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Hard Lefschetz for simplicial polytopes and intersection theory on tropical isolated surface singularities

Tropical geometry. Every k-dimensional affine algebraic variety $V \subset \mathbb{C}^n$ gives rise to its tropicalization, which is a certain k-dimensional piecewise linear set in $\mathbb{R}^n$, reflecting amazingly many topological and geometric properties of V. For instance, the tropicalization of a complex line in the plane is the union of rays in the directions $\uparrow, \rightarrow$ and $\swarrow$ from the common initial point. The combinatorial fact that any two distinct tripods of this kind intersect by one point reflects the algebraic-geometric fact that two generic lines in the plane intersect by one point.

Intersection theory. If two curves in the plane intersect at an isolated point, then slightly shifted copies of these curves intersect at finitely many nearby points. The maximal possible number of such points is called the intersection index of the curves. If the two curves live in a singular surface, rather than in the plane, then applications still require to define their intersection index. However, we cannot shift curves in a singular surface, and thus we need another idea for the definition, which is a classical topics for intersection theory. The simplest way is to try to represent the curve A (aka Weil divisor) as the zero set of a function f (aka Cartier divisor), and to define the intersection index of the curves A and B as the multiplicity of the root of the restriction $f|_B$. What is, for instance, the intersection index of two lines in the cone $z = xy$?

Motivation from enumerative geometry. Both tropical geometry and intersection theory emerged as approaches to enumerative geometry, whose topics is illustrated by the following classical problem: count how many curves of given degree and genus pass through a given set of points in $\mathbb{C}^2$. The approach of intersection theory is to consider the space, parameterizing all curves of given degree and genus. All curves, passing through a given point, form a hypersurface in this space, and we are looking for the intersection index of several such hypersurfaces. The approach of tropical geometry is to count tropical curves of given tropical degree and tropical genus through a given set of points in $\mathbb{R}^2$. It is a purely combinatorial problem, and Mikhalkin proved in 2003 that the tropical count will give the same number as the conventional one. This solution of a classical problem so much advertised the new field of study, that tropical geometry has already earned more than 400 publications and a personal line in the MSC subject classification.

Tropical intersection theory. Motivated by Mikhalkin's achievement, it is now considered desirable to construct a tropical counterpart for every important piece of algebraic geometry. In particular, many authors work on constructing intersection theory of tropical varieties. What we now call the intersection index of tropical curves in the plane was constructed in 1990's under another name by Fulton and Sturmfels. The intersection index of tropical curves in a smooth tropical surface was defined by Allerman, Francois, Raw and Shaw in 2010. We aim at constructing intersection index of tropical curves in a tropical surface with an isolated singularity.

Statement of the problem (in plain words). For linearly independent vectors a and b in the Euclidean plane $\mathbb{R}^2$, define the linear function l on $\mathbb{R}^2$ by the conditions $l(a) = p$ and $l(b) = q$, define the unit vector v by the conditions $v \cdot b = 0$ and $v \cdot a > 0$, and denote the derivative of l along v by $\frac{a,b}{p,q}$.

Let $P \subset \mathbb{R}^3$ be a convex polytope, let f be a function, assigning a real number $f(B)$ to every face $B \subset P$, and let n_B be the unit external normal vector of B. We define another function f' on the set of faces of P by the equality $f'(B) = \sum_A \frac{n_A, n_B}{f(A), f(B)}$, where A runs over all faces of P, adjacent to B. Let $\mathbb{R}^P$ be the space of all functions on the set of faces of P, then the rule $f \mapsto f'$ defines a self-map $D : \mathbb{R}^P \to \mathbb{R}^P$, and one can note that $\dim \ker D = \operatorname{codim} \operatorname{im} D$ is at least 3 (it is a good exercise to describe explicitly a three-dimensional subspace in the kernel and a codimension 3 subspace, containing the image). We can now precisely formulate our question:

$$\dim \ker D = \operatorname{codim} \operatorname{im} D = 3?$$

Note that, instead of working with geometric definition of D, we can explicitly write down its matrix.

Where is tropical geometry in this question? Actually, instead of the polytope P, we should consider the set of all external normal vectors to its faces and edges: this will be the aforementioned tropical surface $\mathcal{S}$. Instead of the function f, we should consider external normal rays to faces of P, with the multiplicity $f(B)$ attached to the ray generated by n_B: the union of these rays will be a tropical curve $\mathcal{C}$ on $\mathcal{S}$. The fact that f is in the image of D means that the "tropical Weil divisor" $\mathcal{C}$ can be represented as a "tropical Cartier divisor", and we can develop the tropical analogue of intersection theory on $\mathcal{S}$.

Suggestion on how to solve. The first objective is to study the case when P is the convex hull of finitely many "generic" points $p_0, \ldots, p_N$ in $\mathbb{R}^3$ (so that all the faces are triangles). For this, we can fix $p_1, \ldots, p_N$, move p_0, and trace what happens to the matrix D as the convex hull of $p_0, \ldots, p_N$ suffers a transformation. Actually, there are only two types of transformations that cannot be avoided by a slight modification of the trajectory of p_0: (1) one triangular face of P splits into three, or (2) two triangular faces of P merge into one quadrilateral, which then breaks again (along another diagonal) into two triangular faces. If we manage to prove that the rank of D is preserved during these transformations, then we can eventually move the point p_0 to the interior of the convex hull of $p_1, \ldots, p_N$, where the rank of D is known by induction on N.

Further objectives. The aforementioned suggestion allows (hopefully) to prove that D is non-degenerate, if the polytope P is the convex hull of a "sufficiently generic" finite collection of points. It would be interesting to prove non-degeneracy of D for all P with triangular faces, or to find a counterexample.

We can also relax the aforementioned condition, allowing faces of P to be triangles or parallelograms. This case seems important: e. g. the intersection of the corresponding tropical surface $\mathcal{S}$ with a plane is exactly the kind of a tropical curve participating in Mikhalkin's theorem. The same suggestion as above might help in this case, with a slight difference that P should be now represented as the convex hull of generic points and parallelograms.

By some reasons outside the scope of this abstract, the problem of non-degeneracy of D becomes irrelevant (although simpler) for polytopes P, whose faces are more complicated than triangles and parallelograms. For instance, it is well known as the "combinatorial hard Lefschetz theorem" for simple polytopes (a polytope is called simple if all faces of its dual are triangular, and the proof is by hard Lefschetz theorem for the toric variety, corresponding to P).

Instead of a polytope in $\mathbb{R}^3$ and its faces, we can consider a polytope in $\mathbb{R}^n$ and its facets. This setting is more general, but should not be harder to solve.

Introduction to tropical geometry: arxiv.org/abs/math/0601041

Stratification of sparse determinantal varieties

Stratification of determinantal varieties. A *stratification* of a set $S \subset \mathbb{C}^k$ is a sequence of closed subsets $S_0 \subset S_1 \subset \ldots \subset S_k = \mathbb{C}^k$, such that the set $S_i \setminus S_{i-1}$ (which is called the i-dimensional stratum) is empty or homeomorphic to an i-dimensional manifold for every i, and S is the union of some of these strata. The determinantal variety $S_{m,n}$ is the set of all complex $(m \times n)$-matrices of rank smaller than $\min(m, n)$; the name comes from the fact that such matrices are characterised by vanishing of maximal minors. It is a closed subset in the space of all complex $(m \times n)$-matrices, and admits the stratification, whose non-empty strata are the sets of the form $\{\text{matrices of rank } r\}$ for $r = 0, \ldots, \min(m, n)$. Exercise: find the dimension of each stratum.

Sparse determinantal varieties. Choose a set $K \subset \{1, \ldots, m\} \times \{1, \ldots, n\}$ and let M_K be the set of all complex $(m \times n)$-matrices, whose (i, j)-entries vanish for $(i, j) \in K$. This is a vector subspace in the space of complex $(m \times n)$-matrices, and the first (easy) problem is as follows:

describe low codimension strata of the *sparse determinantal variety* $S_{m,n} \cap M_K$.

Example: the space of degenerate square triangular 2×2 matrices is the union of two planes, whose intersection is a line L, therefore the two strata are L and its complement.

Effectively degenerate matrices. A matrix of size $m \times n$ with $m \leqslant n$ is said to be *effectively degenerate*, if its rows $a_1, \ldots, a_m$ admit a linear relation $\sum_{i=1}^m \lambda_i a_i = 0$ with $\lambda_i \neq 0$ for every $i = 1, \ldots, m$. The set $T_{m,n}$ of all effectively degenerate complex $(m \times n)$-matrices is dense in $S_{m,n}$, but not closed itself. Thus, the following problem is different from the first one (and is more complicated):

describe low codimension strata of the set $T_{m,n} \cap M_K$.

Example: the space of effectively degenerate square triangular 2×2 matrices is a plane minus its origin O, therefore the two strata are O and its complement.

Motivation. The Kouchnirenko-Bernstein theorem gives an exact upper bound for the number of common roots of n polynomials on $(\mathbb{C} \setminus 0)^n$ with given Newton polytopes. Considering the collection of these polytopes as a matrix of size $1 \times n$, a natural generalization of this question is as follows. Given an $(m \times n)$-matrix A, whose entries are polynomials on $(\mathbb{C} \setminus 0)^k$ with given Newton polytopes, find an exact upper bound for the number of points $x \in (\mathbb{C} \setminus 0)^k$ such that $\operatorname{rk} A(x) \leqslant r$. This problem is solved for the maximal and the minimal possible value of r (i.e. for $\min(m, n)$ and 1 respectively), and the study of other cases leads to the aforementioned problems. For instance, the simplest of the remaining cases $m = n = r + 2$ requires understanding of codimension 3 strata of $T_{m,n} \cap M_K$.

General problem.

Describe stratifications of $S_{m,n} \cap M_K$ and $T_{m,n} \cap M_K$.

It is difficult to estimate how complicated this problem is, until any strata of low codimension are well understood. The solution of this problem might rely upon theory of matroids in order to classify sparse degenerate matrices with respect to combinatorial ways of their degeneration, and require construction of normal forms of such matrices in order to verify that the resulting classes are smooth. It would also be important to check that the resulting stratification is a Whitney stratification, or at least a topological stratification.

Literature: "Determinantal variety", "Matroid" – see Wikipedia and references therein. Normals forms – see Arnold, Varchenko, Gusein-Zade, "Singularities of differentiable maps".