CURRICULUM VITAE

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Invited Lectures: Special Colloquium, University of California at San Diego, November 1994.
Fifth Annual Great Lakes K-theory Conference, Urbana, Illinois, March 1999.
Meeting on Algebraic K-theory, Oberwolfach, Germany, September—October 1999.
Conference on Quadratic Algebras in honor of Prof. Jan-Erik Roos, Stockholm, October 2000.
ESI program on Mathematical aspects of String Theory, Vienna, September 2001.
Conference of Pierre Deligne and Dynasty Foundation Competitions Winners, Moscow, January 2009.
Algebra Seminar, Institut Henri Poincaré, Paris, April 2009.
Special Session on Algebras and Coalgebras, First International Conference on Mathematics and Statistics, American University of Sharjah, UAE, March 2010.

Service: Reviewer for Functional Analysis and its Applications, Journal of the American Mathematical Society, International Mathematics Research Notices/Papers, Advances in Mathematics, Moscow Mathematical Journal, Mathematical Notes, Mathematische Annalen.

Publications:

1. Local Plücker formulas for a semisimple Lie group. *Funct. Anal. Appl.* **25**, #4, p. 291–292, 1991.
2. Nonhomogeneous quadratic duality and curvature. *Funct. Anal. Appl.* **27**, #3, p. 197–204, 1993.
3. The relation between the Hilbert series of dual quadratic algebras does not imply Koszulity. *Funct. Anal. Appl.* **29**, #3, p. 213–217, 1995.
4. All strictly exceptional collections in $\mathcal{D}^b_{\mathrm{coh}}(\mathbb{P}^m)$ consist of vector bundles. Electronic preprint `arXiv:alg-geom/9507014`, 6 pp, 1995.
5. (with A. Vishik) Koszul duality and Galois cohomology. *Math. Research Letters* **2**, #6, p. 771–781, 1995.

6. Mixed Tate motives with finite coefficients and conjectures about the Galois groups of fields. Abstracts of talks at the conference "Algebraische K-theorie", Tagungsbericht 39/1999, September–October 1999, Oberwolfach, Germany, p. 8–9. Available from `http://www.mfo.de/programme/schedule/1999/39/Report_39_99.ps` or `http://www.math.uiuc.edu/K-theory/0375/`.
7. (with R. Bezrukavnikov and A. Braverman) Gluing of abelian categories and differential operators on the basic affine space. *Journ. Inst. Math. Jussieu* **1**, #4, p. 543–557, 2002.
8. Koszul property and Bogomolov's conjecture. *Intern. Math. Research Notices* **2005**, #31, p. 1901–1936, 2005.
9. (with A. Polishchuk) Quadratic Algebras. University Lecture Series, 37. American Mathematical Society, Providence, RI, 2005. xii+159 pp.
10. Galois cohomology of certain field extensions and the divisible case of Milnor–Kato conjecture. *K-Theory* **36**, #1–2, p. 33–50, 2005.
11. Homological algebra of semimodules and semicontramodules: Semi-infinite homological algebra of associative algebraic structures. Appendices C–D in collaboration with D. Rumynin and S. Arkhipov. Monografie Matematyczne, vol. 70, Springer/Birkhäuser Basel, 2010. xxiv+349 pp.
12. (with R. Bezrukavnikov) On semi-infinite cohomology of finite-dimensional graded algebras. *Compositio Math.* **146**, #2, p. 480–496, 2010.
13. Two kinds of derived categories, Koszul duality, and comodule-contramodule correspondence. *Memoirs Amer. Math. Soc.* **212**, #996, 2011. v+133 pp.
14. Mixed Artin–Tate motives with finite coefficients. *Moscow Math. Journal* **11**, #2, p. 317–402, 2011.
15. The algebra of closed forms in a disk is Koszul. Electronic preprint `arXiv:1007.5010 [math.KT]`, 7 pp., 2010.
16. Galois cohomology of a number field is Koszul. Electronic preprint `arXiv:1008.0095 [math.KT]`, 23 pp., 2010.
17. (with A. Polishchuk) Hochschild (co)homology of the second kind I. Electronic preprint `arXiv:1010.0982 [math.CT]`, 67 pp., 2010–11.
18. Artin–Tate motivic sheaves with finite coefficients over a smooth variety. Electronic preprint `arXiv:1012.3735 [math.KT]`, 17 pp., 2010.
19. Coherent analogues of matrix factorizations and relative singularity categories. Electronic preprint `arXiv:1102.0261 [math.CT]`, 16 pp., 2011.

Research Summary:

1. Plücker formulas. The standard local Plücker formulas [23][1] for a germ of holomorphic curve in $\mathbb{CP}^N$ connect the (1,1)-forms corresponding to certain natural metrics on this germ of curve and their curvatures. The classical (global) Plücker formulas can be obtained from the local formulas by integration over the curve. In

[1]References in square brackets refer to the list of References below; the ones in round brackets refer to the Publication list above. The boldfaced references point to subsections of this Summary.

the paper [19], A. Givental proposed a generalization of the local Plücker formulas to the case of a curve in the flag manifold of any complex semi-simple Lie group.

Namely, let F be the flag manifold of a such a group G of rank n. There is a natural n-dimensional tangent distribution $\mathcal{N}$ on F corresponding to the direct sum of all simple negative root subspaces. Let $C \subset F$ be a germ of holomorphic curve tangent to $\mathcal{N}$. Consider the fundamental line bundles $\mathcal{L}_i$ on F and the corresponding morphisms to projective spaces $\pi_i\colon F \longrightarrow \mathbb{P}^{r_i}$. Choose a maximal compact subgroup K in G. Let ϕ_i be the Hermitian metric on C induced from the K-invariant Fubini–Studi metric on $\mathbb{P}^{r_i}$ by means of π_i and let θ_i be the curvature of ϕ_i. Note that on a complex curve C both metrics and their curvatures can be considered as elements of the vector space $\Omega^{1,1}(C)$ of smooth (1,1)-forms.

The local Plücker formulas conjectured by Givental have the form $(\theta) = A(\phi)$, where $(\phi) = (\phi_i)_{i=1}^n$ is the vector of metrics, $(\theta) = (\theta_i)_{i=1}^n$ is the vector of curvatures, and A is the Cartan matrix corresponding to G. (For $G = SL(n, \mathbb{C})$ this gives the formulas from [23].) I proved these generalized local Plücker formulas in my paper (1).

2. Koszul algebras, deformations. A graded algebra $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ over a field $\Bbbk$ with $A_0 = \Bbbk$ is called quadratic if it is generated by A_1 and defined by relations of degree 2. A graded algebra A is called Koszul [37, 4] if the only nonzero component of the graded vector space $\operatorname{Ext}^i_A(\Bbbk, \Bbbk)$ is the one with the grading i for each $i \geq 0$; any Koszul algebra is quadratic. The basic deformation property of Koszul algebras established in my book with A. Polishchuk (9) states that a deformation of a Koszul algebra in the class of quadratic algebras is flat provided that it is flat in the degrees 1, 2, and 3. This is a generalization of a result of V. Drinfeld [13].

In particular, it follows that there is only finite number of different Hilbert series $h_A(q) = \sum(\dim A_i)q^i$ of Koszul algebras with $\dim A_1$ fixed. We also discuss other versions of this deformation principle. One of them applies to nonhomogeneous quadratic deformations of quadratic algebras; the corresponding result generalizes the classical Poincaré–Birkhoff–Witt theorem on the structure of universal enveloping algebras. Another result of this kind concerns the properties of bases in quadratic algebras formed of monomials over a base in the generator space. (The latter is in fact a particular case of the Diamond Lemma.)

3. Nonhomogeneous quadratic duality. The category of quadratic algebras is self-dual: to a quadratic algebra A with the space of generators $A_1 = V$ and the space of relations $R \subset V^{\otimes 2}$ one can assign the quadratic algebra $A^!$ with the dual space of generators $A^!_1 = V^*$ and the space of relations $R^\perp \subset V^{*\otimes 2}$. In the paper (2) and the book (9) we discuss the extension of this duality to algebras with nonhomogeneous quadratic relations (like the universal enveloping algebras).

A nonhomogeneous Koszul algebra can be defined as an algebra A with an increasing filtration F such that the graded algebra $\operatorname{gr}_F A$ is Koszul. In (2) I construct a duality between the category of nonhomogeneous Koszul algebras and the category of Koszul algebras B endowed with a derivation d of degree $+1$ and an element $h \in B_2$ satisfying the equations $d^2 = [h, \cdot]$ and $d(h) = 0$ and defined up to the transformation $(d, h) \longrightarrow (d + [\alpha, \cdot],\ h + d\alpha + \alpha^2)$, where $\alpha \in B_1$. These formulas remind of the formal properties of the curvature of a vector bundle with respect to the choice of a connection; so graded algebras with such a structure are called *curved DG-algebras*, or *CDG-algebras*. I also define the corresponding analogues of the Chern classes.

4. Koszul algebras and probability. The dimensions of the graded components of Koszul algebras are algebraically independent, but satisfy a huge family of polynomial inequalities (generalizing the classical Golod–Shafarevich inequalities). Namely, let $a_i = \dim A_i$; then for a Koszul algebra A one has:

$$a_i \geq 0; \quad a_i a_j - a_{i+j} \geq 0; \quad a_i a_j a_k - a_{i+j} a_k - a_i a_{j+k} + a_{i+j+k} \geq 0;$$
$$a_i a_j a_k a_l - a_{i+j} a_k a_l - a_i a_{j+k} a_l - a_i a_j a_{k+l} + a_{i+j+k} a_l + a_{i+j} a_{k+l} + a_i a_{j+k+l}$$
$$- a_{i+j+k+l} \geq 0;$$

and so on. This family of inequalities has a very nice probabilistic interpretation. Namely, the inequalities mean that starting from a Koszul algebra one can construct a stationary stochastic 0-1 sequence with the so-called "one-dependence" property. A random sequence $\dots, \xi_{-1}, \xi_0, \xi_1, \xi_2, \dots$ is called one-dependent if for any integer n the collection of random variables $(\dots, \xi_{n-2}, \xi_{n-1})$ is independent from the collection $(\xi_{n+1}, \xi_{n+2}, \dots)$. Such stochastic sequences were studied lately by probabilists in [1, 17, 43]. This work of mine is included in the book (9).

5. Koszul algebras, counterexample. The Hilbert series $h_A(q) = \sum(\dim A_i)q^i$ and $h_{A^!}(q)$ of quadratic dual Koszul algebras A and $A^!$ (see **3**) satisfy the identity $h_A(q)h_{A^!}(-q) = 1$. It was an open question for a long time if any quadratic algebra A satisfying this identify is Koszul [2, 4]. Counterexamples were found independently (and in completely different ways) by me (3) and J.-E. Roos [39]. My construction is based on the properties of Koszul algebras with respect to the operation of Segre product $A \circ B = \bigoplus A_i \otimes B_i$ and the quadratic dual operation $A \bullet B$ invented in [32].

6. Exceptional collections. A collection of objects $E_1, \dots, E_n$ in a $\Bbbk$-linear triangulated category $\mathcal{D}$ is called exceptional if one has $\operatorname{Hom}(E_i, E_j[*]) = 0$ for $i > j$ and $\operatorname{Hom}(E_i, E_i[*]) = \Bbbk$. And exceptional collection is called strictly exceptional if $\operatorname{Hom}(E_i, E_j[s]) = 0$ unless $s = 0$ for any i and j. Exceptional collections in the derived categories of coherent sheaves on algebraic varieties were studied by Rudakov, Gorodentsev, Kapranov, Bondal [21, 22, 9, 10] and others.

In (4), I show that any strictly exceptional collection generating the derived category of coherent sheaves on a smooth projective variety with $\operatorname{rk} K_0(X) = \dim X + 1$ (such as a projective space or an odd-dimensional quadric) consists of locally free sheaves (not complexes) up to a common shift. The condition $\operatorname{rk} K_0(X) = \dim X + 1$ seems to be necessary, since otherwise the desired class of exceptional collections is not preserved by mutations.

7. Galois cohomology and Koszulity. Let l be a prime number, F be a field containing a primitive l-root of unity, and $G_F = \operatorname{Gal}(\overline{F}/F)$ be the Galois group of F. Then there is a homomorphism $K^{\mathrm{M}}_*(F)/l \longrightarrow H^*(G_F, \mathbb{Z}/l)$, called the norm residue homomorphism, from the Milnor K-theory ring of the field F modulo l to the Galois cohomology ring. The famous Milnor–Kato conjecture [34, 26] (known also as the

Bloch–Kato conjecture) claims that it is an isomorphism. A complete proof of that was recently obtained by by V. Voevodsky, M. Rost, et. al. [44].

In my joint paper with A. Vishik (5) we show that the whole conjecture follows from its (mostly long known due to Merkurjev–Suslin and Rost) low-degree part provided that the Milnor K-theory algebra $K_*^{\mathrm{M}}(F)/l$ is Koszul. Namely, it is enough to consider the case when the Galois group is a pro-l-group. Let G be pro-l-group and $H = H^*(G, \mathbb{Z}/l)$ be its cohomology algebra. Assume that H^2 is multiplicatively generated by H^1, there are no nontrivial cubic relations in H^3, and the "quadratic part" of the algebra H is Koszul. Then we prove that the algebra H is quadratic.

This is true for the cohomology of an arbitrary pro-nilpotent algebra (or nilpotent coalgebra). This result is yet another version of the basic deformation property of Koszul algebras, this time applied to nonhomogeneous relations with terms of the degrees 2, 3, 4, 5, … (see **2**).

8. Noncommutative gluing. In the paper [27], D. Kazhdan and G. Laumon constructed certain abelian categories of importance in the representation theory. The categories are obtained by gluing together several copies of the categories of perverse sheaves on the principal affine space of a semi-simple Lie group. The gluing procedure they used is a "noncommutative" analogue of the one that recovers the category of coherent sheaves on a variety covered with a set of open subvarieties from the categories of sheaves on the open pieces. This kind of construction was also studied by A. Rosenberg [40] in the context of Noncommutative Algebraic Geometry.

In order to be able to define certain Eiler characteristics, Kazhdan and Laumon had to make a conjecture that their category has a finite homological dimension. Because of the above analogy, it was expected for a long time that the gluing procedure should preserve the finite homological dimension property in general. In the end of 1996, A. Polishchuk and I finally found that this is not true. Moreover, even the localization does not always preserve finite homological dimension in the noncommutative case (7). On the other hand, we were able to prove [36] that each individual Ext-space in the Kazhdan–Laumon category is finite-dimensional, which was also conjectured in [27].

9. Bogomolov's conjecture and Koszulity. It is expected that the absolute Galois groups of fields $G_F = \mathrm{Gal}(\overline{F}/F)$ should have very special homological properties, the statement of the Milnor–Kato conjecture (see **7**) being only a part of them. In particular, a conjecture of F. Bogomolov [8] claims that the commutator subgroup of the maximal quotient pro-l-group of G_F should be a free pro-l-group for any prime l and any field F containing an algebraically closed subfield. In the paper (8) (which is an enhanced version of my Ph. D. thesis) I propose an extension of Bogomolov's conjecture to arbitrary fields.

First of all, I present counterexamples showing that the algebraically closed subfield condition cannot be simply dropped. However, it seems to be OK to replace it with the weaker condition that F contains the roots of unity of orders l^n for all n. For arbitrary fields, the conjecture is formulated as follows. Let $R_l(F)$ denote the field obtained by adjoining to F all roots of orders l^n, for all n, from all elements of F. The strong version of the conjecture claims that the Sylow pro-l-subgroup of the absolute Galois group of $R_l(F)$ is a free pro-l-group. The weaker version, closer to the original Bogomolov's conjecture, states that the maximal quotient pro-l-subgroup of the same absolute Galois group is a free pro-l-group.

In the paper (8), I propose a hypothesis about the Milnor K-theory ring of a field which implies both the Milnor–Kato and extended Bogomolov's conjecture (weaker version). Here I assume that the field F contains a primitive l-root of unity if l is odd, and contains a square root of -1 if $l = 2$. The hypothesis says that the ideal $J_l(F)$ generated by the Steinberg symbols in the exterior algebra $\bigwedge^*_{\mathbb{Z}/l}(F^*/F^{*l})$ (i. e., the kernel of the map from the exterior algebra to the Milnor algebra modulo l) should be a Koszul module over the exterior algebra. I also notice that the analogous condition on the ideal $J_{\mathbb{Q}}(F)$ generated by the Steinberg symbols in the exterior algebra with rational coefficients of a field F of finite characteristic is equivalent to the combination of Goncharov's conjecture that the subalgebra $L_{\geq 2}(F)$ of the graded pro-Lie algebra $L(F)$ describing the category of mixed Tate motives over F with rational coefficients is a free pro-Lie algebra [20] with a corollary to a well-known Beilinson–Parshin conjecture [18] saying that the Milnor algebra with rational coefficients of a field of finite characteristic is a Koszul algebra.

10. Tate motives with finite coefficients. The Beilinson–Lichtenbaum conjecture [3, 31, 42] describes the Ext-spaces between the Tate motives with torsion coefficients $\mathbb{Z}/m(i)$ in terms of the Galois cohomology of the basic field F. In view of this conjecture, it is natural to expect that the category of mixed Tate motives with torsion coefficients should also admit a description in terms of the Galois group; such a problem was posed by A. Beilinson in [3]. The following results, announced in my conference talk (6), are conditional modulo the Beilinson–Lichtenbaum conjecture. Their generatizations to Artin–Tate motives were published in (14).

Unlike for the rational coefficients, in the torsion coefficient case the category of all the successive extensions of the Tate objects is never abelian. However, it is has a canonical structure of an exact category in Quillen's sense; so one can define the Ext-spaces with respect to this category. Furthermore, I show that this exact category is equivalent to the categoryof filtered G_F-modules (M, V) over $\mathbb{Z}/m$ such that the quotient modules $\mathrm{gr}^i_V M$ are direct sums of the cyclotomic modules $\mu_m^{\otimes i}$.

The next natural question is whether the Ext-spaces between the Tate objects computed in this exact category are isomorphic to those in the triangulated category of motives, i. e., those predicted by the Beilinson–Lichtenbaum conjectures. Assuming that F contains the m-roots of unity, I prove that the Ext-spaces coincide if and only if the Galois cohomology algebra $H^*(G_F, \mathbb{Z}/l)$, where $m = l^r$, is Koszul (cf. **7**). If this is the case, then the triangulated category of Tate motives is equivalent to the derived category of the above exact category of filtered modules.

11. Koszulity and triangulated categories. Using the techniques of exact categories of filtered objects developed for the study of motives with torsion coefficients, I was able to prove the following very general result. Let E be the heart of a t-structure on a triangulated category D. Assume that any object of E is a finite extension of irreducibles (has a finite length). Then if the Ext-algebra of the set of all irreducible objects of E defined in the category D is Koszul, then the Ext-spaces between the objects of E computed in the abelian category E are the same as in D. This is a corollary of a result generalizing the main theorem of (5) from nilpotent coalgebras to "nilpotent" abelian categories (14, Section 8).

12. Milnor–Kato conjecture for $l > 2$ in the divisible case. The following result is the first step of Voevodsky's proof of the Milnor–Kato conjecture for $l = 2$: if a field F has no extensions of degree prime to l and $K_n^{\mathrm{M}}(F)/l = 0$, and if the Milnor–Kato conjecture is true in the degree $n-1$, then $H^n(G_F, \mathbb{Z}/l) = 0$.

Voevodsky proves this for any l, but the argument for $l > 2$ is much more complicated, using the results on motivic cohomology from [42]. In (10), I present a shorter and clearer elemenary version of his reasoning, not using the notion of motivic cohomology at all. This method also provides a proof of the exact sequence of Galois cohomology in an arbitrary cyclic field extension conjectured by Bloch and Kato [6] and the exact sequence for a biquadratic field extension conjectured by Merkurjev–Tignol [33] and Kahn [25]. In addition, I obtain exact sequences of Galois cohomology for dihedral field extensions.

Besides, it is conjectured in (10) that the ideal generated by any element of degree 1 in $K_n^{\mathrm{M}}(F)/l = 0$ is a Koszul module over $K_n^{\mathrm{M}}(F)/l = 0$. Evidence in support of this conjecture is presented based on an extension of the classical argument known as the "Bass–Tate lemma".

13. Contramodules and Nakayama's Lemma. Contramodules over coalgebras have been defined in [15], but almost forgotten for the subsequent four decades. Generally, contramodules can be thought of as modules with infinite summation operations. Typically, for a category of "discrete", "smooth", or "torsion" modules there is a related category of contramodules which contains the objects "dual" to the objects of the former, together with some other objects.

A contramodule over a coalgebra C over a field k is a vector space V together with a contraaction map $\mathrm{Hom}_k(C, P) \longrightarrow P$ satisfying the natural contraassociativity and counity axioms. In (11, Appendix A), I prove the following version of Nakayama's Lemma for contramodules: if D is a conilpotent coalgebra without counit and P is a contramodule over D such that the contraaction map is surjective, then $P = 0$. Here a coalgebra D is called conilpotent if for each its element x there exists a positive integer n such that x is annihilated by the iterated comultiplication map $D \longrightarrow D^{\otimes n}$.

I also define contramodules over certain topological rings, topological Lie algebras, and certain topological groups.

14. Comodule-contramodule correspondence. The addditive category of projective contramodules over a coalgebra C is equivalent to the category of injective comodules over C. One would like to extend this correspondence by assigning a complex of C-contramodules to a not necessarily injective C-comodule and vice versa. The problem is, the complexes one obtains in this way are often acyclic. The solution is to introduce certain "exotic" derived categories of comodules and contramodules, called the *coderived* and *contraderived* categories (see **15** for the definitions). The terminology of "coderived categories" is due to K. Lefèvre-Hasegawa and B. Keller [29].

These are certain quotient categories of the homotopy categories of complexes of comodules and contramodules, defined in such a way that the coderived category of comodules is equivalent to the homotopy category of complexes of injective comodules, and the contraderived category of contramodules is equivalent to the homotopy category of complexes of projective contramodules. So the coderived category of C-comodules is equivalent to the contraderived category of C-contramodules. The objects of the coderived category can be thought of as complexes having, in addition to the conventional cohomology in the finite degrees, some kind of "cohomology in the cohomological degree $-\infty$"; while the objects of the contraderived category can be considered as having "cohomology in the degree $+\infty$".

15. Derived categories of the second kind. Generally speaking, a complex can be thought of in two ways: as a deformation of its cohomology and as a deformation of itself considered without the differential. To the former point of view, the conventional derived categories (the unbounded derived category of an abelian/exact category, the derived category of DG-modules over a DG-ring [28], etc.) correspond. These can be called the *derived categories of the first kind*. To the latter point of view, the coderived and contraderived categories correspond; these can be called the *derived categories of the second kind*. The terminology comes from the classical distinction between the differential derived functors of the first and the second kind [24].

Suppose that we have the category of complexes over an exact category, or the category of DG-modules over a DG-ring, or the category of DG-comodules over a DG-coalgebra, etc. The corresponding coderived category D^{co} is defined as the quotient category of the homotopy category of complexes, DG-modules, etc. by its minimal triangulated subcategory, contraining all the totalizations of short exact sequences of complexes, DG-modules, etc. and closed under infinite direct sums. The objects in this minimal triangulated subcategory are called *coacyclic*. To obtain the definition of the contraderived category $\mathsf{D}^{\mathsf{ctr}}$, one has to replace infinite direct sums with infinite products (the objects in the corresponding minimal triangulated subcategory are called *contraacyclic*). For these definitions to work well, the operations of infinite direct sums/products have to preserve exactness; so one is not supposed to apply the definition of the contraderived category to DG-comodules, but rather to DG-contramodules. Then any coacyclic or contraacyclic complex, DG-module, etc. is acyclic; the converse is not true in general.

The definitions of the derived categories of the second kind make perfect sense for CDG-modules, CDG-comodules, and CDG-contramodules; while the conventional derived categories (of the first kind) do not, because the CDG structures have no cohomology (see **3**). In many contexts, the (conventional) derived categories are most suitable for modules, the coderived categories for comodules, and the contraderived categories for contramodules. However, the derived categories of the second kind for CDG-modules can be useful, too (13, 17, 19).

16. Koszul duality or "triality". One would like, e. g., given a Lie algebra $\mathfrak{g}$, define a version of the derived category of DG-comodules over the standard homological complex $C_*(\mathfrak{g}, k)$ in such a way that it would be equivalent to the derived category of $\mathfrak{g}$-modules. The problem is, the standard homological complex of $\mathfrak{g}$ with coefficients in a nonzero $\mathfrak{g}$-module can well be acyclic (e. g., for a nontrivial irreducible module over a reductive Lie algebra). The solution is to consider the coderived category.

To any CDG-coalgebra C, one can assign its cobar-construction, which is a CDG-algebra A. Then the coderived category of CDG-comodules over C, the contraderived category of CDG-contramodules over C, and the coderived=contraderived category of CDG-modules over A are all equivalent (13). The coderived and contraderived categories of A-modules coincide, since the underlying graded algebra of A has a finite homological dimension. When C is coaugmented and conilpotent, the CDG-algebra A is a DG-algebra for which the coderived, contraderived, and the conventional derived categories all coincide. So the derived category of A-modules is equivalent to the two exotic derived categories associated with C.

The somewhat more difficult, relative situation of DG-modules over the de Rham complex is of a special interest. Let M be a smooth algebraic variety and E be a vector bundle over it; consider the sheaf of rings of differential operators $D_{M,E}$ acting in the sections of E. Choosing (perhaps only locally) an algebraic connection ∇ in E, one can construct the de Rham differential on the graded algebra $\Omega(M, \mathrm{End}(E))$ of differential forms with coefficients in the endomorphisms of E. This makes $\Omega(M, \mathrm{End}(E))$ a sheaf of CDG-algebras. Then the derived category of right $D_{M,E}$-modules is equivalent to the coderived category of right CDG-modules over $\Omega(M, \mathrm{End}(E))$. When M is affine, the derived category of left $D_{M,E}$-modules is also equivalent to the contraderived category of left CDG-modules over $\Omega(M, \mathrm{End}(E))$.

17. Semi-infinite homology of associative algebraic structures. The key idea of the monograph (11) is that the associative version of semi-infinite homology arises in the following situation. Let C be a coassociative coalgebra; consider the category of C-C-bicomodules; it is an (associative, noncommutative) tensor category with respect to the operation of cotensor product over C. Let S be an associative, unital algebra object in this tensor category; such an algebraic structure I call a *semialgebra*. Semialgebras are the natural objects dual to corings (cf. [11]).

One can consider S-module structures on left or right C-comodules; these I call left and right *S-semimodules*. The category of left S-semimodules is abelian whenever S is an injective right C-comodule and vice versa; so I suppose that S is injective over C both from the left and from the right. The *semitensor product* over S is a natural functor mixing the cotensor product over C and the tensor product over S relative to C; it is neither left, nor right exact. Its double-sided derived functor, denoted by $\mathrm{SemiTor}^S$, is the associative semi-infinite homology.

Defining double-sided derived functors of functors of two arguments is simple enough provided that one has the appropriate versions of exotic derived categories at hand. The exotic derived category of semimodules that one needs is called the *semiderived category*; it is a mixture of the coderived category over C and the derived category over S relative to C. The semiderived category of S-semimodules is defined as the quotient category of the homotopy category of complexes of S-semimodules by the thick subcategory formed by all the complexes that are *coacyclic as the complexes of C-comodules*. The functor $\mathrm{SemiTor}^S$ is defined on the Carthesian product of the semiderived categories of right and left S-semimodules.

This definition of the semi-infinite homology agrees well with the semi-infinite homology of Tate Lie algebras as defined in [5]. It allows also to define, e. g., the semi-infinite homology of locally compact totally disconnected topological groups.

18. Semimodule-semicontramodule correspondence. One can define semicontramodules over a semialgebra S over a coalgebra C as C-contramodules with certain additional structures related to S. One can define the semiderived category of S-semicontramodules just like it was done in the semimodule case above, and establish an equivalence between it and the semiderived category of S-semimodules, generalizing the comodule-contramodule correspondence (see **14**). This allows to formulate the classical duality between complexes of representations of an infinite-dimensional Lie algebra with the complementary central charges, e. g., c and $26 - c$ in the Virasoro case [16, 38], as a covariant equivalence of triangulated categories.

Using semicontramodules, one also defines the semi-infinite cohomology functor for associative algebraic structures, denoted $\mathrm{SemiExt}_S$. Its first argument is a complex of S-semimodules, while the second argument is a complex of S-semicontramodules.

19. Morphisms through a common subcategory. A very different approach to the semi-infnite cohomology of associative algebras was developed originally by R. Bezrukavnikov; later I joined his effort to generalize it. In (12), the semi-infinite cohomology of finite-dimensional graded algebras of a certain type is interpreted as morphisms between objects of two different derived categories of modules through their common triangulated subcategory.

20. Nonflat Koszulity. The Koszul property is traditionally defined either for positively graded algebras $A = k \oplus A_1 \oplus A_2 \oplus \cdots$ over a field k (5, 9), or for nonnegatively graded algebras $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ such that A_0 is a semisimple algebra [4]. It is pretty straightforward to generalize this definition to the case of an arbitrary ring A_0, assuming that all A_n are flat right A_0-modules (11, Section 0.4 and Chapter 11). For the purposes of the theory of Artin–Tate (as opposed to simply Tate, cf. **10**) motives with finite coefficients one needs, however, to define the Koszul property of nonnegatively graded rings without any flatness assumptions.

The following definition is introduced in my paper (14). A graded ring $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is said to be *Koszul* if there exists an exact category G together with a shift functor $X \longmapsto X(1)$ on G and an embedding of the additive category of finitely generated projective right A_0-modules into G satisfying the following conditions. The shift functor is an auto-equivalence of the exact category G. Let G_0 denote the image of the above embedding; set $\mathsf{G}_i = \mathsf{G}_0(i)$ for $i \in \mathbb{Z}$. Then all objects of G must be obtainable from objects of G_i using iterated extensions. Finally, one should have $\mathrm{Ext}^i_{\mathsf{G}}(A_0, A_0(j)) = 0$ for $i \neq j$ and there should exist multiplicative isomorphisms

$\mathrm{Ext}^n_{\mathsf{G}}(A_0, A_0(n)) \simeq A_n$, where, by an abuse of notation, A_0 denotes the object of G_0 corresponding to the right A_0-module A_0.

When A is a flat right A_0-module, A is a Koszul ring if and only if $\mathrm{Tor}^A_{ij}(A_0, A_0) = 0$ for $i \neq j$. In the general case, the Koszul property does not depent on the zero-degree component of a graded ring A. Given a morphism of rings $A'_0 \longrightarrow A_0$, the graded ring $A' = A'_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is Koszul if and only if the graded ring A is.

21. Integral Tate motives. Among Beilinson's conjectures [3] about mixed motives, there is the conjecture that the Ext spaces between Tate motives with rational coefficients (over a field) can be computed in the abelian subcategory of mixed Tate motives. Bloch and Kriz [7] call it the $K(\pi, 1)$-*conjecture.* Another name for it is the *silly filtration conjecture* (14).

In fact, this conjecture can be considered independently of the existence of an abelian category of mixed Tate motives (i. e., of the Beilinson–Soulé vanishing conjectures). Let us discuss it in the case of Tate motives with integral coefficients. Let M denote the minimal full subcategory of the triangulated category of motives with integral coefficients over a field F containing the Tate objects $\mathbb{Z}(i)$ and closed under extensions. The $K(\pi, 1)$-conjecture claims that any morphism $X \longrightarrow Y[n]$ with X, $Y \in \mathsf{M}$ and $n \geq 2$ can be factorized into a composition of similar morphisms of the degree $n = 1$ between objects of M.

Assume that the silly filtration conjecture holds for a field F, as do the vanishing conjectures and the Beilinson–Lichtenbaum conjecture. Then there is a natural internally graded DG-coalgebra C over $\mathbb{Z}$ with torsion-free components and the cohomology concentrated in the internal degree 0 such that the triangulated category of Tate motives with integral coefficients over F is equivalent to the full subcategory of the derived category of DG-comodules over C generated by the trivial DG-comodules $\mathbb{Z}(i)$. The DG-coalgebra C is to be considered up to quasi-isomorphism in the class of positively internally graded DG-coalgebras with torsion-free components. It is important that the zero cohomology of C is not torsion-free over $\mathbb{Z}$, which does not allow to recover the quasi-isomorphism class of C from its cohomology coalgebra.

22. Artin–Tate motivic sheaves. The problem of constructing an abelian category of mixed motives over with finite $\mathbb{Z}/m$-coefficients over a scheme X in terms of the étale topology of X was posed by Beilinson in [3]. In my paper (18), I propose a candidate exact category of mixed Artin–Tate motivic sheaves with coefficients $\mathbb{Z}/m$ over a smooth variety X over a field K of characteristic prime to m.

Namely, let E^m_X be the category of constructible étale sheaves of $\mathbb{Z}/m$-modules over X whose stalks over the scheme points of X form permutational representations of the respective absolute Galois groups with coefficients in $\mathbb{Z}/m$. A short sequence in E^m_X is said to be exact if the related short sequences of stalks at the scheme points are split exact as the Galois group representations. The objects of the exact category F^m_X are étale sheaves of $\mathbb{Z}/m$-modules M over X endowed with a finite decreasing filtration F such that the étale sheaves $F^iM/F^{i+1}M$ are objects of E^m_X twisted with the cyclotomic sheaves $\mu_m^{\otimes i}$. A short sequence with zero composition in F^m_X is exact if the related sequences of twisted quotient sheaves are exact in E^m_X.

For any variety Y quasi-finite over X one can define its relative cohomological motive with compact supports as an object of E_X^m. Several versions of the assertion that E_X^m is generated by such objects are proven in (18). Furthermore, let $\pi\colon \acute{E}t \longrightarrow Nis$ denote the natural map between the big étale and Nisnevich sites of varieties over K. Then there are natural maps

$$\mathrm{Ext}^*_{\mathsf{F}_X^m}(\mathbb{Z}/m, \mathbb{Z}/m(i)) \longrightarrow \mathbb{H}_{Nis}(X, \tau_{\le i}\mathbb{R}\pi_*\mu_m^{\otimes i}).$$

The Nisnevich hypercohomology in the right hand side is the motivic cohomology of X with coefficients in $\mathbb{Z}/m(i)$ as predicted by the Beilinson–Lichtenbaum conjecture, proven by Voevodsky, Rost, et. al. [42, 44]. It is shown in (18) that the above maps are isomorphisms for all varieties étale over X if and only if they are isomorphisms for all the scheme points of such varieties. When K contains a primitive m-root of unity, the latter property can be interpreted as a Koszulity hypothesis (14).

The above results of mine do not in fact depend on the smoothness assumption on X, but the truth of the Beilinson–Lichtenbaum conjecture, or of its presently existing interpretation, does.

23. Algebra of closed forms. Let D be either an affine space over a field of characteristic zero, or a formal disk, or a complex analytic disk, or an affine space with divided powers over a field of any characteristic, or a formal disk with divided powers. Let $z_1, \dots, z_u$ be the coordinates in D; fix $0 \le v \le u$. Consider the algebra Z of closed differential forms in D, regular outside the first v coordinate hyperplanes $\{z_s = 0\}$, $1 \le s \le v$, and having at most logarithmic singularities along these hyperplanes. It is shown in my note (15) that the algebra Z is Koszul.

Furthermore, in some of the above settings it is more natural to consider Z as a topological algebra. An appropriate purely algebraic setting for topological Koszulity is suggested in (15), making the algebra Z topologically Koszul, too.

This solves a problem posed to me by A. Levin in 2003 in connection with his preprint [30], where the algebras of closed forms appear in the description of the abelian category of real Hodge–Tate sheaves on a complex algebraic variety.

24. Galois cohomology of global fields. Let F be a one-dimensional local or global field, i. e., an algebraic field extension of $\mathbb{Q}_p$, $\mathbb{Q}$, $\mathbb{F}_p((z))$, $\mathbb{F}_p(z)$, or $\mathbb{R}$. It is shown in my paper (16) that the Galois cohomology algebra $\bigoplus_n H^n(G_F, \mu_l^{\otimes n})$ is Koszul for any prime number l, confirming the Koszulity conjecture from **7** in the case of one-dimensional fields.

Assuming that either l is odd, or F contains a square root of -1, or F is a local field, it is also shown that the module Koszulity conjectures from **9** and **12** hold for the Milnor K-theory/Galois cohomology algebra of F. The proofs are based on the Class Field Theory (including certain arguments from Chevalley's proof of the Second inequality and particular cases of the Chebotarev Density Theorem) and constructions of commutative PBW-bases (quadratic commutative Groebner bases).

This work was originally started in collaboration with A. Vishik in 1995 and later finished by myself in 2010.

25. Coherent analogues of matrix factorizations. Let X be a separated Noetherian scheme with enough vector bundles, L be a line bundle on X, and $w \in \mathcal{L}(X)$ be a nonzero-dividing section, i. e., the map $w\colon \mathcal{O}_X \longrightarrow \mathcal{L}$ is an embedding of sheaves. A matrix factorization of w is a pair of vector bundles $\mathcal{U}^0$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ on X together with morphisms $\mathcal{U}^0 \longrightarrow \mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2}$ and $\mathcal{U}^1 \otimes \mathcal{L}^{\otimes 1/2} \longrightarrow \mathcal{U}^0 \otimes \mathcal{L}$, both of whose compositions are equal to the multiplication with w. Replacing vector bundles with coherent sheaves, one obtains the definition of a coherent (analogue of) matrix factorization.

Matrix factorizations form a $\mathbb{Z}/2$-graded DG-category with a natural additional exact category structure, so the construction of the derived category of the second kind (see **15**) is applicable to them. The absolute derived category of (coherent) matrix factorizations is defined as the quotient category of their homotopy category by the minimal thick subcategory containing the totalizations of short exact sequences of (coherent) matrix factorizations. It is shown in my paper (19) (in the generality of arbitrary Noetherian quasi-coherent CDG-algebras) that the absolute derived category of conventional (locally free) matrix factorizations is a full triangulated subcategory of the absolute derived category of coherent matrix factorizations. When X is regular, these two categories coincide.

Let X_0 denote the closed subscheme of X defined locally by the equation $w = 0$. The triangulated category $\mathcal{D}^b_{Sg}(X_0)$ of singularities of X_0 is defined as the quotient category of the bounded derived category of coherent sheaves on X_0 by the bounded derived category of vector bundles on X_0. Orlov [35] constructs a fully faithful functor from the absolute derived category of (locally free) matrix factorizations to the triangulated category of singularities $\mathcal{D}^b_{Sg}(X_0)$. When X is regular, this functor is an equivalence of triangulated categories.

I explain in (19) that Orlov's functor can be transformed into an equivalence of triangulated categories even in the singular case by replacing the source category with a "larger" one and the target category with a "smaller" one. The triangulated category $\mathcal{D}^b_{Sg}(X_0/X)$ of relative singularities of X_0 in X is defined as the quotient category of the bounded derived category of coherent sheaves on X_0 by the thick subcategory generated by the derived inverse images of coherent sheaves from X. The absolute derived category of coherent matrix factorizations of w is equivalent to the triangulated category of relative singularities $\mathcal{D}^b_{Sg}(X_0/X)$.

Moreover, the images of locally free matrix factorizations under the Orlov's functor are both left and right orthogonal to the derived inverse images of coherent sheaves from X inside the triangulated category $\mathcal{D}^b_{Sg}(X_0)$.

26. Hochschild (co)homology of the second kind. In spirit of the general philosophy of differential derived functors of the first and the second kind, one can define two kinds of Hochschild (co)homology groups for a DG-algebra or a DG-category (at least, over a field). The difference between the two constructions lies in two ways of constructing the total complex of a bicomplex: one can either take infinite direct sums along the diagonals, or infinite products. For CDG-algebras or CDG-categories, it is only the Hochschild (co)homology of the second kind that are interesting, while the (co)homology of the first kind tend to vanish whenever the curvature elements are nonzero. Hochschild homology of the second kind are otherwise known as Borel–Moore Hochschild homology, and Hochschild cohomology of the second kind as compactly supported Hochschild cohomology [12].

To a CDG-ring or CDG-category B, one can assign the DG-category C of right CDG-modules over B whose underlying graded B-modules are projective and finitely generated. In connection with the recent work on the Hochschild cohomology of the DG-categories of matrix factorizations [41, 14], the question on the relation between the Hochschild (co)homology of the second kind of B and the Hochschild (co)homology of the first kind of C arose. In my paper with A. Polishchuk (17), we work out the foundations of the theory while developing an approach to this comparison problem based on the constructions of two kinds of derived categories of DG-modules (see **15**).

First of all, at least for any CDG-category B over a field, the CDG-category B and the DG-category C have naturally isomorphic Hochschild (co)homology of the second kind. Secondly, there are natural maps between the two kinds of Hochschild (co)homology of a DG-category, and sufficient conditions for these maps to be isomorphisms can be formulated in terms of a comparison between the two kinds of derived categories of DG-bimodules. In particular, for the DG-category C as above, a kind of "resolution of the diagonal" condition for the diagonal CDG-bimodule B over B guarantees an isomorphism of the two kinds of Hochschild (co)homology of C.

Applying these results to the case of the $\mathbb{Z}/2$-graded CDG-algebra B describing matrix factorizations of a function w on a smooth affine variety, we conclude that the Hochschild (co)homology of the second kind for B and the Hochschild (co)homology of the first kind for C are naturally isomorphic provided that w has no other critical values but zero.

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