Games with restricted cooperation

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A classical cooperative game is a pair (N, v), where N is a finite set of players, and $v : 2^N \to \mathbb{R}$ with the convention $v(\emptyset) = 0$. For any $S \subseteq N$, the quantity $v(S)$ represents the best amount of money the coalition S can achieve by itself without the help of the other players (*worth* of S). Cooperative game theory primarily aims at defining rational ways (called *solution concepts*) of sharing the worth of the grand coalition N among the players in N, when it is supposed that eventually all players cooperate to form the coalition N. Most classical solution concepts are the core, the Shapley value and the nucleolus (see, e.g., the following textbooks: [5, 6, 7]).

The classical framework of cooperative game theory tacitly assumes that every coalition $S \subseteq N$ can form. In most real situation however, this is too strong an assumption. The formation of coalition may be constrained by hierarchy relations, affinity and compatibility (e.g., of political nature) between players, size restrictions, etc. It is therefore of interest to generalize the definition of a cooperative game as follows: it is a triple $(N, \mathcal{F}, v)$, where $\mathcal{F}$ is a nonempty collection of subsets (coalitions) of N, and $v : \mathcal{F} \to \mathbb{R}$. Such games are called *games with restricted cooperation*. Once this is done, it remains the difficult task to redefine and study the properties of the main solution concepts, like the core and the Shapley value. Many researchers have worked on this topic, and this course will present a synthesis of the work done, with an emphasis on the core (see especially the survey [3] where many references can be found, and [4, 2] for further studies).

In decision making, capacities introduced by Choquet can be seen as particular cooperative games: they are mappings $\mu : 2^N \to \mathbb{R}$, $\mu(\emptyset) = 0$, with the additional requirement that they are monotone w.r.t. inclusion: $S \subseteq T$ implies $\mu(S) \leq \mu(T)$. In decision making, capacities often play the rôle of a generalized probability, and therefore the question of how to compute an expected value w.r.t. a capacity arises. This is usually achieved by the *Choquet integral*, a generalization of the Lebesgue integral.

Going back to restricted cooperation, one can imagine as well capacities defined on a subcollection $\mathcal{F}$ of 2^N. Then arises the problem of a proper definition of the Choquet integral. We propose here a general definition, which can be reached by several ways (see [1] for details).

The structure of the course is as follows.

- **Basic notions of cooperative game theory and mathematical prerequisites (2h)**
  - cooperative games: basic definitions; convex games
  - the core of a game; classical results on the structure of the core; the Weber set; balanced games

- The Shapley value
- basic notions on partially ordered sets (posets); main families of posets: lattices, regular set systems, weakly union-closed systems; relations between these families

- **The core of games with restricted cooperation (2h)**
  - basic notions on convex polyhedra
  - general results for arbitrary set systems: balancedness, vertices and rays of the core
  - the case of distributive lattices
  - the case of regular set systems
  - the case of weakly union-closed set systems: the positive core and Monge extensions
- **The Shapley value for games on regular set systems (0h30)**
- **How to make the core bounded for games with restricted cooperation (1h30)**
  - the general idea: add efficiency constraints (notion of restricted core)
  - the case of distributive lattices; the Weber set
  - the general case
- **The Choquet integral for capacities defined on set systems (2h)**
- prerequisites on the Choquet integral: definition, characterization results, Möbius transform, belief functions
- upper integrals
- the Monge algorithm
- the case of weakly union-closed systems
- supermodularity and super additivity

(durations are indicative)

References

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