HSE/Math in Moscow 2012-2013// Topology 2 // Problem sheet 1

General topology

Recall that a *topological space* is a couple $(X,\mathcal{U})$ where X is a set and $\mathcal{U}$ is a collection of subsets of X such that

- $\varnothing, X \in \mathcal{U}$,
- $\mathcal{U}$ is closed under taking *finite* intersections and *arbitrary* unions.

If such a collection $\mathcal{U}$ is given it is called a *topology* on X and its elements are called *open subsets*. Subsets of X of the form $X \setminus U, U \in \mathcal{U}$ are called *closed*. Finite unions and arbitrary interesections of closed subsets are again closed. In the sequel, when it is clear which topology on X is meant we will simply write X instead of $(X,\mathcal{U})$.

Example. Every set (with more than 1 element) carries at least two topologies: the *trivial* topology, in which the only open sets are $\varnothing$ and X itself and the *discrete* topology, in which all subsets are open. The resulting topological spaces will be denoted X^{tr} and X^{disc}. They are not very interesting though.

A more interesting example is the following:

Example. One of the most important topological spaces is the set $\mathbb{R}$ of real numbers equipped with the following topology: a set $X \subset \mathbb{R}$ is open iff for any $x \in X$ there are $a, b \in \mathbb{R}$ such that $x \in (a, b) \subset X$. The Euclidean space $\mathbb{R}^n$ can be made into a topological space in a similar way: a set $X \subset \mathbb{R}^n$ is open iff for any $x = (x_1, \ldots, x_n) \in X$ there are $a_1, b_1, \ldots, a_n, b_n \in \mathbb{R}$ such that $x \in \Pi_{i=1}^n (a_i, b_i) \subset X$. (Here $\Pi_{i=1}^n (a_i, b_i)$ stands for the set of all $(t_1, \ldots, t_n) \in \mathbb{R}^n$ such that $a_i < t_i < b_i$ for all $i = 1, \ldots, n$; this is the Cartesian product of the intervals (a_i, b_i).)

Question 1. Check that this procedure indeed gives a topology on $\mathbb{R}^n$.

If X is a topological space and $Y \subset X$ then we can introduce *the induced topology* on Y by declaring a set $U \subset Y$ open iff $U = Y \cap V$ where V is open in X. We will often refer to Y with the induced topology as a *subspace* of X. So, starting from the previous example we can construct many more. In particular, we get topologies on $(0, 1)$ (an open interval), $[0, 1]$ (a closed interval) and $[0, 1)$.

Question 2. a) Let X be a topological space, $U \subset X$ an open subset and $V \subset U$ a subset that is open in the induced topology. Prove that V is open in X.

b) Let X be a topological space, $Z \subset X$ a closed subset and $W \subset Z$ a subset that is closed in the induced topology. Prove that W is closed in X.

A map $f : X \to Y$ of topological spaces is called *continuous* if one of the following equivalent conditions is satisfied.

1. If $U \subset Y$ is open, than $f^{-1}(U)$ is open in X.
2. If $Z \subset Y$ is closed, than $f^{-1}(Z)$ is closed in X.
3. For any $x \in X$ and any open $U \subset Y$ that contains $f(x)$ there is an open set $V \subset X$ such that $x \in V$ and $f(V) \subset U$.

Note that in the case $X \subset Y = \mathbb{R}$ the third condition is precisely the definition of continuity that can be found in most undergraduate analysis books.

Question 3. Prove that these conditions are indeed equivalent.

In practice it is often convenient to glue continuous maps from maps defined on subspaces. The following shows when this is possible.

Question 4. a) Suppose a topological space X is represented as a union $X = \bigcup_{i \in I} X_i$ of its subspaces. Let $f : X \to Y$ be a map from X to another topological space Y. Suppose $f|X_i : X_i \to Y$ is continuous for all $i \in I$. Prove that then f is itself continous in each of the following cases:

(i) all X_i are open;

(ii) all X_i are closed and there are only finitely many of them (i.e., $\#I < \infty$).

b) Can the condition $\#I < \infty$ in part a) (i) be removed?

In algebra it often happens that if a bijection preserves some structure, so does the inverse bijection. E.g., if $f : G_1 \to G_2$ is a group homomorphism, then $f^{-1} : G_2 \to G_1$ is again a group homomorphism. In topology this fails quite spectacularly.

Question 5. Construct a continuous bijection $f : [0, 1) \to S^1$ where $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ with the topology induced from $\mathbb{R}^2$. Show that $f^{-1} : S^1 \to [0, 1)$ is not continuous.

Let X and Y be topological spaces. A continuous bijection $f : X \to Y$ is called a *homeomorphism* if f^{-1} is continuous. If such a map exists we say that X and Y are *homeomorphic*.

Question 6. a) Show that $(0,1), (0,\infty), \mathbb{R}$ are all homeomorphic.

b) Show that $[0,1)$ and $[0,\infty)$ are homeomorphic.

A topological space X is said to be

- *pathwise connected* if for all $x, y \in X$ there is a continuous map $f : [0,1] \to X$ such that $f(0) = x, f(1) = y$.
- *connected* if for any two open $U, V \subset X$ such that $U \cap V = \varnothing$ and $U \cup V = X$ we have either $U = X$ and $V = \varnothing$, or vice versa.

Intermediate Value Theorem: If $a, b \in \mathbb{R}$ and $f : [a,b] \to \mathbb{R}$ is a continuous function, then for all $y \in [min(f(a), f(b)), max(f(a), f(b))]$ there is an $x \in [a,b]$ such that $f(x) = y$.

We will assume the IVT; it is not too hard to prove it using one's favourite completeness principle of the real numbers.

Question 7. a) Using the IVT show that $[0,1]$ is connected. Deduce from this that any pathwise connected space is connected.

b) (*) Let $X \subset \mathbb{R}^2$ be the union of the segment joining the points $x = 0, y = -1$ and $x = 0, y = 1$, and the graph of $f(x) = \sin \frac{1}{x}$ for $x \in (0,1)$. Show that X is connected but not path connected.

Question 8. Show that for any $n > 1$ the spaces $(0,1), [0,1), [0,1], \mathbb{R}^n, n > 1$ are pairwise non-homeomorphic.

Simplicial homology

Recall that a *simplicial complex* is an ordered set V and a collection S of finite subsets of V such that if $F \in S$, then any non-empty subset of F also $\in S$. Given a simplicial complex (V, S) we can construct a chain complex $C_*(V,S)$ with C_n being a free abelian group with basis $\triangle_F, F \in S, \#F = n+1$ and differential $\partial_n : C_n(V,S) \to C_{n-1}(V,S)$ defined as

$$\partial_n(\triangle_{\{i_0,\ldots,i_n\}}) = \sum_{j=0}^{n} (-1)^j \triangle_{\{i_0,\ldots,\widehat{i_j},\ldots,i_n\}}.$$

Set

$$\triangle^k = \{(x_0, \ldots, x_k) \in \mathbb{R}^{k+1} \mid \text{all } x_i \geq 0, \Sigma x_i = 1\}.$$

If $V = \{0, \ldots, k\}$ then to each $F = \{i_0, \ldots, i_n\} \subset V$ we can associate a face $\triangle(F)$ of $\triangle^k$, namely, the convex hull of $e_{i_0}, \ldots, e_{i_n}$ where e_i is the i-th element of the standard basis of $\mathbb{R}^{k+1}$. The *geometric realisation* of (V,S) is obtained by taking the union of all $\triangle(F), F \in S$.

Question 9. Draw the geometric realisation of (V,S) and compute the homology of $C_*(V,S)$ when

a) $V = \{0,1,2\}$ and S is formed by all non-empty subsets of V.

b) $V = \{0,1,2\}$ and S is formed by all non-empty subsets of V apart from V itself.

Exact sequences

Recall that a sequence of abelian groups and homomorphisms

$$(\cdots \to C_{n+1} \overset{f_{n+1}}{\to} C_n \overset{f_n}{\to} C_{n-1} \to \cdots)$$

is *exact* if $\ker f_n = Im f_{n+1}$ for all n.

Question 10. Below A, B are abelian groups.

a) Show that $0 \to A \overset{i}{\to} B$ is exact iff i is injective.

b) Show that $A \overset{p}{\to} B \to 0$ is exact iff p is surjective.

c) Show that $0 \to A \overset{f}{\to} B \to 0$ is exact iff f is an isomorphism.

d) Show that $0 \to A \to 0$ is exact iff $A = 0$.

e) How many abelian groups A are there, up to isomorphism, such that there is an exact sequence $0 \to \mathbb{Z} \to A \to \mathbb{Z}/2 \to 0$?