

# Moduli spaces of framed instanton bundles on $\mathbb{C}\mathbb{P}^3$ and twistor sections of moduli spaces of instantons on $\mathbb{C}^2$

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## Abstract

We show that the moduli space  $\mathcal{M}$  of framed instanton bundles on  $\mathbb{C}\mathbb{P}^3$  is isomorphic (as a complex manifold) to a subvariety in the moduli of rational curves of the twistor space of the moduli space of framed instantons on  $\mathbb{R}^4$ . We then use this characterization to prove that  $\mathcal{M}$  is equipped with a torsion-free affine connection with holonomy in  $Sp(2n, \mathbb{C})$ .

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# 1 Introduction

The operation of complexifying a real algebraic variety is well known. It is a special case of a procedure known as “extension of scalars”, and is given essentially by tensoring of all the relevant rings with  $\mathbb{C}$ . In the converse direction, there is an operation called “Weil’s restriction of scalars”, producing a real algebraic variety (with the same topological space) from a complex one. This operation is given by taking real and imaginary parts of all algebraic functions.

A composition of these two operations is called **complexification**: given a complex algebraic variety of dimension  $k$ , it produces another complex algebraic variety, of dimension  $2k$ .

In complex analytic category, this operation is defined only locally. Starting with a complex analytic  $n$ -manifold  $X$ , it produces another complex analytic manifold  $X_{\mathbb{C}}$  of dimension  $2n$ , equipped with an anticomplex involution  $\iota$ , such that the fixed point set  $X_{\mathbb{R}} := St(\iota)$  of  $\iota$  is equivalent to  $X$  as a real analytic manifold. The manifold  $X_{\mathbb{C}}$  is defined non-uniquely; only the germ of a neighborhood of  $X_{\mathbb{R}}$  in  $X_{\mathbb{C}}$  is unique. One of the ways to produce a complexification is to take  $X_{\mathbb{C}} := X \times \bar{X}$ , where  $\bar{X}$  is the “complex conjugate” of  $\mathbb{C}$ , that is, the same manifold with complex conjugate action of  $\mathbb{C}$ .

Recall that a **hyperkähler manifold** is a Riemannian manifold  $M$  with a triple of complex structures  $I, J, K \in \text{End}(TM)$  which are Kähler and satisfy quaternionic relations. As shown by D. Kaledin [K] and B. Feix [F2], a complexification of a Kähler manifold is equipped with a canonical hyperkähler structure in a neighborhood of the fixed point set. To be more precise, Kaledin and Feix (independently) constructed a hyperkähler structure on a neighborhood of the zero section in the cotangent space  $T^*M$ . It is also easy to show that this cotangent space is naturally isomorphic to the complexification of  $M$ . This claim is implicit in Kaledin’s work, and found in explicit form in Feix’s Ph. D. thesis, [F1], Section 2.2.2.

Therefore, it is natural to ask what happens if one complexifies a hyperkähler manifold. Some results in this direction are known.

Recall that a **twistor space**  $\text{Tw}(M)$  of a hyperkähler manifold  $M$  is  $\mathbb{C}\mathbb{P}^1 \times M$  equipped with a complex structure which is defined as follows. Embed the sphere  $S^2 = \mathbb{C}\mathbb{P}^1 \subset \mathbb{H}$  into the quaternion algebra  $\mathbb{H}$  as the set of all quaternions  $J$  with  $J^2 = -1$ . For every point  $x = \{m\} \times \{J\} \in M \times S^2$ , the tangent space  $T_x \text{Tw}(M)$  is canonically decomposed as  $T_x X = T_m M \oplus T_J \mathbb{C}\mathbb{P}^1$ . Let  $I_J : T_J \mathbb{C}\mathbb{P}^1 \rightarrow T_J \mathbb{C}\mathbb{P}^1$  be the usual complex structure operator, and  $I_m : T_m M \rightarrow T_m M$  be the complex structure on  $M$  induced by  $J \in S^2 \subset \mathbb{H}$ .

The operator  $I_x = I_m \oplus I_J : T_x \text{Tw}(M) \rightarrow T_x \text{Tw}(M)$  satisfies  $I_x \circ I_x = -1$ . Moreover, it depends smoothly on the point  $x$ , hence it defines an almost complex structure on  $\text{Tw}(M)$ . This almost complex structure is known to be integrable (see [Sal]).

It is well-known that a complexification of a hyperkähler manifold is naturally identified with a component called  $\text{Sec}(M)$  in the moduli of rational curves on its twistor space (see e.g. [V2] and Subsection 2.2 below). In fact, it is possible to characterize hyperkähler manifolds in terms of the geometric structures

on the space  $\text{Sec}(M)$ . This approach is quite useful, because (as suggested by Deligne and Simpson) it allows to define singular hyperkähler varieties (see [V2] for a precise definition and a desingularization theorem).

In this paper we study the geometry of  $\text{Sec}(M)$  in some detail, and apply this to obtain a description of geometry of the space of mathematical instantons on  $\mathbb{C}\mathbb{P}^3$ .

More precisely, we show that there exists a family of integrable foliations  $S_v \subset T\text{Sec}(M)$ , parametrized by  $v \in \mathbb{C}\mathbb{P}^1$ , where  $\dim S_v = \frac{1}{2} \dim \text{Sec}(M)$ . For each  $C \in \text{Sec}(M)$ , the tangent space  $T_C \text{Sec}(M)$  is naturally identified with the space of sections of  $NC$ , where  $C$  is identified with  $\mathbb{C}\mathbb{P}^1$ , and its normal bundle  $NC$  (in generic  $C$ ) with a direct sum of several copies of  $\mathcal{O}(1)$ . We define  $S_v|_C$  as the space of all  $x \in NC$  vanishing at  $v \in \mathbb{C}\mathbb{P}^1 = C$ .

This family of foliations form what is known as a *3-web*, see Subsection 2.1 below for a precise definition. Theory of 3-webs was developed in 1930-ies by S. S. Chern ([C]), who chose it as a subject of his doctoral dissertation under Blaschke. Among other things, Chern constructed a natural torsion-free connection on a manifold with a 3-web, called **the Chern connection**.

The 3-webs occurring on  $\text{Sec}(M)$  are of a particular kind, which we call an *SL(2)-web*. For such an *SL(2)-web*, we obtain that the Chern connection is flat on the leaves of the foliation  $S_v \subset T\text{Sec}(M)$ , and its holonomy lies in the centralizer of  $\mathfrak{sl}(2, \mathbb{C}) = \mathfrak{su}(2) \otimes \mathbb{C}$  acting on  $\mathbb{C}^{4n} = (\mathbb{C}^2)^{2n}$ , as a direct sum of  $2n$  weight 1 representations. As a consequence of Schur's lemma, this centralizer is isomorphic  $GL(2n, \mathbb{C})$ .

On the manifold  $\text{Sec}(M)$ , which is a complexification of a hyperkähler manifold  $M$ , it is easy to see that the Chern connection is in fact a complexification of the Levi-Civita connection  $\nabla_{LC}$  on  $M$  (it follows immediately from the uniqueness of the Chern connection on 3-webs). Since the holonomy of  $\nabla_{LC}$  is  $Sp(n)$ , it follows that the holonomy of the Chern connection lies in its complexification  $Sp(n, \mathbb{C})$ .

Recall now that the de Rham algebra of a hyperkähler manifold  $M$  is equipped with a natural multiplicative action of the group  $SU(2)$  of unitary quaternions. A connection (not necessarily Hermitian) on a vector bundle  $B$  is called **autodual**, or **NHYM-autodual** if its curvature is  $SU(2)$ -invariant as an  $\text{End}(B)$ -valued 2-form.

Now let  $\sigma : \text{Tw}(M) \rightarrow M$  denote the natural projection. It is well known that a pullback to the twistor space of an autodual connection on  $B \rightarrow M$  defines a holomorphic structure on  $\sigma^*B$  (see 3.3 below and [KV, Lemma 5.1]). In [KV], the converse statement was proved (see 3.5 below): a holomorphic bundle on  $\text{Tw}(M)$  which is trivial on the rational curves of form  $\{m\} \times \mathbb{C}\mathbb{P}^1 \subset M \times \mathbb{C}\mathbb{P}^1 = \text{Tw}(M)$  is obtained as a pullback of a NHYM-autodual bundle  $(B, \nabla)$  on  $M$ , defined uniquely.

When  $M$  is compact, more can be said. Each of the holomorphic bundles on  $\text{Tw}(M)$  obtained this way corresponds uniquely to a section of the twistor projection  $\text{Tw}(W) \rightarrow \mathbb{C}\mathbb{P}^1$ , where  $W$  is a connected component of the moduli space of holomorphic bundles on  $M$ , equipped with a natural hyperkähler

structure ([V1]).

One of the main goals of this paper is to show that a similar correspondence can be defined for the case  $M = \mathbb{R}^4$ , and  $W$  a connected component of the moduli space of framed instantons. More precisely, we prove that the space of holomorphic sections of the twistor fibration  $\text{Tw}(W) \rightarrow \mathbb{C}\mathbb{P}^1$  is biholomorphically equivalent to moduli space of instanton bundles on  $\mathbb{C}\mathbb{P}^3$  framed at a line, thus we conclude that this moduli space has the structure of a  $SL(2)$ -web.

This result can be used to obtain many kinds of geometrical information about the instanton spaces. As a quick application, we show that the moduli space of instanton bundles on  $\mathbb{C}\mathbb{P}^3$  framed at a given complex line has no complex subvarieties. Also, we prove that at least one irreducible component of this moduli space has the expected dimension (this result is already known from [T]).

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## 2 Holomorphic 3-webs on complex manifolds

### 2.1 $SL(2)$ -webs and Chern connection

The following notion is based on a classical notion of a 3-web, developed in the 1930-ies by Blaschke and Chern, and much studied since then.

**Definition 2.1.** *Let  $M$  be a complex manifold,  $\dim_{\mathbb{C}} M = 2n$ , and  $S_t \subset TM$  a family of  $n$ -dimensional holomorphic sub-bundles, parametrized by  $t \in \mathbb{C}\mathbb{P}^1$ . This family is called a **holomorphic  $SL(2)$ -web** if the following conditions are satisfied*

- (i) *Each  $S_t$  is involutive (integrable), that is,  $[S_t, S_t] \subset S_t$ .*
- (ii) *For any distinct points  $t, t' \in \mathbb{C}\mathbb{P}^1$ , the foliations  $S_t, S_{t'}$  are transversal:  $S_t \cap S_{t'} = \emptyset$ .*
- (iii) *Let  $P_{t,t'} : TM \rightarrow S_t$  be a projection of  $TM$  to  $S_t$  along  $S_{t'}$ . Then  $P_{t,t'} \in \text{End}(TM)$  generate a 4-dimensional sub-bundle in  $\text{End}(TM)$ .*

**Remark 2.2.** *The classical definition of 3-webs (see e.g. [Nag]) is quite similar: one is given three integrable foliations  $S_0, S_1$  and  $S_\infty$  which are pairwise transversal. The  $SL(2)$ -webs defined above can be obtained as a special case of a 3-web.*

**Remark 2.3.** *The operators  $P_{t,t'} \in \text{End}(M)$  generate a Lie algebra isomorphic to  $\mathfrak{gl}(2)$*

**Definition 2.4.** ([A]) Let  $B$  be a holomorphic vector bundle over a complex manifold  $M$ . A **holomorphic connection** on  $B$  is a holomorphic differential operator  $\nabla : B \rightarrow B \otimes \Omega^1 M$  satisfying  $\nabla(fb) = b \otimes df + f\nabla(b)$ , for any holomorphic function  $f$  on  $M$ .

**Remark 2.5.** Let  $\nabla$  be a holomorphic connection on a holomorphic bundle, considered as a map  $\nabla : B \rightarrow B \otimes \Lambda^{1,0} M$ , and  $\bar{\partial} : B \rightarrow B \otimes \Lambda^{0,1} M$  the holomorphic structure operator. The sum  $\nabla_f := \nabla + \bar{\partial}$  is clearly a connection. Since  $\nabla$  is holomorphic,  $\nabla\bar{\partial} + \bar{\partial}\nabla = 0$ , hence the curvature  $\nabla_f^2$  is of type  $(2,0)$ . The converse is also true: a  $(1,0)$ -part of a connection with curvature of type  $(2,0)$  is always a holomorphic connection.

The following claim is well known (in the smooth setting). Its holomorphic version is no different.

**Claim 2.6.** (See [Nag], Theorem 3.2) Let  $S_1, S_2, S_3$  be a holomorphic 3-web on a complex manifold  $M$ . Then there exists a unique holomorphic connection  $\nabla$  on  $M$  which preserves the foliations  $S_i$ , and such that its torsion  $T$  satisfies  $T(S_1, S_2) = 0$ .

**Definition 2.7.** This connection is called **the Chern connection** of a 3-web. It was constructed by Chern in 1936 in his doctoral dissertation under direction of Blaschke ([C]).

**Remark 2.8.** Please notice that this definition is not symmetric on  $S_1, S_2, S_3$ .

**Proposition 2.9.** Let  $S_t, t \in \mathbb{C}\mathbb{P}^1$  be an  $SL(2)$ -web, and  $\nabla$  the Chern connection associated with  $S_0, S_1, S_\infty$ . Then  $\nabla$  is torsion-free.

**Proof:** Let  $S_t, t \in \mathbb{C}\mathbb{P}^1$  be an  $SL(2)$ -web, and  $\nabla$  its Chern connection associated with  $S_0, S_1, S_\infty$ . Clearly, the torsion  $T : TM \times TM \rightarrow TM$  preserves the sub-bundles  $S_t \subset TM$ . Indeed, these bundles are integrable, and  $\nabla$  preserves the projection maps  $P_{t,t'}$ . Consider a decomposition  $\Lambda^2 TM = \Lambda_{inv}^2 TM \oplus \Lambda_+^2 TM$  onto its invariant part and weight 2 part under the  $\mathfrak{sl}(2)$ -action. From a computation relating the action of  $\mathfrak{sl}(2)$  on  $TM$  and the condition  $T(S_t, S_t) \subset S_t$  we infer that  $T$  vanishes on  $\Lambda_+^2 TM$  (by Lemma 2.10 below). However, an  $\mathfrak{sl}(2)$ -invariant form which vanishes on  $S_0 \otimes S_1$  must be zero, by Lemma 2.11 below.

**Lemma 2.10.** Let  $V$  be a weight 1 representation of  $\mathfrak{sl}(2)$ ,  $R$  the set of all  $r \in \mathfrak{sl}(2)$  which satisfy  $\dim(rV) = \frac{1}{2} \dim V^1$ , and  $T : \Lambda^2 V \rightarrow V$  a linear map which satisfies

$$T(\text{Im } r, \text{Im } r) \subset \text{Im } r, \quad \forall r \in R. \quad (2.1)$$

Then  $T|_{\Lambda_{\pm}^2 V} = 0$ , where  $\Lambda^2 V = \Lambda_{inv}^2 V \oplus \Lambda_+^2(V)$  is the decomposition of  $\Lambda^2 V$  onto the  $\mathfrak{sl}(2)$ -invariant and weight 2 component.

<sup>1</sup>This is equivalent to  $q(r, r) = 0$ , where  $q$  is a Killing form on  $\mathfrak{sl}(2)$ .

**Proof:** Let  $x, y$  be a standard basis in an irreducible weight 1 representation  $H$  of  $\mathfrak{sl}(2) = \langle f, g, h \rangle$ , with  $h(x) = x, h(y) = -y, f(x) = y, g(y) = x$ , and  $\text{Sym}^2 H$  be its symmetric square. As a representation of  $\mathfrak{sl}(2)$ , the space  $\Lambda_+^2(V)$  is isomorphic to a direct sum of several copies of  $\text{Sym}^2 H$ . On each of these summands,  $T$  induces a map  $T_0 : \text{Sym}^2 H \rightarrow H$  satisfying (2.1). Let  $T_0^* : H \rightarrow \text{Sym}^2 H$  be its dual map (with respect to a natural  $\mathfrak{sl}(2)$ -invariant pairing). The condition (2.1) now implies that  $T_0^*(ax + by)$  is proportional to  $(ax + by)^2 \in \text{Sym}^2 H$ , for any  $a, b \in \mathbb{C}$ . The set of vectors of form  $\lambda(ax + by)^2$  is a quadric in  $\text{Sym}^2 H$ . Since the quadric contains no 2-dimensional planes, the image of  $T_0^*$  is one-dimensional unless  $T_0^* = 0$ . This is impossible, because  $T_0^*(ax + by)$  is proportional to  $(ax + by)^2$  for all  $a, b \in \mathbb{C}$ . We proved that  $T_0^* = 0$ . ■

**Lemma 2.11.** *Let  $V$  be a representation of  $\mathfrak{sl}(2)$  of weight 1,  $R$  the set of all  $r \in \mathfrak{sl}(2)$  which satisfy  $\dim(rV) = \frac{1}{2} \dim V$ ,  $x, y \in R$ , and  $\xi \in \Lambda^2 V$  an  $\mathfrak{sl}(2)$ -invariant 2-form which vanishes on  $S_x \otimes S_y$ , where  $S_x = xV$ ,  $S_y = yV$ . Then  $\xi = 0$ .*

**Proof:** It is easy to see that the projectivization  $\mathbb{P}R$  is isomorphic to  $\mathbb{CP}^1$ , and the adjoint action of  $SL(2)$  on  $\mathfrak{sl}(2)$  induces the standard action of  $SL(2)$  on  $\mathbb{P}R = \mathbb{CP}^1$ . Since this action is bitransitive, the pair  $S_x, S_y$  can be transformed to  $S_x, S_z$  for any  $z \in R$  distinct from  $x$ , by an appropriate action of  $SL(2)$ . Since  $\xi$  is  $SL(2)$  invariant and vanishes on  $S_x \otimes S_y$ , it also vanishes on  $S_x \otimes S_z$ , for all  $z \in R$ . Since a sum of all  $S_z$  is  $V$ , this implies that  $\xi$  vanishes on  $S_x \otimes V$ . ■

**Remark 2.12.** *From Proposition 2.9 it follows immediately that the Chern connection on a manifold with  $SL(2)$ -web does not depend from the choice of three points  $S_0, S_1, S_\infty$  in  $\mathbb{CP}^1$ . Indeed, the torsion-free connection preserving  $P_{t,t'} \subset \text{End}(M)$  is unique by Claim 2.6.*

**Theorem 2.13.** *Let  $M$  be a manifold equipped with a holomorphic  $SL(2)$ -web. Then its Chern connection is a torsion-free affine holomorphic connection with holonomy in  $GL(n, \mathbb{C})$  acting on  $\mathbb{C}^{2n}$  as a centralizer of an  $SL(2)$ -action, where  $\mathbb{C}^{2n}$  is a direct sum of  $n$  irreducible  $SL(2)$ -representations of weight 1. Conversely, every connection with such holonomy preserves a holomorphic  $SL(2)$ -web.*

**Proof:** Let  $\nabla$  be a Chern connection associated with a holomorphic  $SL(2)$ -web. Then  $\nabla$  is holomorphic and commutes with an action of  $\mathfrak{gl}(2)$  generated by the projection maps  $P_{t,t'} \subset \text{End}(M)$ . Conversely, for every such connection,  $\text{Hol}(\nabla) \subset GL(n, \mathbb{C})$  implies that  $\nabla$  preserves a family of  $n$ -dimensional sub-bundles of  $TM$  parametrized by  $\mathbb{CP}^1$ . These sub-bundles are integrable because  $\nabla$  is torsion-free. ■

## 2.2 An example: rational curves on a twistor space

The basic example of holomorphic  $SL(2)$ -webs comes from hyperkähler geometry. Let  $M$  be a hyperkähler manifold, and  $\text{Tw}(M)$  its twistor space.

Denote by  $\text{Sec}(M)$  the space of holomorphic sections of the twistor fibration  $\text{Tw}(M) \xrightarrow{\pi} \mathbb{C}\mathbb{P}^1$ .

We consider  $\text{Sec}(M)$  as a complex variety, with the complex structure induced from the Douady space of rational curves on  $\text{Tw}(M)$ . Clearly, for any  $C \in \text{Sec}(M)$ ,  $T_C \text{Sec}(M)$  is a subspace in the space of sections of the normal bundle  $NC$ . This normal bundle is naturally identified with  $T_\pi \text{Tw}(M)|_C$ , where  $T_\pi \text{Tw}(M)$  denotes the vertical tangent space.

For each point  $m \in M$ , one has a horizontal section  $C_m := \{m\} \times \mathbb{C}\mathbb{P}^1$  of  $\pi$ . The space of horizontal sections of  $\pi$  is denoted  $\text{Sec}_{hor}(M)$ ; it is naturally identified with  $M$ . It is easy to check that  $NC_m = \mathcal{O}(1)^{\dim M}$ , hence some neighborhood of  $\text{Sec}_{hor}(M) \subset \text{Sec}(M)$  is a smooth manifold of dimension  $2 \dim M$ . It is easy to see that  $\text{Sec}(M)$  is a complexification of  $M = \text{Sec}_{hor}(M)$  considered as a real analytic manifold (see [V2]).

Let  $\text{Sec}_0(M)$  be a part consisting of all rational curves  $C \in \text{Sec}(M)$  with  $NC = \mathcal{O}(1)^{\dim M}$ . Clearly,  $\text{Sec}_0(M)$  is a smooth, open subvariety in  $\text{Sec}(M)$ .

On  $\text{Sec}_0(M)$ , there is an  $SL(2)$ -web constructed as follows. For each  $C \in \text{Sec}_0(M)$  and  $t \in \mathbb{C}\mathbb{P}^1 = C$ , define  $S_t \subset T_C \text{Sec}_0(M) = \Gamma_C(NC)$  as a space of all sections of  $NC$  vanishing at  $t \in C$ .

It is not difficult to check that this is an  $SL(2)$ -web. Transversality of  $S_t$  and  $S_{t'}$  is obvious, because a section of  $\mathcal{O}(1)$  vanishing at two points is zero. Integrability of  $S_t$  is also obvious, because the leaves of  $S_t$  are fibers of the evaluation map  $ev_t : \text{Sec}(M) \rightarrow (M, t)$ , mapping  $C : \mathbb{C}\mathbb{P}^1 \rightarrow \text{Tw}(M)$  to  $C(t)$ . The last condition is clear, because  $\Gamma_{\mathbb{C}\mathbb{P}^1}(V \otimes_{\mathbb{C}} \mathcal{O}(1)) = V \otimes_{\mathbb{C}} \mathbb{C}^2$ , and the projection maps  $P_{t,t'}$  act on  $V \otimes_{\mathbb{C}} \mathbb{C}^2$  through the second component.

## 3 Autodual NHYM-bundles on quaternionic projective spaces

### 3.1 NHYM autodual bundles on hyperkähler manifolds

Let  $M$  be a hyperkähler manifold; recall that the quaternionic action on  $TM$  naturally induces a multiplicative action of  $SU(2)$  on  $\Lambda^*(M)$ .

**Definition 3.1.** *Let  $(B, \nabla)$  be a complex vector bundle with connection, not necessarily Hermitian, on a hyperkähler manifold  $M$ . The connection  $\nabla$  is called **non-Hermitian Yang-Mills autodual** (NHYM autodual) if its curvature  $R \in \Lambda^2(M, \text{End } B)$  is  $SU(2)$ -invariant.*

**Remark 3.2.** *The relation between this notion and the notion of hyperholomorphic bundles defined in [V1] is somewhat intricate. A hyperholomorphic bundle on a compact hyperkähler manifold is a stable bundle with first two Chern classes  $SU(2)$ -invariant. It is known that the Yang-Mills connection of such a bundle (which exists by Donaldson-Uhlenbeck-Yau, and is unique) has  $SU(2)$ -invariant*

curvature, [V1], hence it is “NHYM autodual” in the sense of the above definition. Moreover, every stable bundle admitting a NHYM autodual connection has  $SU(2)$ -invariant Chern classes, hence it is hyperholomorphic. However, the NHYM condition does not necessarily imply stability.

Autodual connections on  $M$  give rise to holomorphic bundles on the twistor space  $\text{Tw}(M)$  by means of a construction known as *twistor transform*. This construction turns out to be essentially invertible. More precisely, let us first recall the following result.

**Lemma 3.3.** ([KV, Lemma 5.1]) *Let  $(B, \nabla)$  be a complex vector bundle with connection on a hyperkähler manifold, and  $\text{Tw}(M) \xrightarrow{\sigma} M$  the standard projection. The connection  $\nabla$  is NHYM autodual if and only if the connection  $\sigma^*\nabla$  has curvature of Hodge type  $(1, 1)$ .*

The holomorphic bundle  $(\sigma^*B, \sigma^*\nabla^{0,1})$  is called **the twistor transform** of the autodual bundle  $(B, \nabla)$ .

Now let  $\mathcal{B}$  be a holomorphic bundle on  $\text{Tw}(M)$ . We say that  $\mathcal{B}$  is **trivial on horizontal curves** if the restriction of  $\mathcal{B}$  to any  $C \in \text{Sec}_{hor}(M)$  is trivial (see Subsection 2.2 for notation).

**Remark 3.4.** *Let  $M$  be a compact hyperkähler manifold. Since a trivial bundle is polystable, a small deformation of a bundle which is trivial on  $\mathbb{CP}^1$  is again trivial. Since the set of horizontal curves is identified with  $M$ , it follows that it is compact. Therefore, a small deformation of a bundle which is trivial on horizontal curves is again trivial on horizontal curves.*

**Theorem 3.5.** ([KV, Theorem 5.12]) *The twistor transform gives an equivalence between the category of autodual NHYM-bundles on  $M$  and the category of holomorphic bundles on  $\text{Tw}(M)$  which are trivial on horizontal curves.*

For Hermitian autodual bundles on quaternionic-Kähler manifolds, an analogue of Theorem 3.5 is proved by T. Nitta in [Nit], and the NHYM-version of his theorem can be obtained in the same way as [KV, Theorem 5.12]. The space  $\text{Sec}_{hor}(M)$ , in this case, is the set of all holomorphic lines of form  $\sigma^{-1}(m)$ , where  $m \in M$  is a point, and  $\sigma : \text{Tw}(M) \rightarrow M$  a standard projection, and  $\text{Sec}(M)$  the space of all rational curves in  $\text{Tw}(M)$  obtained by deforming curves in  $\text{Sec}_{hor}(M)$ .

For  $\mathbb{HP}^k$ , the  $k$ -dimensional quaternionic projective space ( $k \geq 1$ ), one has  $\text{Tw}(M) = \mathbb{CP}^{2k+1}$ . By Nitta’s result mentioned above, there exists a 1-1 correspondence between NHYM autodual bundles on  $\mathbb{HP}^k$  and holomorphic bundles on  $\mathbb{CP}^{2k+1}$  which are trivial on horizontal curves.

In what follows, we will provide a linear algebraic description of holomorphic bundles on  $\mathbb{CP}^{2k+1}$  which are framed on a horizontal curve.

### 3.2 ADHM description of framed bundles on complex projective spaces

Let  $V$  and  $W$  be complex vector spaces, with dimensions  $c$  and  $r$ , respectively. Fix  $d \geq 0$ , and consider the following data ( $k = 0, \dots, d$ ):

$$A_k, B_k \in \text{End}(V)$$

$$I_k \in \text{Hom}(W, V) \quad , \quad J_k \in \text{Hom}(V, W) \quad .$$

Choose homogeneous coordinates  $[z_0 : \dots : z_d]$  on  $\mathbb{C}\mathbb{P}^d$  and define

$$\tilde{A} := A_0 \otimes z_0 + \dots + A_d \otimes z_d \quad \text{and} \quad \tilde{B} := B_0 \otimes z_0 + \dots + B_d \otimes z_d \quad . \quad (3.1)$$

These can be regarded as sections of  $\text{Hom}(V, V) \otimes \mathcal{O}_{\mathbb{P}^d}(1)$ . Define also:

$$\tilde{I} = I_0 \otimes z_0 + \dots + I_d \otimes z_d \quad \text{and} \quad \tilde{J} = J_0 \otimes z_0 + \dots + J_d \otimes z_d \quad . \quad (3.2)$$

Similarly,  $\tilde{I}$  can be regarded as a section of  $\text{Hom}(W, V) \otimes \mathcal{O}_{\mathbb{P}^d}(1)$ , while  $\tilde{J}$  can be regarded as a section of  $\text{Hom}(V, W) \otimes \mathcal{O}_{\mathbb{P}^d}(1)$ .

A  $d$ -dimensional ADHM datum is a quadruple  $\tilde{X} = (\tilde{A}, \tilde{B}, \tilde{I}, \tilde{J})$ , and can be thought as a point in the affine space  $\tilde{\mathbf{B}}_d := \mathbf{B} \otimes H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(1))$ , where

$$\mathbf{B} := \text{End}(V) \oplus \text{End}(V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W).$$

Given a point  $p \in \mathbb{C}\mathbb{P}^d$ , we have a natural evaluation map  $\text{ev}_p : \tilde{\mathbf{B}}_d \rightarrow \mathbf{B}$ ; for simplicity, we denote  $\tilde{X}(p) := \text{ev}_p(\tilde{X})$ ; note that  $\tilde{X}(p)$  is a 0-dimensional ADHM datum.

Recall that a 0-dimensional ADHM datum is said to be:

1. *stable*, if there is no subspace  $S \subsetneq V$  such that  $A(S), B(S) \subset S$  and  $I(W) \subset S$ ;
2. *costable*, if there is no nontrivial subspace  $S \subset V$  such that  $A(S), B(S) \subset S$  and  $S \subset \ker J$ ;
3. *regular*, if it is both stable and costable.

The most relevant definition for this paper is the following.

**Definition 3.6.** *A datum  $\tilde{X} = (\tilde{A}, \tilde{B}, \tilde{I}, \tilde{J}) \in \tilde{\mathbf{B}}_d$  is said to be globally regular if  $\tilde{X}_p$  is regular for every  $p \in \mathbb{C}\mathbb{P}^d$ .*

We consider here the following generalization of the ADHM equation, which we call the *d-dimensional ADHM equation*:

$$[\tilde{A}, \tilde{B}] + \tilde{I}\tilde{J} = 0 \quad . \quad (3.3)$$

Now consider the following action of the group  $G = GL(V)$  on  $\tilde{\mathbf{B}}_d$ :

$$g \cdot (A_0, \dots, A_d, B_0, \dots, B_d, I_0, \dots, I_d, J_0, \dots, J_d) = \quad (3.4)$$

$$(gA_0g^{-1}, \dots, gA_dg^{-1}, gB_0g^{-1}, \dots, gB_dg^{-1}, gI_0, \dots, gI_d, J_0g^{-1}, \dots, J_dg^{-1}) .$$

Note that

$$g \cdot (\tilde{A}, \tilde{B}, \tilde{I}, \tilde{J}) = (g\tilde{A}g^{-1}, g\tilde{B}g^{-1}, g\tilde{I}, \tilde{J}g^{-1})$$

where

$$g\tilde{A}g^{-1} = gA_0g^{-1} \otimes z_0 + \dots + gA_dg^{-1} \otimes z_d$$

and so on. In particular, we have  $(g \cdot \tilde{X})(p) = g \cdot \tilde{X}(p)$ . It is easy to see that such action preserves global regularity and the set of solutions of the  $d$ -dimensional ADHM equation (3.3).

**Definition 3.7.**  $\mathcal{M}_d(r, c)$  denotes the set of globally regular solutions of the  $d$ -dimensional ADHM equation modulo the action of  $G$ .

The corresponding geometric objects are instanton bundles on projective spaces. Recall that an *instanton bundle* on  $\mathbb{C}\mathbb{P}^n$  ( $n \geq 2$ ) is a locally free coherent sheaf  $E$  on  $\mathbb{C}\mathbb{P}^n$  with  $c_1(E) = 0$  satisfying the following cohomological conditions:

- (i) for  $n \geq 2$ ,  $H^0(E(-1)) = H^n(E(-n)) = 0$ ;
- (ii) for  $n \geq 3$ ,  $H^1(E(-2)) = H^{n-1}(E(1-n)) = 0$ ;
- (iii) for  $n \geq 4$ ,  $H^p(E(k)) = 0$ ,  $2 \leq p \leq n-2$  and  $\forall k$ ;

The integer  $c = -\chi(E(-1)) = h^1(E(-1)) = c_2(E)$  is called the *charge* of  $E$ .

Moreover, a locally free coherent sheaf  $E$  on  $\mathbb{C}\mathbb{P}^n$  is said to be of trivial splitting type if there is a line  $\ell \subset \mathbb{C}\mathbb{P}^n$  such that the restriction  $E|_\ell$  is the free sheaf, i.e.  $E|_\ell \simeq \mathcal{O}_\ell^{\oplus \text{rk} E}$ . A *framing* on  $E$  is the choice of an isomorphism  $\phi : E|_\ell \rightarrow \mathcal{O}_\ell^{\oplus \text{rk} E}$ . A *framed bundle* is a pair  $(E, \phi)$  consisting of a locally free coherent sheaf  $E$  of trivial splitting type and a framing  $\phi$ . Two framed bundles  $(E, \phi)$  and  $(E', \phi')$  are isomorphic if there exists a bundle isomorphism  $\Psi : E \rightarrow E'$  such that  $\phi' = \phi \circ (\Psi|_\ell)$ .

For  $n = 2$ , every locally free coherent sheaf of trivial splitting type is automatically instanton, since the vanishing condition (i) is satisfied.

The following result, announced in [J2], establishes the relation between framed instanton bundles and solutions of the  $d$ -dimensional ADHM equations, generalizing the well-known result due to Donaldson [D1].

**Theorem 3.8.** *There exists a 1-1 correspondence between equivalence classes of globally regular solutions of the  $d$ -dimensional ADHM equations and isomorphism classes of instanton bundles on  $\mathbb{C}\mathbb{P}^{d+2}$  framed at a fixed line  $\ell$ , where  $\dim W = \text{rk}(E)$  and  $\dim V = c_2(E)$ .*

The case  $d = 0$  is originally due to Donaldson [D1], who also showed that  $\mathcal{M}_0(r, c)$  is isomorphic to the moduli space of rank  $r$  framed instantons (i.e. finite energy connections with anti-self-dual curvature) on  $\mathbb{R}^4$  of charge  $c$ .

The case  $d = 1$  was considered in detail in [FJ]. In general, the odd case  $d = 2k+1$  admits the following differential geometric interpretation. Fix  $\ell \subset \mathbb{C}\mathbb{P}^{2k+1}$

to be a horizontal line. By Remark 3.4, one may regard the set of (framed) holomorphic bundles on  $\mathbb{C}\mathbb{P}^{2k+1} = \text{Tw}(\mathbb{H}\mathbb{P}^k)$  which are trivial on horizontal curves as an open subset of  $\mathcal{M}_{2k-1}(r, c)$ .

In other words, for each  $k \geq 1$ , there is an open subset of  $\mathcal{M}_{2k-1}(r, c)$  whose points may be interpreted, via the twistor correspondence, as autodual NHYM-bundles on  $\mathbb{H}\mathbb{P}^k$  which are framed at a point.

We will finish this paper by taking a closer look at the case  $d = 1$ .

### 3.3 Geometric structures on $\mathcal{M}_1(r, c)$

Let  $W$  be a component of the moduli space of Hermitian autodual bundles on a compact hyperkähler manifold  $M$ . Then  $W$  can be identified with a component of the moduli space of stable holomorphic bundles on  $M$ ; in this case,  $W$  is hyperkähler, as follows from [V1]. Indeed, in [V1] it was shown that the moduli space of stable holomorphic structures on a bundle  $B$  with  $SU(2)$ -invariant Chern classes  $c_1(B)$ ,  $c_2(B)$  is hyperkähler. The invariance of the Chern classes is clear because the Chern classes are expressed through the curvature using the Gauss-Bonnet formula, and the curvature is  $SU(2)$ -invariant.

In this case, one may define its twistor space  $\text{Tw}(W)$ . In [KV, Theorem 7.2] it was shown that the space  $\text{Sec}(W)$  of twistor sections can be identified with an open subset of the moduli of holomorphic bundles on  $\text{Tw}(M)$ .

In this section we establish an analogue to this result for framed instantons on euclidean space. As it is also well-known (see for instance [Nak]),  $\mathcal{M}_0(r, c)$  has the structure of a smooth hyperkähler manifold with  $\dim_{\mathbb{R}} \mathcal{M}_0(r, c) = 4rc$ ; let  $\text{Tw}(\mathcal{M}_0(r, c))$  denote its twistor space; let also  $\mathcal{S}(r, c) := \text{Sec}(\mathcal{M}_0(r, c))$ , the set of holomorphic sections of the twistor fibration  $\text{Tw}(\mathcal{M}_0(r, c)) \rightarrow \mathbb{C}\mathbb{P}^1$ .

**Theorem 3.9.** *The moduli space  $\mathcal{M}_1(r, c)$  is biholomorphically equivalent to the space of sections  $\mathcal{S}(r, c)$ .*

In particular, it follows that  $\dim \mathcal{M}_1(r, c) = 4rc$ , and that the smooth locus of  $\mathcal{M}_1(r, c)$  has the structure of a holomorphic  $SL(2)$ -web.

*Proof.* Given a framed instanton bundle on  $\mathbb{C}\mathbb{P}^3$ , let  $\tilde{X} = (\tilde{A}, \tilde{B}, \tilde{I}, \tilde{J})$  be the associated 1-dimensional ADHM datum; let  $[\tilde{X}]$  denote its  $G$ -orbit. Then define a map  $\sigma : \mathbb{C}\mathbb{P}^1 \rightarrow \mathcal{M}_0(r, c)$  in the following way:

$$\sigma(p) = [\tilde{X}(p)].$$

One easily sees that this map is a well-defined, holomorphic map.

Conversely, giving a (holomorphic) map  $\sigma : \mathbb{C}\mathbb{P}^1 \rightarrow \mathcal{M}_0(r, c)$  is the same as giving a family of 0-dimensional ADHM data  $(A(p), B(p), I(p), J(p))$  which is a regular solution of the ADHM equation at each  $p$ . But this is precisely a globally regular solution of the 1-dimensional ADHM equation, and therefore a point of  $\mathcal{M}_1(r, c)$ .

This establishes a bijection between the sets  $\mathcal{M}_1(r, c)$  and  $\mathcal{S}(r, c)$ . To check that this correspondence is a biholomorphic, we use [KV, Theorem 7.2], where

the space of fiberwise stable holomorphic bundles on a twistor space  $\text{Tw}(M)$  for a hyperkähler manifold  $M$  is shown to be biholomorphic with a space  $\text{Sec}(W)$  of twistor sections, associated with the moduli of stable bundles on  $M$ . However, this result cannot be applied directly, because in our case  $M$  is  $\mathbb{C}^2$ , which is non-compact, and  $W$  is the moduli of framed instantons. Nevertheless, it is easy to check that the proof of [KV, Theorem 7.2] carries through to framed instantons without any changes. In fact, the proof of [KV, Theorem 7.2] is based on an equivalence between holomorphic bundles on a twistor space  $\text{Tw}(M)$  and NHYM autodual bundles on  $M$  ([KV, Theorem 5.12]), which is a local result, valid for non-compact  $M$ . ■

As an immediate application of Theorem 3.9, we obtain the following corollary.

**Corollary 3.10.** *Let  $\mathcal{M}_1(r, c)$  be the space of framed instantons on  $\mathbb{C}\mathbb{P}^3$ . Then  $\mathcal{M}_1(r, c)$  has no compact complex subvarieties of positive dimension.*

**Proof:** In [KV, Proposition 8.15], a strictly plurisubharmonic function was constructed on  $\text{Sec}(M)$  for any hyperkähler manifold  $M$ . Then,  $\mathcal{M}_1(r, c) \cong \text{Sec}(\mathcal{M}_0(r, c))$  admits a strictly plurisubharmonic function. ■

Holomorphic rank 2 bundles  $E \rightarrow \mathbb{C}\mathbb{P}^3$  with vanishing first Chern class and satisfying  $H^0(E) = H^1(E(-2)) = 0$  are known in the literature as *mathematical instanton bundles*, see for instance [CTT]. These objects have been intensively studied since the 1980's by various authors, see for instance the survey in the Introduction of [CTT] for several references. In particular, every such bundle is (slope and Gieseker) stable.

Let  $\mathcal{I}(c)$  denote the moduli space of such bundles; the following is an important open question (c.f. [CTT, Conjecture 1.2]):

**Conjecture 3.11.**  *$\mathcal{I}(c)$  is an irreducible, non-singular, quasi-projective variety of (complex) dimension  $8c - 3$ .*

This conjecture is known to be true for  $c \leq 5$  [CTT], and each case was originally proved by different sets of authors.

By [J1, Theorem 3], instanton bundles are precisely those obtained as cohomology of a linear monad. Moreover, [J1, Proposition 11] tells us that if  $E$  is a rank  $n - 1$  instanton bundle on  $\mathbb{C}\mathbb{P}^n$ , then  $H^0(E) = 0$ . In particular, it follows that rank 2 instanton bundles on  $\mathbb{C}\mathbb{P}^3$  are mathematical instanton bundles.

Moreover, the converse is also true: if  $E$  is a mathematical instanton bundle, then there is a (unique up to a scalar) symplectic isomorphism between  $E$  and its dual  $E^*$ ; one can then use Serre duality to show that  $H^2(E(-2)) = H^3(E(-3)) = 0$ , thus  $E$  is a rank 2 instanton bundle.

Therefore, there exists a forgetful map  $\psi : \mathcal{M}_1(2, c) \rightarrow \mathcal{I}_\ell(c)$ , where  $\mathcal{I}_\ell(c)$  is the open subset of  $\mathcal{I}(c)$  consisting of mathematical instanton bundles restricting trivially to a fixed  $\ell$ . The fibers of  $\psi$  are the sets of all possible framings (up to equivalence), thus  $\mathcal{M}_1(2, c)$  becomes a principal  $SL(2)$ -bundle over  $\mathcal{I}_\ell(c)$ .

Now it follows from Theorem 3.9 and the observations in Section 2.2 that the irreducible component of  $\mathcal{M}_1(2, c)$  containing the complexification of  $\mathcal{M}_0(2, c)$  has dimension  $2 \dim \mathcal{M}_0(2, c) = 8c$ ; we can then conclude that at least one component of  $\mathcal{I}_\ell(c)$  has dimension  $8c - 3$ .

The following observation seems to be well-known (see e.g. [HL]).

**Lemma 3.12.** *If  $E$  is a semistable rank 2 coherent torsion free sheaf on  $\mathbb{C}\mathbb{P}^3$  with  $c_1(E) = 0$ , then there is a line  $\ell$  such that  $E|_\ell \simeq \mathcal{O}_\ell^{\oplus 2}$ .*

On the other hand, notice that if  $E$  is nontrivial, then there exist a line  $\ell'$  such that the restricted sheaf  $E|_{\ell'}$  is nontrivial. So for any given line  $\ell \subset \mathbb{C}\mathbb{P}^3$ , there are semistable rank 2 coherent torsion free sheaves which are trivial at  $\ell$  as well as sheaves that are nontrivial at  $\ell$ .

*Proof.* The lemma is a direct application of the famous Grauert-Müllich theorem ([GM]). The restriction of  $E$  to a generic line  $\ell \subset \mathbb{P}^3$  yields a torsion free (hence locally free) sheaf on  $\ell$ . If  $E|_\ell = \mathcal{O}_\ell(a) \oplus \mathcal{O}_\ell(b)$  with  $a \geq b$ , Theorem 3.1 of [M] implies that  $0 \leq a - b \leq 1$ . But  $a + b = c_1(E) = 0$ , thus we must have  $a = b = 0$ .

■

Now let  $\mathcal{G}(c)$  denote the moduli space of S-equivalence classes of semistable torsion-free sheaves  $E$  of rank 2 on  $\mathbb{P}^3$  with  $c_1(E) = 0$ ,  $c_2(E) = c$  and  $c_3(E) = 0$ ; it is a projective variety.  $\mathcal{I}(c)$  can be regarded as the open subset of  $\mathcal{G}(c)$  consisting of those locally free sheaves satisfying  $H^1(E(-2)) = 0$ .

Two important facts follow from our previous lemma. First, for any fixed line  $\ell \subset \mathbb{P}^3$ ,  $\mathcal{I}(c)$  is contained in  $\overline{\mathcal{I}_\ell(c)}$ , where the closure is taken within  $\mathcal{G}(c)$ , thus  $\mathcal{I}(c)$  is irreducible if and only if  $\mathcal{I}_\ell(c)$  is. Second,  $\mathcal{I}(c)$  is covered by open subsets of the form  $\mathcal{I}_\ell(c)$ , but it is not contained within any such sets, thus  $\mathcal{I}(c)$  and  $\mathcal{I}_\ell(c)$  must have the same dimension, and one is nonsingular if and only if the other is as well.

Summing up our conclusions, we have proved that the Conjecture is true if and only if, for some line  $\ell \subset \mathbb{C}\mathbb{P}^3$ , the quasi-projective variety  $\mathcal{I}_\ell(c)$  is irreducible, non-singular and of dimension  $8c - 3$ . In particular, we have established that  $\mathcal{I}(c)$  possesses an irreducible component of dimension  $8c - 3$  for all  $c \geq 1$ ; existence of an irreducible component of the expected dimension was already known, since there are several examples of mathematical instanton bundles that are unobstructed, i.e.  $H^2(\text{End}(E)) = 0$ ; see also [T] for a more elaborate result.

Moreover,  $\mathcal{I}_\ell(c)$  is irreducible and nonsingular if and only if  $\mathcal{M}_1(2, c)$  is. We hope that the geometric and ADHM-type descriptions of this space given in this paper will be valuable tools in an attempt to prove the Conjecture.

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