - |E_d| + 1}\right\}\right)}{(s - |E_d| - |E'_{i-d}| + 1)} \right\rfloor \leq q'_{i-d} - q'_0.$$

In the sum in the nominator only the first component is negative, and

$$q'_0 - q'_0 - 1 + \left\{\frac{|V'_0|}{s - |E_d| + 1}\right\} \geq -1.$$

Thus,

$$-1 \leq \left\lfloor \frac{\sum_{j \in J'_{i-d}}\left(q'_0 - q'_j - 1 + \left\{\frac{|V'_0|}{s - |E_d| + 1}\right\}\right)}{(s - |E_d| - |E'_{i-d}| + 1)} \right\rfloor \leq q'_{i-d} - q'_0.$$

We obtain $-1 \leq q_i - q_0 \leq 0$, which is true for the final iteration as well ■

Theorem 2. The only method satisfying Axioms 1-4 is the random equiprobable method of selection of the winning coalition on each iteration with the re-evaluation of the quota according to the formula $q_i = \left\lfloor \frac{|V_i|}{s-|E_i|+1} \right\rfloor + 1$.

Proof. Since the choice does not depend of the subsequent alternatives (Axiom 2), and on each iteration the choice is made similarly to the choice on the initial iteration (Axiom 1) which does not have a history, then the voters differ only by their names and by their first preferable alternative on i-th iteration. Anonymity (Axiom 3) implies that each voter, hence, each coalition *a priori* (before the procedure begins) have equal chances to be winning. By Axiom 1 the initial and subsequent iterations do not differ, i.e., the equal independent chances are preserved on each iteration. The only method which gives equal chances is the equiprobable on each iteration method of coalition selection. For the size of winning coalition does not depend on the iteration itself, it is necessary and sufficient to re-evaluate the quota according to the formula $q_i = \left\lfloor \frac{|V_i|}{s-|E_i|+1} \right\rfloor + 1$. Thus the only method satisfying Axioms 1-3 is this method. By construction it is neutral as well, i.e. Axiom 4 is obeyed as well. ■

The method introduced can be called the probabilistic version of the weighted inclusive Gregory method with the re-evaluation of the quota on each iteration.

We will show next that the Axioms 1-3 are independent. To do this we will construct examples which violate only one axiom out of three.

Method 1. At the initial iteration the voters are randomly ranked. According to this ordering, lexicographic order is defined on the set of coalitions. We choose the coalition with the minimal number. Axioms 2-4 are satisfied but Axiom 1 is violated.

Method 2. On each iteration the quota is re-evaluated, and the candidates are randomly ranked. The coalition is selected from those voters for which the second best candidate is closest to the chosen one. Among coalitions which are incomparable with respect to this criterion we select the one by random mechanism. Obviously, Axioms 1, 3, 4 are obeyed, but Axiom 2 is violated.

Method 3. Rank coalitions lexicographically with respect to the names of voters. On each iteration re-evaluate the quota and choose the coalition with the smallest rank. It can be checked that the Axioms 1, 2, 4 are satisfied, but Axiom 3 is violated.

Theorem 2 implies that there is no a method satisfying Axioms 1-3 but violating Axiom 4.

We will show now that Gregory method with proportional selection or with random selection of winning coalitions does not satisfy Axiom 1.

To this end consider Example 1. After the distribution of the surplus of the candidate **a** the following situation is observed

5 votes $b > c > d$ (received from a),
 10 voters $b > d$,
 10 voters c ,
 10 voters d .

The quota is equal to 14. The candidate **b** is elected. According to Gregory Method one vote is transferred to the candidate **c**.

If such situation is observed at the first iteration, then votes have been selected with equal probability among all ballots. Thus with the probability 5/15 the votes of the first group have been re-distributed, and with the probability 10/15 the votes of the second group have been re-distributed, i.e., the selection of votes to transfer to the next candidate has been changed. This means that Gregory Method does not satisfy Axiom 1 – the independence of the previous iterations. In proportional re-distribution of ballots as it can be seen from this example, Gregory Method violates Axiom 1. The inclusive Gregory Method violates Axiom 1 as well but the weighted inclusive Gregory Method completed with the re-evaluation of the quota on each iteration satisfies Axiom 1.

Gregory Method with random selection of ballots satisfies Axiom 2 – the independence of consequent preferences because any change of the preference profile does not change the probabilities of the selection of winning coalitions on previous iteration. Similarly, all extended versions of Gregory Method satisfy Axiom 2.

However, Gregory Method with proportional selection of ballots does not satisfy Axiom 2. Indeed, consider the following example.

Example 3. Let the numbers of voters and candidates are such that the quota is equal to 6. In Table 3 one can see the situation on the first iteration with respect to the preferences of voters voting for the candidate **a**. Other votes are such that the candidate **a** receives the maximal number of votes.

Table 3. The selection of winning coalition

	Votes for a								Other votes
	1	2	3	4	5	6	7	8	
First preferences	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	...
Second preferences	<u>c</u>	<u>c</u>	<u>c</u>	<u>c</u>	<u>d</u>	<u>d</u>	<u>d</u>	<u>d</u>	...

The procedure chooses a winning coalition (votes {1,2,3,6,7,8} - underlined in the Table 3) proportionally to the second preferences. Since the number of the ballots in which **c** and **d** take second preferences are equal, then in the coalition the respective voters should be represented in equal parts. Votes 4 and 5 are transferred to candidates **c** and **d**. However, if we change second preferences in this profile, then in the previous coalition the distribution of the second preferences will not be proportional which can be seen from Table 4.

Table 4. A change of second preferences

	Votes for a								Other votes
	1	2	3	4	5	6	7	8	
First preferences	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	...
Second preferences	<u>d</u>	<u>c</u>	<u>c</u>	<u>c</u>	<u>c</u>	<u>d</u>	<u>d</u>	<u>d</u>	...

After the change of second preferences the winning coalition has to be changed which is shown in Table 5. This demonstrates the violation of Axiom 2.

Table 5. A selection of the winning coalition

	Votes for a								Other votes
	1	2	3	4	5	6	7	8	
First preferences	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	<u>a</u>	...
Second preferences	<u>d</u>	<u>c</u>	<u>c</u>	<u>c</u>	<u>c</u>	<u>d</u>	<u>d</u>	<u>d</u>	...

The same reasoning can be applied to the extended Gregory Methods for they do not differ on the initial iteration.

4. Fractional votes case

The remaining votes on the iteration i will be presented by a vector $v_i = (v_{i1}, \dots, v_{in})$, $v_i \in [0, 1]$. The unit (1) denotes complete vote, and $v_0 = (1, \dots, 1)$.

A characteristic vector of a coalition is denoted as $w_{ij} = (w_{ij1}, \dots, w_{ijn})$, $w_{ijk} \in [0, 1]$. If on the iteration i k -th voter belongs to the winning coalition for the candidate j , then $w_{ijk} = 1$. In addition, we put $w_{ijk} = 0$ if $v_{ik} = 0$. Let w_{ij}^{\max} be a maximal coalition in favor of j , i.e., all other voters vote for other candidates. Note that v_{ik} and w_{ijk} can be fractional.

A coalition is winning if the number of votes (dot product of these vectors) is equal to quota, i.e.,

$$v_i w_{ij} = q(n, s). \quad (4)$$

Now we describe the procedure.

At the initial iteration the quota is defined. The quota can be defined as in the case of integer votes by $q_0 = \left\lfloor \frac{n}{s+1} \right\rfloor + 1$, but it can be defined as $q_0 = \frac{n}{s+1} + \varepsilon$, with $\varepsilon > 0$ being an arbitrary small real number. As before, on the initial iteration we put $i := 0$, $E_0 = \emptyset$.

An iteration $i \geq 0$.

a) If there is a winning coalition w_j for some candidate c_j , then the candidate c_j supported by this coalition is elected. Then

$$E_{i+1} := E_i \cup \{c_j\}.$$

If $|E_{i+1}| < s$, then

$$v_{i+1,k} := v_{ik} \cdot (1 - w_{ijk}),$$

$$C_{i+1} := C_i \setminus \{c_j\},$$

$$i := i + 1,$$

and the new iteration begins, otherwise the procedure stops.

b) If such winning coalition does not exist, then the procedure continues as follows. If $s - |E_i| = |C_i|$, then all candidates are elected $E_{i+1} := E_i \cup C_i$, and the procedure terminates. In other case, i.e., if $s - |E_i| < |C_i|$ the candidate $c_j \in C_i$ with minimal value $v_i \cdot w_{ij}^{\max}$ is erased from the list and

$$E_{i+1} := E_i,$$

$$v_{i+1,k} := v_{ik},$$

$$C_{i+1} := C_i \setminus \{c_j\},$$

$$i := i + 1,$$

then the new iteration begins.

In addition to equiprobable Gregory Method, weighted inclusive Gregory Method satisfies Axioms 1-3 if the quota is re-evaluated. If the quota is defined as in the case with integer votes with $q_i = \left\lfloor \frac{|V_i|}{s - |E_i| + 1} \right\rfloor + 1$, it does change to not more than 1. If it is defined with $q_i = \frac{|V_i|}{s - |E_i| + 1} + \varepsilon$, it changes to not more than ε . Weighted inclusive Gregory Method distributes equal parts of each vote, namely, from each complete or fractional vote it gives to the winning coalition that fraction of k-th vote that defines the ratio of quota in all winning coalition, i.e.,

$$w_{ijk} = \frac{q_i}{w_{ij}^{\max} v_i}. \quad (5)$$

This definition implies that the construction of coalition does not depend on further preferences (Axiom 2) and does not depend on the number of iteration (Axiom 1). Since all voters are taken into account in equal parts, the procedure satisfies Anonymity (Axiom 3).

Theorem 3. In fractional votes framework there exists non-Neutral Method satisfying Axioms 1-3.

Proof. We construct this method. The candidate x is selected at the beginning of the procedure. All winning coalitions can be described by (4) subject to $w_{ijk} \leq w_{ijk}^{\max}$. If the candidate x wins, then select the winning coalition equiprobably from the set of all winning

coalitions. If some other candidate wins, then define winning coalition by (5) transferring equal fractions of votes. Obviously, Axioms 1-3 are obeyed but not 4. ■

Axioms 1-4 define the class of methods which combine equiprobable distribution and a transfer of equal parts of votes either completely independently or being dependent of the first votes in the voters preferences.

Meek's Method [5, 6] as well as other methods constructed on its basis cannot be formalized by the model described above since their specific feature is a permanent reconstruction of winning coalitions of candidates already elected. Meek's Method is used now only in the elections in New Zealand [9].

5. Concluding remarks

We construct an extension of different methods based on STV as an iterative procedure. The existing methods can be considered as particular cases of this procedure.

We provide an axiomatic for this procedure which determines none of the components of definition of winners – neither the names of candidates, nor the names of voters, nor the name of a coalition, nor the number of iteration. This immediately leads to the random selection of winning coalition on each iteration which is proved in Theorem 2.

We have constructed a new method of STV which we call the weighted inclusive Gregory Method with re-evaluation of the quota on each iteration with votes transfer either equiprobably or in equal fractions if fractional votes are allowed. We provide an axiomatic justification of this method.

The re-evaluation of the quota is not significant indeed since we proved that the quota cannot increase and can only decrease to no more than 1. If the quota is not re-evaluated this implies difference between the initial iteration of the procedure and the subsequent ones. Thus the choice of the procedure depends on the number of iteration. This reason led to the Axiom 1 (The independence of the previous iterations). In real elections this specific addition will not play an important role, but the re-evaluation of the quota will increase a transparency of the procedure since it decreases the difference between the first and the subsequent iterations.

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