Asset Pricing with Heterogeneous Investors and Portfolio Constraints

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Motivation

- Heterogeneity in preferences and portfolio constraints is widespread
- Models with heterogeneity and constraints widely used to explain various phenomena in financial markets (Detemple and Murthy, 1997; Basak and Cuoco, 1998; Basak and Croitoru, 2000, 2006; Kogan, Makarov and Uppal, 2007; Gallmeyer and Hollifield, 2008)
- Literature mainly looks at single-asset economies with logarithmic constrained investors, which impedes the evaluation of impact of constraints on equilibrium
- Incorporating multiple assets, heterogeneity in preferences and constraints into equilibrium analysis is challenging
- *Our objective*: evaluate impact of heterogeneity and constraints on equilibrium in one-asset and two-asset economies with general CRRA preferences

Main Results

- Methodology for solving problems with constraints such as limited stock market participation, margin requirements, short-sale prohibition, etc.
  - closed form solutions for the unconstrained benchmark
  - closed form solution in the case of leverage constraint
  - develop efficient numerical method for the general case
- Provide full picture of the dependence of equilibrium on the tightness of constraints
  - cases when constraints decrease/increase equilibrium processes
  - cases when constraints make them pro-/counter-cyclical
  - cases when constraints can generate excess volatility
  - model can match dynamic patterns in the data, and some levels
- Extend model to the case of two Lucas trees
  - derive extension of Black's CAPM with leverage constraint
  - evaluate impact of constraints on stock return correlations

Related Literature

- Unconstrained investors and two Lucas trees: Menzly, Santos, and Veronesi (2004), Cochrane, Longstaff, and Santa-Clara (2008), Buraschi, Trojani, and Vedolin (2010), Martin (2011), Ehling and Heyerdahl-Larsen (2011)
- Heterogeneous investors, multiple trees, and constraints: Pavlova and Rigobon (2008)
- Models with heterogeneous investors, one Lucas tree, and constraints in continuous time: Detemple and Murthy (1997), Basak and Cuoco (1998), Basak and Croitoru (2000, 2006), Kogan, Makarov and Uppal (2003), Gallmeyer and Hollifield (2008), Prieto (2010)
- Unconstrained heterogeneous investors, one tree: Wang (1996), Longstaff and Wang (2008), Yan (2008), Bhamra and Uppal (2009, 2010), Cvitanic and Malamud (2010), Gârleanu and Panageas (2011)

Economy

- Continuous-time infinite horizon Markovian economy with one consumption good, uncertainty generated by Brownian motion $w = (w_1, w_2)^\top$
- Two CRRA investors, $i \in \{A, B\}$, with risk aversions γ_A and γ_B, $\gamma_A \geq \gamma_B$
- Two exogenous streams of dividends (Lucas trees) following GBM:

$$dD_{jt} = D_{jt}[\mu_{D_j} dt + \sigma_{D_j} dw_{jt}], \qquad j = 1, 2$$

- Aggregate dividend/consumption $D = D_1 + D_2$ follows process

$$dD_t = D_t \left[\mu_{Dt} dt + \sigma_{Dt}^\top dw_t \right]$$

Investment Opportunities

- Investment opportunities:
  - riskless bond B_t in zero net supply with return r_t
  - two stocks, each in net supply of 1, are claims to dividends
- We study Markovian equilibria in which bond and stock prices evolve as

$$dB_t = B_t r_t dt,$$

$$dS_{jt} + D_{jt}dt = S_{jt}[\mu_{jt}dt + \sigma_{jt}^\top dw_t], \qquad j = 1, 2$$

where $\mu = (\mu_1, \mu_2)^\top$ and $\sigma = (\sigma_1, \sigma_2)^\top$ are determined in equilibrium

- Endowments at $t = 0$: investor 1 has $(n_1, n_2)^\top$ units of stock and $-b$ units of bond while investor 2 has $(1 - n_1, 1 - n_2)^\top$ units of stock and b units of bond
- Investors allocate fraction of wealth α_i to bonds and $\theta_i = (\theta_{i1}, \theta_{i2})^\top$ to stocks

Investor Optimization

- Investor i's dynamic optimization is given by:

$$\max_{c_i,\,\theta_i} \mathbb{E}\Big[\int_0^\infty e^{-\rho t}\frac{c_{it}^{1-\gamma_i}-1}{1-\gamma_i}dt\Big]$$

$$\text{s.t.}\quad dW_{it} = \Big[W_{it}\Big(r_t + \theta_{it}^\top(\mu_t - r_t)\Big) - c_{it}\Big]dt + W_{it}\theta_{it}^\top\sigma_t dw_t$$

$$\theta_{it}\in\Theta_i,\quad W_{it}\geq 0,\quad i = A, B$$

- Investor A is unconstrained, investor B faces margin/leverage constraint [e.g., Brunnermeier and Pedersen (2009); Gromb and Vayanos (2009)]:

$$\Theta_A = \mathbb{R}^2,\quad \Theta_B = \{\theta\in\mathbb{R}^2 : m^\top\theta \leq 1\}$$

  where $m = (m_1, m_2)^\top$ is vector of margins, where $0 \leq m_i \leq 1$

- Constraint can be rewritten as $\theta_1 + \theta_2 \leq 1 + (1 - m_1)\theta_1 + (1 - m_2)\theta_2$

- Special cases: $m = (0,0)^\top$ (no constraints) and $m = (1,1)^\top$ (leverage constraint)

- c_{it}^* and θ_{it}^* solve optimization subject to budget and portfolio constraints

Definition of Equilibrium

- An *equilibrium* is a set of processes $\{r_t, \mu_{jt}, \sigma_{jt}\}_{j\in\{1,2\}}$ and of consumption and investment policies $\{c^*_{it}, \alpha^*_{it}, \theta^*_{it}\}_{i\in\{A,B\}}$ that solve each investor's dynamic optimization problem given processes $\{r_t, \mu_{jt}, \sigma_{jt}\}_{j\in\{1,2\}}$, and consumption and financial markets clear, i.e.,

$$\begin{aligned} c^*_{At} + c^*_{Bt} &= D_t \\ \alpha^*_{At}W^*_{At} + \alpha^*_{Bt}W^*_{Bt} &= 0 \\ \theta^*_{At}W^*_{At} + \theta^*_{Bt}W^*_{Bt} &= (S_{1t}, S_{2t})^\top \end{aligned} \tag{1}$$

  where W^*_{At} and W^*_{Bt} denote optimal wealths of investors A and B

- Derive equilibrium parameters as functions of $x = D_1/(D_1 + D_2)$ and $y = c^*_B/(c^*_A + c^*_B)$, following Markovian dynamics:

$$\begin{aligned} dx_t &= \quad x_t[\mu_{xt}dt + \sigma^\top_{xt}dw_t] \\ dy_t &= -y_t[\mu_{yt}dt + \sigma^\top_{yt}dw_t] \end{aligned}$$

Characterization of Equilibrium

- Following Cvitanic and Karatzas (1992) solve constrained optimization in equivalent fictitious unconstrained economy with adjusted dynamics:

$$dB_t = B_t\Big(r_t + f(\tilde{\nu}_t)\Big)dt$$

$$dS_{jt} + D_{jt}dt = S_{jt}\Big[\Big(\mu_{jt} + \tilde{\nu}_{jt} + f(\tilde{\nu}_t)\Big)dt + \sigma_t dw_t\Big] \qquad j = 1, 2$$

where $\tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2)^\top$ solves *dual problem* and $f(\cdot)$ is *support function* for Θ_B

- It can be shown that $\tilde{\nu}$ and $f(\cdot)$ have the following structure:

$$\tilde{\nu} = \nu^*(m_1, m_2)^\top, \quad f(\tilde{\nu}) = -\nu^*$$

where $\nu^* \in \mathbb{R}$ and $\nu^* \leq 0$

Characterization of Equilibrium (cont'd)

- State price densities for investors A and B follow processes

$$d\xi_t = -\xi_t[r_t dt + \kappa_t^\top dw_t]$$
$$d\xi_{Bt} = -\xi_{Bt}[(r_t + f(\tilde{\nu}_t))dt + \kappa_{Bt}^\top dw_t]$$

where $\kappa_t = \sigma_t^{-1}(\mu_t - r_t)$, $\kappa_{Bt} = \sigma_t^{-1}(\mu_t - r_t + \tilde{\nu}_t)$

- Optimal consumptions satisfy first order conditions

$$e^{-\rho t}(c_{At}^*)^{-\gamma_A} = \psi_A \xi_t, \quad e^{-\rho t}(c_{Bt}^*)^{-\gamma_B} = \psi_B \xi_{Bt}$$

- Equilibrium processes can be found by applying Itô's Lemma to both sides of market clearing condition $c_{At}^* + c_{Bt}^* = D_t$ and matching dt and dw_t terms

Characterization of Equilibrium (cont'd)

- In equilibrium $\kappa = \sigma^{-1}(\mu - r)$, r, μ_y, σ_y are given by:

$$\kappa_t = \Gamma_t \sigma_{Dt} - \frac{\Gamma_t y_t}{\gamma_B} \nu_t^* \sigma_t^{-1} m$$

$$r_t = \rho + \Gamma_t \mu_{Dt} - \frac{\Gamma_t \Pi_t}{2} \sigma_{Dt}^\top \sigma_{Dt} + \frac{\Gamma_t y_t \nu_t^*}{2} + \left(\nu_t^* \sigma_t^{-1} m\right)^\top \left(a_1(y_t)\sigma_{Dt} + a_2(y_t)\nu_t^* \sigma_t^{-1} m\right)$$

$$\sigma_{yt} = \frac{\Gamma_t (1 - y_t)}{\gamma_A \gamma_B}\left((\gamma_B - \gamma_A)\sigma_{Dt} - \nu_t^* \sigma_t^{-1} m\right)$$

$$\mu_{yt} = \mu_{Dt} - \sigma_{Dt}^\top \sigma_{yt} - \frac{1 + \gamma_B}{2}(\sigma_{Dt} - \sigma_{yt})^\top (\sigma_{Dt} - \sigma_{yt}) - \frac{r_t - \nu_t^* - \rho}{\gamma_B}$$

  where $a_1(y)$ and $a_2(y)$ are functions available in closed form

- Γ_t and Π_t denote relative risk aversion and prudence parameters of a representative investor:

$$\Gamma_t = \frac{1}{(1 - y_t)/\gamma_A + y_t/\gamma_B}, \qquad \Pi_t = \Gamma_t^2 \left(\frac{1 + \gamma_A}{\gamma_A^2}(1 - y_t) + \frac{1 + \gamma_B}{\gamma_B^2} y_t\right)$$

Characterization of Equilibrium (cont'd)

- Adjustment ν_t^* satisfies complementary slackness condition

$$(m^\top \theta_{Bt}^*(x, y; \nu_t^*) - 1)\nu_t^* = 0, \quad \nu_t^* \leq 0$$

- Adjustments are available in closed form in two special cases:
  - unconstrained benchmark, i.e. $m = (0, 0)^\top$:

$$\nu_t^* = 0, \quad \nu_t^* \sigma_t^{-1} m = 0$$

  - leverage constraint, i.e. $m = (1, 1)^\top$:

$$\nu_t^* = \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}, \qquad \nu_t^* \sigma_t^{-1} m = \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}\Big(\frac{1}{\sigma_{D_1}}, \frac{1}{\sigma_{D_2}}\Big)^\top$$

- General case: adjustments obtained as functions of price-dividend and wealth-consumption ratios and their derivatives

Consumption CAPM

- Easy to derive consumption CAPM in terms of adjustment:

$$\mu_t - r_t = \Gamma_t \sigma_t \sigma_D - \frac{\Gamma_t y_t}{\gamma_B} \nu^* m$$

- Multiplier $(\Gamma y / \gamma_B)\nu^*$ the same for all stock $\Rightarrow$ can be expressed in terms of market risk premium

- Consequently, we obtain consumption CAPM with margin constraints:

$$\mu_t - r_t = \left(I - \frac{m\theta_{Mt}^\top}{\theta_{Mt}^\top m} \right) \beta_{Ct} + \frac{m}{\theta_{Mt}^\top m} (\mu_{Mt} - r_t)$$

  where μ_M and θ_M are mean-return and weights of the market portfolio, respectively, $\mu - r = (\mu_1 - r, \mu_2 - r)^\top$, and β_C is vector of consumption betas:

$$\beta_{Ct} = \Gamma_t \left(\frac{\mathrm{cov}(dS_{1t}/S_{1t}, dD_t)}{dt}, \frac{\mathrm{cov}(dS_{2t}/S_{2t}, dD_t)}{dt} \right)^\top$$

Consumption CAPM (cont'd)

- Leverage constraint: $m = (1,1)^\top \Rightarrow \nu^*$ and $\nu^* \sigma^{-1} m$ available in closed form
- Heterogeneity and constraints amplify each other $\Rightarrow$ interaction is important
- C-CAPM with leverage constraint:

$$\mu_{jt} - r_t = \Gamma_t \frac{\mathrm{cov}_t(dS_{jt}/S_{jt}, dD_t)}{dt} - \frac{\Gamma_t y_t}{\gamma_B} \nu_t^* m$$

$$= \Gamma_t \frac{\mathrm{cov}_t(dS_{jt}/S_{jt}, dD_t)}{dt} - \frac{\Gamma_t (1 - y_t)}{\gamma_B} \frac{\gamma_B - \gamma_A}{1/\sigma_{D_1}^2 + 1/\sigma_{D_2}^2}, \quad j = 1, 2$$

- Risk premia higher than in standard C-CAPM, both terms have approximately the same magnitude
- This C-CAPM extends Breeden's (1979) C-CAPM and Black's (1972) CAPM with leverage constraints

Computation of Equilibrium

- In complete markets wealth-consumption ratios solve linear PDEs (e.g., Liu (2007))
- *Wealth-consumption ratios* Φ_A, and *price-dividend ratios* Ψ_1 and Ψ_2 solve linear PDEs if ν^* and $\nu^* \sigma^{-1} m$ are known
- In general adjustments are functions of Ψ_1, Ψ_2, and $\Phi_B \Rightarrow$ get quasilinear PDEs solved by method of iterations.
- If $m = (0, 0)^\top$, or $m = (1, 1)^\top \Rightarrow$ get Ψ_1 and Ψ_2 in closed form
- Equilibrium processes derived as functions of $x = D_1/(D_1 + D_2)$, and $y = c_B^*/(c_A^* + c_B^*)$

Computation of Equilibrium (con't)

- Closed form Ψ_j have the following structure:

$$\Psi_1(x,y) = \Psi(x,y;\{\mu_{D_1},\sigma_{D_1}\},\{\mu_{D_2},\sigma_{D_2}\})$$
$$\Psi_2(x,y) = \Psi(1-x,y;\{\mu_{D_2},\sigma_{D_2}\},\{\mu_{D_1},\sigma_{D_1}\})$$

where in the unconstrained case

$$\Psi(x,y;\{\mu_{D_1},\sigma_{D_1}\},\{\mu_{D_2},\sigma_{D_2}\}) =$$

$$\int_0^1 \int_0^1 \left(\frac{s}{x}\frac{y}{z}\right)^{\gamma_B} \frac{\gamma_B(1-z)+\gamma_A z}{s(1-s)z(1-z)} e^{q^\top \Sigma^{-1} u(s,z;x,y)} \times$$

$$\frac{K_0\left(p\sqrt{u(s,z;x,y)^\top \Sigma^{-1} u(s,z;x,y)}\right)}{\pi \gamma_A \sqrt{\det(\Sigma)}} ds\, dz,$$

where $K_0(\cdot)$ is McDonald's function $K_0(z) = \int_0^\infty e^{-z\cosh(s)} ds$,

$$u(s,z;x,y) = \left(\ln\left(\frac{1-z}{1-y}\right) - \frac{\gamma_B}{\gamma_A}\ln\left(\frac{z}{y}\right) + \frac{\gamma_B - \gamma_A}{\gamma_A}\ln\left(\frac{s}{x}\right), \ln\left(\frac{1-s}{1-x}\frac{x}{s}\right)\right)^\top$$

Computation of Equilibrium (con't)

- From price-dividend ratios Ψ_1 and Ψ_2 obtain volatilities and correlations
- Applying Itô's Lemma to $S_i = \Psi_i D_i$ and matching dw terms we obtain:

$$(\sigma_{j1t}, \sigma_{j2t})^\top = e_j \sigma_{D_j} + \sigma_{xt} \frac{\partial \Psi_{jt}}{\partial x_t} \frac{x_t}{\Psi_{jt}} - \sigma_{yt} \frac{\partial \Psi_{jt}}{\partial y_t} \frac{y_t}{\Psi_{jt}}, \quad j = 1, 2$$

$$\text{corr}\Big(\frac{dS_{1t}}{S_{1t}}, \frac{dS_{2t}}{S_{2t}}\Big) = \frac{\sigma_{11t}\sigma_{21t} + \sigma_{21t}\sigma_{21t}}{|\sigma_{1t}||\sigma_{2t}|}$$

where $e_1 = (1,0)^\top$, $e_2 = (0,1)^\top$

- New effects due to investor heterogeneity come from the third term in the expression for volatilities
- The results are applied to study correlations under heterogeneous preferences and portfolio constraints

Unconstrained Benchmark

- Unconstrained case: $m = (0,0)^\top$, $\nu^* = 0$
  - r, $\sigma^{-1}(\mu - r)$, μ_y, and σ_y available in closed form
  - price-dividend ratios Ψ_1 and Ψ_2 available in closed form
- Consumption share $y = c_B^*/(c_A^* + c_B^*)$ is procyclical:

$$\mathrm{corr}_t(dy_t, dD_t) = 1$$

- Intuition:
  - $\gamma_B < \gamma_A \Rightarrow B$ invests more in stocks
  - in good times B has more wealth $\Rightarrow y$ high in good times
  - Similarly, y is low in bad times

Unconstrained Benchmark (cont'd)

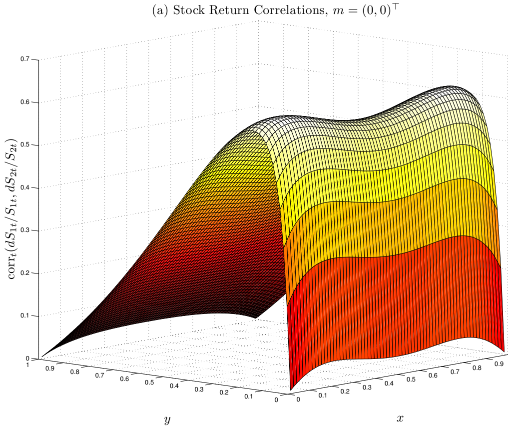


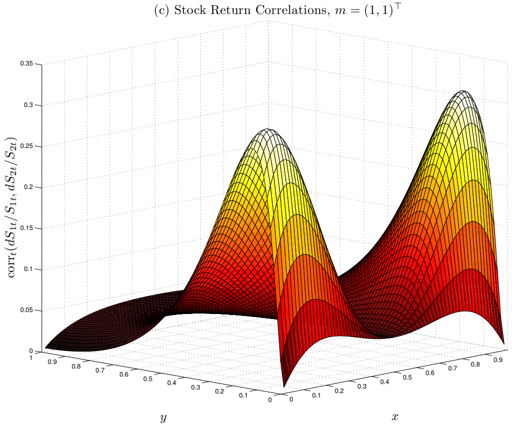
Figure: Stock Return Correlations

Correlation as function of $x = D_1/(D_1 + D_2)$ and $y = c_B^*/(c_A^* + c_B^*)$. Parameters: $\mu_{D_1} = \mu_{D_2} = 1.8\%$, $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, $\gamma_A = 10$, $\gamma_B = 2$, $\rho = 0.01$.

Unconstrained Benchmark (cont'd)

- Limits $y \to 0$ and $y \to 1$ correspond to one-investor economies populated by A and B, respectively (e.g., Logstaff et al (2008), Martin (2011))
- Heterogeneity amplifies correlations, correlations countercyclical for wide range of y, procyclical in very bad times
- Intuition:
  - $\gamma_B < \gamma_A \Rightarrow B$ borrows from A to invest in stocks
  - amount of liquidity for borrowing is time-varying
  - liquidity $\uparrow \Rightarrow B$ invests more in stocks, liquidity $\downarrow \Rightarrow B$ invests less in stocks
  - correlations $\uparrow$
- Intuition (cont'd): two effects working in opposite directions:
  - $y \downarrow$ share of A increases, more funds for borrowing $\Rightarrow$ correlations high
  - $y \downarrow$ impact of B diminishes $\Rightarrow$ correlations low
  - get hump-shaped correlations

Leverage Constraint



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Leverage Constraint (cont'd)

- Constraints significantly decrease correlations and make them less countercyclical
- Relative size of industries (Lucas trees) becomes more important ⇒ saddle-shaped correlations
- Intuition: 1) constraints homogenize investors ⇒ decrease correlations consistently with the intuition on the role of leverage 2) common discount factor effect stronger when $x = 0$ or $x = 1$
- Homogenization effect is stronger around $x = 0.5$
  - at $x = 0.5$ stocks and trees look "symmetric" ⇒ $\theta_{i1} = \theta_{i2}$, $i \in \{A, B\}$
  - because of leverage constraints $\theta_{i1} = \theta_{i2} = 0.5$
  - portfolio choice heterogeneity disappears ⇒ $\sigma_y = 0$ when $x = 0.5$:

$$\begin{aligned}\sigma_{yt} &= \frac{\Gamma_t(1-y_t)}{\gamma_A\gamma_B}\big((\gamma_B-\gamma_A)\sigma_{Dt} - \nu_t^*\sigma_t^{-1}m\big)\\ &= \frac{\Gamma_t(1-y_t)(\gamma_B-\gamma_A)\sigma_{D1}}{\gamma_A\gamma_B}(x-0.5, 0.5-x)^\top\end{aligned}$$

  - correlations decrease more around $x = 0.5$

Margin Constraint

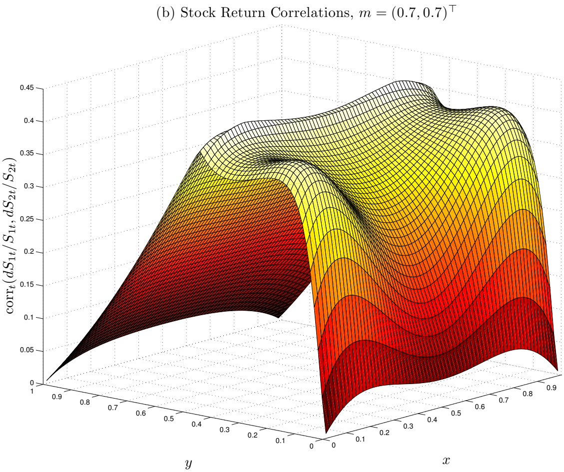

Figure: Stock Return Correlations

Correlation as function of $x = D_1/(D_1 + D_2)$ and $y = c_B^*/(c_A^* + c_B^*)$. Parameters: $\mu_{D_1} = \mu_{D_2} = 1.8\%$, $\sigma_{D_1} = \sigma_{D_2} = 3.6\%$, $\gamma_A = 10$, $\gamma_B = 2$, $\rho = 0.01$.

Limited Participation in One-Asset Economy

- The equilibrium in the one-asset model can be derived analogously
- Start with limited participation constraint $\theta_t \leq \bar{\theta}$, $\bar{\theta} < 1$
  - margin given by: $m = 1/\bar{\theta} > 1$
  - $m = +\infty$ corresponds to limited participation (e.g., Basak and Cuoco (1998))
  - assume $\gamma_A = \gamma_B \Rightarrow$ evaluate pure effect of constraints
- Derive equilibrium parameters as functions of constrained investor's share in aggregate consumption y_t
- Consumption share y increases in bad times and decreases in good times, i.e. $\mathrm{cov}_t(dy_t, dD_t) < 0 \Rightarrow$ countercyclical
- This is because negative shocks to dividends shift wealth distribution towards constrained investors which are less exposed to stock market

Limited Participation (cont'd)

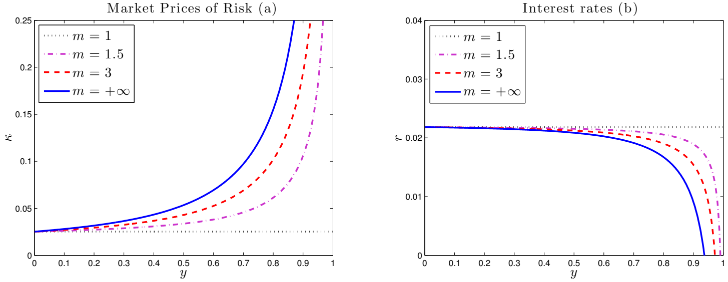

Figure 2: Market Prices of Risk and Interest Rates, $\gamma < 1$.

Interest rates r_t and market prices of risk $\kappa_t = (\mu_t - r_t)/\sigma_t$ as functions of consumption share of constrained investor, y. Parameters: $\mu_D = 1.8\%$ and $\sigma_D = 3.6\%$, $\gamma = 0.7$, $\rho = 0.01$.

- Tighter constraints $\Rightarrow \kappa_t \uparrow$ and $r_t \downarrow$ since constrained invests more in bonds while unconstraint bears more risk
- $y \uparrow \Rightarrow \kappa_t \uparrow$ and $r_t \downarrow$
- Market prices of risk κ_t are higher in bad times (e.g., Ferson and Harvey, 1991)
- Around $y = 0.7$ [e.g., Basak and Cuoco (1998)] 380% increase in κ, but still too small

Limited Participation (cont'd)

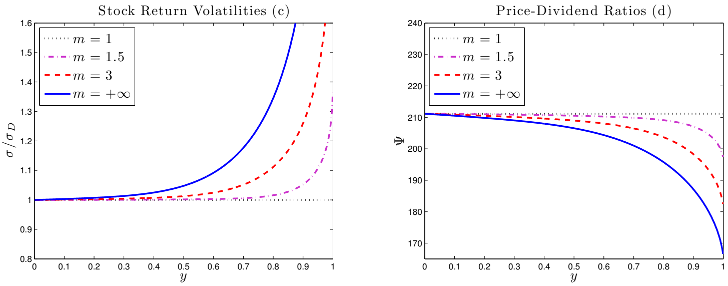

Figure: Price-Dividend Ratio and Stock Return Volatility, $\gamma < 1$.

Price-dividend ratio Ψ_t and stock return volatility σ_t as functions of consumption share of constrained investor, y.

Parameters: $\mu_D = 1.8\%$ and $\sigma_D = 3.6\%$, $\gamma = 0.8$, $\rho = 0.01$.

- Tighter constraints $\Rightarrow$ price-dividend ratio $\Psi_t \downarrow$ and stock return volatility $\sigma_t \uparrow$
- $y \uparrow \Rightarrow$ price-dividend ratio $\Psi_t \downarrow$, stock return volatility $\sigma_t \uparrow$
- In bad times (when y is large) price-dividend ratio is lower and volatility is higher, consistently with literature (Schwert, 1989; Campbell and Cochrane, 1999)
- Lower price-dividend ratios predict higher market prices of risk and risk premia
- $\sigma_t > \sigma_D \Rightarrow$ model generates 20% excess volatility

Limited Participation (cont'd)

- Intuition depends on relative strength of classical income and substitution effects
- For CRRA investor substitution effect dominates for $\gamma < 1$ and income effect dominates for $\gamma > 1$
- As an example consider partial equilibrium unconstrained economy with constant r and $\kappa = (\mu - r)/\sigma \Rightarrow$ wealth-consumption ratio given by:

$$\frac{W}{c} = \frac{\gamma}{\rho - (1-\gamma)(r + 0.5\kappa^2/\gamma)}$$

  - substitution effect dominates ($\gamma < 1$): investment opportunities worsen $\Rightarrow$ $W/c \downarrow$
  - income effect dominates ($\gamma > 1$): investment opportunities worsen $\Rightarrow W/c \uparrow$

Limited Participation (cont'd)

- Consider $\gamma < 1$ when substitution effect dominates
- Price-dividend ratio is given by $\Psi_t = (W^*_{At} + W^*_{Bt})/(c^*_{At} + c^*_{Bt})$
- When constrained investor dominates (y is close to 1) $\Psi_t \approx W^*_{Bt}/c^*_{Bt}$
- With tighter constraints investment opportunities for constrained investor deteriorate $\Rightarrow$ since substitution effect dominates $W^*_{Bt}/c^*_{Bt} \downarrow \Rightarrow \Psi \downarrow$
- For smaller y effect less pronounced because when unconstrained investor dominates price-dividend ratio is close to that in unconstrained economy
- Since $\mathrm{cov}_t(dy_t, dD_t) < 0$ and price-dividend ratio Ψ_t decreases in $y \Rightarrow$ $\mathrm{cov}_t(d\Psi_t, dD_t) > 0 \Rightarrow$ volatility σ_t get excess volatility

Limited Participation (cont'd)

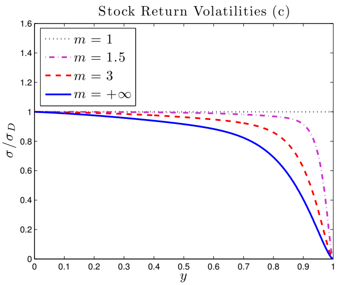

Figure: Price-Dividend Ratio and Stock Return Volatility, $\gamma > 1$.

Price-dividend ratio Ψ_t and stock return volatility σ_t as functions of consumption share of constrained investor, y.

Parameters: $\mu_D = 1.8\%$ and $\sigma_D = 3.6\%$, $\gamma = 3$, $\rho = 0.01$.

- Tighter constraints $\Rightarrow$ price-dividend ratio $\Psi_t \uparrow$ and stock return volatility $\sigma_t \downarrow$
- $y \uparrow \Rightarrow$ price-dividend ratio $\Psi_t \uparrow$
- $y \uparrow \Rightarrow$ stock return volatility $\sigma_t \downarrow$
- Implication: volatility decreases more in bad times (when y is large)

Limited Participation (cont'd)

- Intuition for price-dividend ratios and volatilities driven by relative strength of income and substitution effects
- CRRA preferences cannot separate risk aversion γ from intertemporal elasticity of substitution ψ
- With CRRA having $\gamma > 1$ to match κ, and having $\psi > 1$ to match dynamics is not feasible
- Volatilities are difficult to match in GE models [e.g., Heaton and Lucas (1996)]

Margin Constraints in One-Asset Economy

- Consider margin constraint $\theta_B m \leq 1$, $m < 1$
- $\gamma_A > \gamma_B$ to make constraint binding
- Derive equilibrium parameters as functions of $y_t = c^*_{Bt}/(c^*_{Bt} + c^*_{At})$
- Consumption share y is now procyclical, i.e. $\mathrm{cov}_t(dy_t, dD_t) > 0$
  - intuition: unconstrained investor is more risk averse $\Rightarrow$ less exposed to stocks
  - hence, bad shocks to dividends shift wealth to unconstrained investor A

Margin Constraints (cont'd)

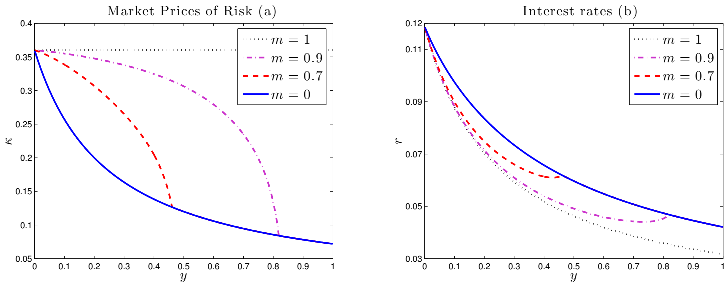

Figure: Market Prices of Risk and Interest Rates.

Market price of risk κ_t and interest rate r_t as functions of consumption share of constrained investor, y.

Parameters: $\mu_D = 1.8\%$ and $\sigma_D = 3.6\%$, $\gamma_A = 10$, $\gamma_B = 2$, $\rho = 0.01$.

- Tighter constraints $\Rightarrow r_t \downarrow$ and $\kappa_t \uparrow$
- Market prices of risk κ_t are higher in bad times (e.g., Ferson and Harvey, 1991)

Margin Constraints (cont'd)

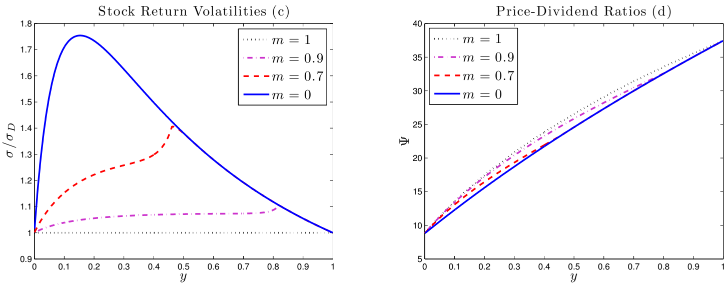

Figure: Price-Dividend Ratio and Stock Return Volatility.

Price-dividend ratio Ψ_t and stock return volatility σ_t as functions of consumption share of constrained investor, y.

Parameters: $\mu_D = 1.8\%$ and $\sigma_D = 3.6\%$, $\gamma_A = 10$, $\gamma_B = 2$, $\rho = 0.01$.

- Tighter constraints $\Rightarrow$ price-dividend ratio $\Psi_t \uparrow$ and stock return volatility $\sigma_t \downarrow$
- Constraints make stock return volatilities less countercyclical
- Intuitively, under tighter constraints investment policies are as if investors were less heterogeneous

Conclusion

- Approach for computing equilibrium with heterogeneous investors and portfolio constraints
- Evaluate impact of constraints on financial markets
- Heterogeneity in preferences and constraints amplify the impact of each other
- Constraints decrease correlations and make them less countercyclical
- Interesting directions for future research include:
  - model with two constrained investors
  - model with recursive preferences