MANY-TO-MANY MATCHING
CHOICE FUNCTIONS and REVEALED PREFERENCE
Alkan 2002, Alkan Gale 2003

College Admissions -> Marriage Problem (Gale Shapley 1961)
Men $M \ni m$, Women $W \ni w$, total orders
$w >_m w'$ $m >_w m'$

Monogamous Matching $\mu$ is a set of pairs $(mw)$
each $m, w$ in at most one pair.
Pair $m, w$ blocks $\mu$ if they prefer each other to

their $\mu$ mates.
$\mu$ is STABLE if there are no blocking pairs.

College Admissions
Each college $C$ has quota $q$ maximum number of students it can admit.
Reduces to marriage problem by "replication".
This is the "classical case"
Problem of "diversity"

First Generalization [Blair 1985]
Each college has a <u>choice function</u> $C$.
Given set $X$ of students, $C(X) \subseteq X$.
For classical case $C(X) = \{q \text{ highest ranked}\}$
if $|X| \geq q$, $C(X) = X$ otherwise.

EXAMPLE
Students : men $m, m'$ women $w, w'$
College with quota 2
Choice function :

| | |
|---|---|
| $mwm'w' -> mw$, | $mwm' -> mw$, |
| $mm'w' -> mw'$ | $mww' -> mw$ |
| $wm'w' -> wm'$ | |

This choice function is not classical,
for, say, $m > w > m' > w'$,
then we would have $C(mm'w') = mm'$.

The relations on the right follow from
DEFINITION. Choice function $C$ is <u>consistent</u> if $C(X) \subseteq X' \subseteq X \implies \subseteq C(X') = C(X)$.

We denote the range of $C$ by $\mathcal{A}$

DEFINITION. If $X \neq Y \in \mathcal{A}$,
$X$ is revealed preferred to $Y$,written $X \succ Y$,
if $$C(X \cup Y) = X.$$

The relation $\succ$ may not be transitive.
EXAMPLE
1. $mwm'w' -> mw$
2. $mm'w' -> mw'$
3. $wm'w' -> m'w'$
$>$From 2. and 3. we have
$mw' \succ m'w' \succ wm'$
but from 1. $mw', m'w$ are non-comparable.

DEFINITION. Choice function is persistent if $x \in X' \subseteq X$ and $x \in C(X) \Longrightarrow x \in C(\mathbf{x}')$.

For college admissions, if a student is chosen from a given pool of applicants she will be chosen from any smaller pool.

A market with no stable matching.
College $A$, quota 2, choice function above.

College $B$ has quota 1.

| | $m$ | | $w$ | $m'$ | $w'$ |
|---|---|---|---|---|---|
| $A$ | $Am$ | | $Aw$ | | |
| | $\downarrow$ | | $\uparrow$ | xx | xx |
| $B$ | $Bm$ | $\longrightarrow$ | $Bw$ | xx | xx |

Recall, we have $$mw' \succ m'w' \succ wm'$$

$\{Amw\}$ blocked by $B, m$,
$\{Amw'\}$ blocked by $(A, w)$,
$\{Am'w\}$ blocked by $(A, w),'$
$\{Am'w'\}, \{Bm\}$ blocked by $(B, w)$,
$\{Am'w'\}, \{Bw\}$ blocked by $(A, m)$.

Second Generalization [Balinski-Baiou 2000]

Schedules.

An agent chooses a schedule $\mathbf{x} = (x(1), .., x(n))$ consisting of amounts of $n$ items, given a a positive $n$-vector $\mathbf{b}$,

$\mathbf{x} \in \mathcal{B} = \{\S : \S \leq \mathbf{b} = (b(1), .., b(n))\ \}$.

Choice function $C$ on $\mathcal{B}$, with $C(\mathbf{x}) \leq \mathbf{x}$.

EXAMPLES

The classical choice function.

Ordered items $i \prec i+1$ and quota $q$.

Choose $i$ so that $z = \sum_1^i x(i) \leq q$, $z + x(i+1) > q$.

$C(\mathbf{x}) = (x(1), .., x(i), x(z-q), 0, 0, .., 0)$

The diversified choice function. $C(\mathbf{x}) = \mathbf{y}$

Choose $c$ so that $\sum_i x(i) \wedge c = q$.

$C_i(\mathbf{x}) = x(i) \wedge\ c$.

DEFINITIONS
$\mathbf{x}$ is revealed preferred to $\mathbf{y}$,
written $\mathbf{x} \succ \mathbf{y}$, if $C(\mathbf{x} \vee \mathbf{y}) = \mathbf{x}$.

The choice function $C$ is

consistent if $C(\mathbf{x}) \leq \mathbf{x}' \leq \mathbf{x} \Longrightarrow C(\mathbf{x}') = C(\mathbf{x})$.

persistent if $\mathbf{x}' \leq \mathbf{x} \Longrightarrow C(\mathbf{x}') \geq \mathbf{x}' \vee C(X)$.

subadditive if $C(\mathbf{x} \vee \mathbf{y}) \leq C(\mathbf{x}) \vee \mathbf{y}$.

Stationary if $C(\mathbf{x} \vee \mathbf{y}) = C(C(\mathbf{x}) \vee \mathbf{y})$.

LEMMA 1. *Persistent* $\Longrightarrow$ *Subadditive*

LEMMA 2. *Subadditive+consistent* $\Longrightarrow$ *stationary.*

Notation. We denote $C(\mathbf{x} \vee \mathbf{y})$ by $\mathbf{x} \curlyvee \mathbf{y}$.

LEMMA 3. *If* $C$ *is stationary then* $\succeq$ *is a partial order and* $A$ *is a lattice.*

■ We first show $\curlyvee$ is associative, for
$\mathbf{x} \curlyvee (\mathbf{y} \curlyvee \mathbf{z}) = C\,(\mathbf{x} \vee (\mathbf{y} \curlyvee \mathbf{z}) = C(\mathbf{x} \vee C(\mathbf{y} \vee \mathbf{z}) =$
$C((\mathbf{x} \vee (\mathbf{y} \vee \mathbf{z}) = C((\mathbf{x} \vee \mathbf{y}) \vee \mathbf{z}) = (\mathbf{x} \curlyvee \mathbf{y}) \curlyvee \mathbf{z}$
Next $\mathbf{x} \succeq \mathbf{y} \succeq \mathbf{z} \Longrightarrow \mathbf{x} \curlyvee \mathbf{y} = \mathbf{x}, \mathbf{y} \curlyvee \mathbf{z} = \mathbf{y}$ so $\mathbf{x} \curlyvee \mathbf{z} = (\mathbf{x} \curlyvee \mathbf{y}) \curlyvee \mathbf{z} = (\mathbf{x} \curlyvee (\mathbf{y} \curlyvee \mathbf{z}) = \mathbf{x} \curlyvee \mathbf{y} = \mathbf{x}$,
so $\mathcal{A}$ is a lattice.■

The Revealed Preference Lattice

In $\mathcal{A}$ we have $\mathbf{x} \curlyvee \mathbf{y} = C(\mathbf{x} \vee \mathbf{y})$.

What is $\mathbf{x} \curlywedge \mathbf{y}$ ?

DEFINITION. For $\mathbf{x} \in \mathcal{A}$ the closure $\bar{\mathbf{x}}$ of $\mathbf{x}$ is given by $\bar{\mathbf{x}} = \sup\{\mathbf{y} : C(\mathbf{y}) = \mathbf{x}\}$.

Since $C$ is continuous we have $C(\bar{\mathbf{x}}) = \mathbf{x}$

For classical college admissions the closure of $X$ is $X+$ students ranked below all of $X$.

Isomorphism Theorem: The mapping $\mathbf{x} \longrightarrow \bar{\mathbf{x}}$ is a lattice isomorphism from $\{\mathcal{A}, \succeq\}$ to $\{\mathcal{B}, \geq\}$.

Corollary $\mathbf{x} \curlywedge \mathbf{y} = C(\bar{\mathbf{x}} \wedge \bar{\mathbf{y}})$.

LEMMA 4 $\mathbf{x} \wedge \bar{\mathbf{y}} \leq \mathbf{x} \curlywedge \mathbf{y}$

■ $\bar{\mathbf{x}} \geq \bar{\mathbf{x}} \wedge \bar{\mathbf{y}}$ so from persistence we have

$\mathbf{x} \curlywedge \mathbf{y} = C(\bar{\mathbf{x}} \wedge \bar{\mathbf{y}}) \geq C(\bar{\mathbf{x}}) \wedge \bar{\mathbf{x}} \wedge \bar{\mathbf{y}} = \mathbf{x} \wedge \bar{\mathbf{x}} \wedge \bar{\mathbf{y}}$
$= \mathbf{x} \wedge \bar{\mathbf{y}}$ ■

Stability

A schedule is "$i$-satiated" if the agent would not choose to increase consumption of item $i$ if it was available. Formally,

Notation Given a schedule $\mathbf{x}$, the vector $\mathbf{x}^{(i)}$
replaces the $i$th entry of $\mathbf{x}$ by upper bound $b(i)$.

$$\mathbf{x}^{(i)} = (x(1), ., x(i-1), b(i), x(i+1), .., x(n)).$$

DEFINITION Item $i$ is stable in $\mathbf{x}$ if $C\left(\mathbf{x}^{(i)}\right) = \mathbf{x}$.Otherwise it is unstable in $\mathbf{x}$.

Classical case, $i$ is stable $\Longleftrightarrow \sum_{j=1}^{i} x(j) = q$.
Diversified case $i$ is stable $\Longleftrightarrow x(i) = \max_j [x(j)]$.

LEMMA 5. $\mathbf{x}$ or $\mathbf{y}$ stable $\Longrightarrow \mathbf{x} \curlyvee \mathbf{y}$ stable.
$\mathbf{x}$ and $\mathbf{y}$ stable $\Longrightarrow \mathbf{x} \curlywedge \mathbf{y}$ stable.

Schedule Matching

Firms $F$, members $f$. Workers $W$, members $w$.

Each $f$ has a choice function $C_f$.

Each $w$ has a choice function $C_w$.

DEFINITION. A matching $\mathbf{x}$ is a $F \times W$ matrix where $x(fw)$ represents the amount of time worker $w$ works for firm $f$.

We assume given a positive $F \times W$ matrix $\mathbf{b}$ such that $x(fw) \leq b(fw)$.

Denote by $\mathbf{x}_f$ the $f$-row, $\mathbf{x}_w$ the $w$-column of $\mathbf{x}$.

DEFINITIONS Matching $\mathbf{x}$ is *$F-$acceptable* if $\mathbf{x}_f \in \mathcal{A}_f$ for all $f$. Similarly for $W$.

The pair $f, w$ blocks the matching $\mathbf{x}$ if

$fw$ is unstable in $\mathbf{x}_f$ and in $\mathbf{x}_w$.

The matching $\mathbf{x}$ is STABLE if there are no blocking pairs.

Existence We define sequence of alternately $F-acceptable$ and $W-acceptable$ matchings $\mathbf{y}^n, \mathbf{z}^n$ which converge to a stable matching.

Initial choice matrix for $F$ is $\mathbf{b}$.
$\mathbf{y}^1$ is defined by $\mathbf{y}^1_f = C_f(\mathbf{b}_f)$.
If $\mathbf{y}^1$ is $W - acceptable$ then stop. It is stable.
If not $\mathbf{z}^1$ is defined by $\mathbf{z}^1_w = C_w(\mathbf{y}^1_w)$.
Define $\mathbf{x}^1$, new choice matrix by,
$x^1(fw) = b(fw)$ if $z^1(fw) = y^1(fw)$,
$\qquad = z^1(fw)$ if $z^1(fw) < y^1(fw)$.
$\mathbf{y}^2$ is defined by $\mathbf{y}^2_f = C_f(\mathbf{x}^1_f)$,etc.
Note, $\mathbf{x}^n$ non increasing so converges to $\tilde{\mathbf{x}}$
so $\mathbf{y}^n \longrightarrow \tilde{\mathbf{y}}$ and $\mathbf{z}^n \longrightarrow \tilde{\mathbf{z}}$ by continuity of $C_f, C_w$.
Also $\mathbf{x}^n \geq \mathbf{y}^n \geq \mathbf{z}^n$
Claim $\tilde{\mathbf{y}} = \tilde{\mathbf{z}}$ because $x^n\,(fw) - x^{n+1}\,(fw) \longrightarrow 0$ so
$y^n - z^n \longrightarrow 0$.
Using consistency and persistence one shows that $\tilde{\mathbf{y}}$ is stable.

The Stable Matching Lattice

The revealed preference ordering for individuals extends naturally to matchings.

We write $\mathbf{x} \succeq_F \mathbf{y}$ if $\mathbf{x} \geq_f \mathbf{y}$ for all $f$.

Define $\mathbf{z}^F = \mathbf{x} \curlyvee_F \mathbf{y}$ if $\mathbf{z}^f = \mathbf{x}_f \curlyvee_f \mathbf{y}_f$ for all $f$.

and similarly for $W$.

We would like to show that the set of stable matchings is a lattice under order $\succ_F$ or $\succ_W$ .

However,

EXAMPLE Firms $A, B, C, D, E$

Workers $a, b, c, d, z$ with preferences,

| $\underline{A}$ | $\underline{B}$ | $\underline{C}$ | $\underline{D}$ | $\underline{E}$ | $\underline{a}$ | $\underline{b}$ | $\underline{c}$ | $\underline{d}$ | $\underline{e}$ | $\underline{z}$ |
|---|---|---|---|---|---|---|---|---|---|---|
| $a*$ | $b^{\#}$ | $c*$ | $d^{\#}$ | $z$ | $C^{\#}$ | $D*$ | $A^{\#}$ | $B*$ | $E^{\#}*$ | $A^{\#}$ |
| $cz^{\#}$ | $dz*$ | $a^{\#}$ | $b*$ | $e^{\#}*$ | $A*$ | $B^{\#}$ | $C*$ | $D^{\#}$ | | $B*$ |
| | | | | | | | | | | $E*$ |

The matching * and # are both stable but

$* \curlyvee_F \# = \{Aa, Bb, Cc, Dd, Ee\}$ is blocked by $E, z$.

Some further condition is needed.

The size of a schedule $\mathbf{x}$ written $|\mathbf{x}|$ is the sum of its entries $\sum_i x(i)$.

DEFINITION (Alkan 2002) $C$ is size monotone if

$\mathbf{x} \leq \mathbf{y}$ implies $|C(\mathbf{x})| \leq |C(\mathbf{y})|$.

Note if $C$ is "quota filling" it is size monotone, so both classical and diversified choice functions are size monotone.

Polarity Theorem. If $\mathbf{x}, \mathbf{y}$ are stable matchings

then $\mathbf{x} \succ_F \mathbf{y}$ if and only if $\mathbf{y} \succ_W \mathbf{x}$.

Method of proof. Let $\mathbf{z}^F = \mathbf{x} \curlyvee_F \mathbf{y}$, $\mathbf{z}_F = \mathbf{x} \curlywedge_F \mathbf{y}$.

Using stability, persistence, we show $\mathbf{z}^F \leq \mathbf{z}_W$

>From size monotone $|\mathbf{z}_f| \leq \left|\mathbf{z}^f\right|$ and $|\mathbf{z}_w| \leq |\mathbf{z}^w|$ so $|\mathbf{z}_F| = \sum_F |\mathbf{z}_f| \leq \sum_F \left|\mathbf{z}^f\right| = \left|\mathbf{z}^F\right| \leq$

$|\mathbf{z}_W| = \sum_W |\mathbf{z}_w| \leq \sum_W |\mathbf{z}^w| = \left|\mathbf{z}^W\right| \leq |\mathbf{z}_F|$

so $|\mathbf{z}_W| = \left|\mathbf{z}^F\right|$ so $\mathbf{z}_W = \mathbf{z}^F$.

Corollary. $|\mathbf{z}_f| = \left|\mathbf{z}^f\right|$ and $|\mathbf{z}_w| = |\mathbf{z}^w|$ for all $f, w$.

MAIN THEOREM The set of stable matchings a lattice $\Lambda$ under $\succ_F$and $\succ_W$ .

Sketch of Proof:

Must show that $\mathbf{z}^F = \mathbf{x} \curlyvee_F \mathbf{y}$ is $W-acceptable$ and $Stable$.

The first follows from the Polarity Theorem.

To prove stability, suppose for some $f$ we have

$fw$ is unstable in $\mathbf{z}^f$. Then by Lemma 5 it is unstable in both $\mathbf{x}_f$ and$\mathbf{y}_f$. Therefore by stability $fw$ is stable in both $\mathbf{x}_w$ and $\mathbf{y}_w$ so by the second part of Lemma 5, $fw$ is stable in $\mathbf{z}_w$, hence it is stable in $\mathbf{z}_W$,but from polarity $\mathbf{z}_W$=$\mathbf{z}^F$ so $\mathbf{z}^F$ is stable.

Properties of the Stable Matching Lattice.

1. The lattice $\Lambda_F$ has max and min elements.

2. "Unisize" : $\mathbf{x}, \mathbf{y} \in \mathbf{\Lambda} \Longrightarrow |\mathbf{x}_f| = |\mathbf{y}_f|$
from the corollary to the Polarity Theorem.

3. If $\mathbf{x}, \mathbf{y} \in \mathbf{\Lambda}$ and $C$ is quota filling and $|\mathbf{x}_f| < q$ then $\mathbf{x}_f = \mathbf{y}_f$.
Proof. If $\mathbf{x} \neq \mathbf{y}$ then $|\mathbf{x} \vee \mathbf{y}| > |\mathbf{x}|$ so, from quota filling, $|\mathbf{x} \curlyvee \mathbf{y}| > |\mathbf{x}|$ but this contradicts unisize.

4. $\mathbf{x} \wedge \mathbf{y} \leq \mathbf{x} \curlyvee \mathbf{y}$ and $\mathbf{x} \wedge \mathbf{y} \leq \mathbf{x} \curlywedge \mathbf{y}$.
For college admissions this says that those students admitted in both $\mathbf{x}$ and $\mathbf{y}$ are admitted in both $\mathbf{x} \curlyvee \mathbf{y}$ and $\mathbf{x} \curlywedge \mathbf{y}$.

5. $\mathbf{x}, \mathbf{y} \in \Lambda, \Longrightarrow \mathbf{x} \vee \mathbf{y} = (\mathbf{x} \curlyvee \mathbf{y}) \vee (\mathbf{x} \curlywedge \mathbf{y})$

6. Classical case $\mathbf{x}, \mathbf{y} \in \mathbf{\Lambda} \Longrightarrow \mathbf{x} \succeq \mathbf{y}$ or $\mathbf{y} \succeq \mathbf{x}$.
Not true for general case.

$A$ prefers $mw$. $B$ prefers $m'w'$

$$\begin{array}{ccccc} & m & w & m\prime & w' \\ A & Am & Aw & Am' & Aw' \\ & \downarrow & \downarrow & \uparrow & \uparrow \\ B & Bm & Bw & Bm' & Bw' \end{array}$$