THE ASSIGNMENT GAME : ONE-TO-ONE MATCHING WITH MONEY

an elementary self-contained exposition

Model & Existence: Koopmans Beckmann 1957, Shapley Shubik 1970 (Roth&Sotomayor Ch 8)

Strategy : Leonard 1983

Auction : Demange Gale Sotomayor 1986, Alkan 1989, 1992

An *assignment game* is a triple $(B, S, w)$ where $B$ and $S$ are two finite sets of agents and $w$ is an array whose entry $w_{bs} \geq 0$ denotes the *worth* of or the *surplus* achievable by the pair $(b, s)$.

Examples: (i) Market for a set of objects on one side and a set of "unit-demander" buyers on the other, (ii) Partnership formation between two sides, e.g., entrepreneurs and venture capitalists.

A **matching** $\mu$ is a subset of $B \times S$ where no agent recurs, i.e., for any $(b, s), (b', s')$ in $\mu$ neither $b = b'$ nor $s = s'$. (With some notational abuse, we will write $b \in \mu$ as well as $(b, s) \in \mu$.)

An **allocation** is a matching $\mu$ and a pair of (nonnegative) **payoff** vectors $u \in R_+^B$, $v \in R_+^S$.

An allocation $(u, v; \mu)$ is **feasible** if $u_b + v_s \leq w_{bs}$ for all $(b, s) \in \mu$ and if $u_b = v_s = 0$ for $b \notin \mu$, $s \notin \mu$.

EQUIVALENCE OF PAIRWISE STABLE, CORE AND COMPETITIVE EQUILIBRIUM OUTCOMES

A feasible allocation $(u, v; \mu)$ is **stable** if $u_b + v_s \geq w_{bs}$ for all $(b, s) \in B \times S$.

Note that if $(u, v; \mu)$ is **stable** then $u_b = w_{b\mu(b)} - v_{\mu(b)}$ for $b \in \mu$ and $u_b = 0$ for $b \notin \mu$; in particular, the payoff $u$ is determined by $(v, \mu)$ (likewise $v$ by $(u, \mu)$) .

For any assignment game $(B, S, w)$, a game in **coalitional form** is obtained as follows : For every coalition $C = B' \cup S'$ where $B' \subset B$ and $S' \subset S$, define its **worth** $W(C)$ to be the maximum of $\sum_{(b,s)\in\eta} w_{bs}$ among all matchings $\eta \subset B' \times S'$. (If $C$ contains no pair $(b, s)$ then $W(C) = 0$, in particular $W(b) = W(s) = 0$ for every $b$ and $s$, and $W(b, s) = w_{bs}$.)

A matching $\mu$ for which $W(B \cup S) = \sum_{(b,s)\in\mu} w_{bs}$ is called an **optimal assignment**.

An **imputation** is a payoff $(u, v) \in R_+^B \times R_+^S$ such that $\sum_{b\in B} u_b + \sum_{s\in S} v_s = W(B \cup S)$. An imputation $(u, v)$ is said to be in the **core**, or a **core payoff,** if $\sum_{b\in B'} u_b + \sum_{s\in S'} v_s \geq W(B' \cup S')$ for every $B' \subset B, S' \subset S$.

**Proposition 1** *If $(u, v)$ is a core payoff and $\mu$ is an optimal assignment then $(u, v; \mu)$ is a stable allocation.*

**Proof.** Let $(u, v)$ be a core payoff and $\mu$ an optimal assignment. Then $W(B \cup S) = \sum_{b \in B} u_b + \sum_{s \in S} v_s$ (since $(u, v)$ is an imputation) $= \sum_{b \notin B} u_b + \sum_{s \notin S} v_s + \sum_{(b,s) \in \mu} (u_b + v_s)$ (regrouping) $\geq \sum_{b \notin \mu} u_b + \sum_{s \notin \mu} v_s + \sum_{(b,s) \in \mu} W(b, s)$ (core inequalities for $(b, s) \in \mu$) $\geq \sum_{(b,s) \in \mu} W(b, s)$ (since imputation payoffs nonnegative) $= \sum_{(b,s) \in \mu} w_{bs} = W(B \cup S)$ (since $\mu$ is optimal). So the inequalities must be equalities. Therefore $u_b + v_s = w_{bs}$ for all $(b, s) \in \mu$ and $u_b = v_s = 0$ for all $b \notin \mu$, $s \notin \mu$; thus $(u, v; \mu)$ is feasible. Of course, the stability inequalities $(b, s) \in B \times S$ are just the core inequalities for $(b, s) \in B \times S$. ■

An *assignment market* is a quadruple $(B, S, r, z)$ where $B$ is a set of *buyers*, $S$ is a set of *sellers* each owning a single object, $r_s$ is the *reservation value* of $s \in S$ for his object, and $z_{bs}$ is the *maximum willingness to pay* of $b \in B$ for object $s \in S$. We assume each buyer has need for at most one object.

At any *price* vector $p \in R^S$ and for any buyer $b \in B$, let $u_{bs}(p) = u_{bs}(p_s) = z_{bs} - p_s$ be $b$'s *utility* for buying object $s \in S$, and define his *demand correspondence* $D_b(p)$ as the set of all $s \in S$ with the *largest* utility $u_{bs}(p)$ among all the objects in $S$ if this utility is nonnegative and as the empty set otherwise. At a price vector $p$, say that $b$ is *active* if $u_{bs}(p)$ is positive for some $s$ and that $s$ is *active* if $p_s - r_s$ is positive.

Call $(p, \mu)$ a **competitive equilibrium** if at prices $p$ the matching $\mu$ equates supply and demand, that is, if $b$ is active then $b \in \mu$ and if $s$ is active then $s \in \mu$ and $\mu(b) \in D_b(p)$ for all $b \in \mu$. (Wlog $p \geq r$.)

Note that $(p, \mu)$ is a competitive equilibrium for $(B, S, 0, w)$ iff $(p + r, \mu)$ is a competitive equilibrium for $(B, S, r, z)$. Also note that an assignment market $(B, S, r, z)$ defines an assignment game $(B, S, w)$ where $w_{bs} = \max\{z_{bs} - r_s, 0\}$ while an assignment game $(B, S, w)$ defines an assignment market $(B, S, r, z)$ for any $r$ by setting $z_{bs} = w_{bs} + r_s$.

Wlog let $r = 0$ and $(B, S, w)$ be an assignment game or market.

**Proposition 2** *If $(u, v; \mu)$ is a stable allocation then $(v, \mu)$ is a competitive equilibrium.*

**Proof.** Let $(u, v; \mu)$ be a stable allocation. Then $u_b = w_{b\mu(b)} - v_{\mu(b)} \geq 0$ for $b \in \mu$ and $u_b = 0$ for $b \notin \mu$ and $v_s = 0$ for $s \notin \mu$. In particular, any $b$ or $s$ active at the price vector $v$ belongs to

$\mu$. For $b \in \mu$, use the stability inequalities to see $u_b = w_{b\mu(b)} - v_{\mu(b)} \geq w_{bs} - v_s$ for all $s \in S$, that is $\mu(b) \in D_b(p)$. ■

**Proposition 3** *If $(v, \mu)$ is a competitive equilibrium then $(u, v)$, where $u_b = w_{b\mu(b)} - v_{\mu(b)}$ for $b \in \mu$ and $u_b = 0$ for $b \notin \mu$, is a core payoff.*

**Proof.** Exercise. ■

**Proposition 4** ***(Corollary)*** *The set of core, stable and competitive equilibrium payoffs are identical.*

*Call a matching $\mu$ a stable matching (a competitive equilibrium matching) if there is a stable allocation $(u, v; \mu)$ (a competitive equilibrium $(v; \mu)$ ).*

***Corollary*** *A matching is a stable matching or a competitive equilibrium matching if and only if it is an optimal assignment.*

*EXISTENCE*

**Theorem 1** *There exists a competitive equilibrium.*

**Proof.** Call $(p, \mu)$ a *seller equilibrium* if the matching $\mu$ contains a $(b, s)$ with $s \in D_b(p)$ for every $s$ active at the price vector $p$. Let $P$ denote the set of all seller equilibrium price vectors. Note $P$ is nonempty (since $0 \in P$), bounded above and closed. Let $p$ be an element of $P$ with maximum coordinate sum. Let $\mu$ be a seller equilibrium matching at $p$ which (among all seller equilibrium matchings) assigns the largest number of active buyers. We claim $(p, \mu)$ is a competitive equilibrium.

Suppose not. Then there is an active buyer $b \notin \mu$. Call $s$ *reachable* from $b$ if there is a sequence $(b_0, s_1), (b_1, s_1), ..., (b_n, s_n)$ such that $b = b_0$, $s_j \in D_{b_{j-1}}(p)$ and $(b_j, s_j) \in \mu$ for $j = 1, ..., n$. Note that all $b_j$ in such a sequence are active, for otherwise the number of active buyers assigned can be increased (by assigning $s_1$ to $b$ and modifying $\mu$ along the sequence.) Now let $T$ be the set of all $s$ reachable from $b$. Note that $q$ defined by $q_s = p_s + \epsilon$ for $s \in T$ and $q_s = p_s$ for $s \notin T$ is a seller equilibrium price vector (under $\mu$) for $\epsilon$ sufficiently small and positive. Contradiction. ■

Exercise : If $p$ and $q$ are two competitive equuilibrium price vectors the so is their coordinatewise maximum $p \vee q$ and minimum $p \wedge q$. In particular, there is a minimum (buyer-optmal) equilibrium price vector, and a maximum (seller-optimal) one.

## STRATEGY

Consider that you are one of a number of individuals, asked to report maximum-willingness-to-pay values for each of a number of objects, in a sealed auction which will award each participant at most one object, according to the (buyer-optimal) minimum price equilibrium for the reported values. Prove that it is a weakly dominant strategy for you to submit your true values. (Generalization of the Vickrey Second Price Auction in the heterogeneous objects unit-demand case. Leonard 1983.)

Proof: Let $(B, S, (w_{bs}))$ be an assignment game. Recall that if $(p, \mu)$ is a competitive price equilibrium then $\mu$ is an optimal assignment, i.e. $\sum_{(b,s)\in\mu} w_{bs} = W(B \cup S)$ where $W(B \cup S)$ is the maximum worth of the grand coalition, $B \cup S$, among all assignments, and $(u, p)$, given by $u_b = w_{b\mu(b)} - p_{\mu(b)}$ for $b \in \mu$ and $u_b = 0$ otherwise, is a core imputation, i.e. $\sum_{b\in B} u_b + \sum_{s\in S} p_s = W(B \cup S)$.

The important observation here is that if $(p,\mu)$ is the minimum price equilibrium, then for any buyer $b \in B$, there is a $\mu - alternating$ path to either an object $s \in \mu$ with $p_s = 0$ or a buyer $b \notin \mu$ with $u_b = 0$. (Otherwise it would be possible to lower $p_s$ to $p_s - \epsilon$ for $\epsilon$ sufficiently small for all $s$ reachable from $b$ by a $\mu - alternating$ path and still have an equilibrium. )

The next observation is that hence $(p,\mu)$ is a competitive equilibrium for $(B\backslash b, S, (w_{bs}))$. ( To see this, modify $\mu$ by its alternate along the $\mu - alternating$ path either leaving the end object $s \in \mu$ with $p_s = 0$ unmatched or assigning the end object to the end buyer $b \notin \mu$ with $u_b = 0$ )

Therefore $W(B\backslash b \cup S) = \sum_{b\in B\backslash b} u_b + \sum_{s\in S} p_s$ , so $u_b = W(B \cup S) - W(B\backslash b \cup S)$. Since $W(B \cup S) = w_{b\mu(b)} + W(B\backslash b \cup S\backslash s)$, one has $p_{\mu(b)} = W(B\backslash b \cup S) - W(B\backslash b \cup S\backslash \mu(b))$. In particular $b$ cannot acquire $\mu(b)$ at a lower price by misreporting. To see that $b$ cannot achieve a utility higher than $u_b$ by getting a different object either, note $W(B \cup S) \geq w_{bs} + W(B\backslash b \cup S\backslash s)$ whence $u_b \geq w_{bs} - (W(B\backslash b \cup S) - W(B\backslash b \cup S\backslash s))$ for any $s$.

## AUCTION

Start at $p = 0$. Increase prices simultaneously over an "Overdemanded Set of Objects" until there occurs a change in demand. Repeat. (Demange Gale Sotomayor "Multiobject Auction" 1986)

For any bipartite graph $G \subset B \times S$ and $b \in B$ denote $G_b = \{s \in S \text{ such that } (b, s) \in G\}$ . A set

$T \subset S$ is "overdemanded" if for every nonempty subset $U$ of $T$, $|U| < |\{b \in B$ such that $G_b \subset T$ and $G_b \cap U \neq$

Note that, given a price vector $p$ and a maximal matching $\mu$ in the buyers' demand graph at $p$, the set of all objects $\mu$-reachable from any single unmatched (active) buyer is an overdemanded set. It can be shown that, picking the overdemanded set in each step, generates a path that stops at the buyer-optimal price vector. The same holds if prices are raised over "the" overdemanded set of objects, $S^o$, identified in the lemma below. (If prices are raised over the maximal overdemanded set $S^o \cup S^e$ then the auction lands at the seller-optimal price vector, etc..)

Maximal Matching Decomposition Lemma (in Alkan "Equilibrium in a matching market with general preferences") : The vertices in $B \cup S$ have a unique partition $B = B^o \cup B^e \cup B^u$ and $S = S^o \cup S^e \cup S^u$ such that $|B^o| > |S^o|, |B^e| = |S^e|, |B^u| < |S^u|$ and $G \cap (B^o \times (S^e \cup S^u)) = G \cap (B^e \times S^o) = \varnothing$. Furthermore, $\mu(B^o) = S^o, \mu(B^e) = S^e, \mu(S^u) = B^u$ for every maximal matching $\mu$ in $G$.

(The lemma above is a special case of the Gallai-Edmonds Theorem for bipartite graphs.)

Exercise (An Alternate Auction Procedure) : Consider a "discrete approximation" of a given assignment market where money is denominated in an indivisible unit. Show the convergence to a stable allocation of the following Price Adjustment cum Deferred Acceptance Procedure : Given prices $p(t)$, each buyer chooses any object in his demand set to "propose", next each object chooses any one top-paying buyer who has proposed to "reject" all others, then $p(t+1)_{bs}$ goes up to $p(t)_{bs}+1$ if $s$ s has rejected $b$ and stays the same otherwise. Note the "rate of convergence" of this auction procedure by considering the case of 3 identical buyers competing for 2 identical objects.