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NON-RESERVATION PRICE EQUILIBRIA AND CONSUMER SEARCH

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When consumers do not know the prices at which different firms sell, they often also do not know production costs. Consumer search models which take asymmetric information about production costs into account continue focusing on reservation price equilibria (RPE) and their properties. We argue that RPE assume specific out-of-equilibrium beliefs that are not consistent with the logic of the D1 refinement criterion. Moreover, RPE suffer from a non-existence problem as they typically do not exist when cost uncertainty is large. We characterize an alternative class of so-called non-RPE. We show these equilibria always exist and do not rely on specific out-of-equilibrium beliefs. Non-reservation equilibria are characterized by active consumer search among consumers. As cost uncertainty facilitates search, more consumers make price comparisons resulting in stronger price competition between firms and higher consumer surplus.

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1 Introduction

Since the fundamental paper of Rothschild (1974) it is known that when consumers do not know from which distribution of offers to select their information, the optimal consumer search rule may well be different from the typical search protocol that is characterized by a reservation price property. The main reason is that on the basis of past search observations, consumers update their beliefs about their search environment. Depending on the environment, it may well be that after observing a relatively good outcome consumers infer that even better outcomes are likely to be observed in the next search round and rationally conclude to continue to search, whereas after observing a relatively bad outcome consumers infer that better outcomes are unlikely and thus stop searching.

The consumer search literature has, however, by and large, neglected this observation. The typical model in the consumer search literature studies consumers searching sequentially in an environment that is fully known to the consumers. For example, the celebrated models by Stahl (1989) and Wolinsky (1986), and much of the literature that takes them as a starting point, have consumers comparing the current price-product offer with the expected price-product offer when continuing to search. In these models consumers can make correct predictions about the prices or price distributions chosen by firms. In this environment, the optimal search rule is indeed characterized by a reservation price property: at or above a certain reservation utility level consumers decide to buy, otherwise they continue to search.

At the heart of the consumer search literature lies the assumption that consumers do not know the prices charged by different firms. In most markets where consumers are indeed imperfectly informed about prices, it is likely they also do not know the underlying costs of firms. Cost information is, however, obviously relevant for the firms' pricing decisions. In such markets, learning is important as consumers cannot have correct predictions about the prices or price distributions that are *unconditional* on the underlying unknown costs.

There is a small amount of literature on learning and consumer search that takes consumer uncertainty about firms' costs into consideration (see, Benabou and Gertner (1993), Dana (1994), Fishman (1996) and more recently, Yang, and Ye (2008), Tappata (2009), Janssen et al. (2011) and Chandra and Tappata (2011)). Most of this literature is inspired by retail gasoline markets where the common wholesale price of crude oil is the most important determinant of the (variation in) costs of retailers, and consumers are uncertain about these costs due to the large fluctuations of this wholesale price on the world market. The observations by Rothschild are of immediate concern to this literature. However, these papers continue to characterize equilibria where the consumer search rule is characterized by a reservation price.

Our paper is the first to systematically incorporate Rothschild's observations on non-reservation price strategies into an equilibrium search model with endogenous firm behaviour.[5] Benabou and Gertner (1993) also mention the fact that in their model reservation price equilibria (RPE) may not exist. They set up the equations that have to be satisfied in a non-RPE. They perform some simulation analysis numerically characterizing non-RPE for some parameter values, but they neither have an analysis characterizing these non-RPE, nor do they show the conditions under which these equilibria exist.[6]

RPE take the form of perfect Bayesian equilibria where consumers update their beliefs about the underlying costs (and therefore on the expected price they encounter on their next search) after observing a price. The above mentioned litera-

[5]Even though our paper focuses on consumer search, similar considerations apply to the labour search literature that uses reservation wage equilibrium (see McCall (1970) for pioneering work in this direction, and subsequent literature as, for example, surveyed in Rogerson et al. (2005)).

[6]Benabou and Gerther (1993, pp. 74) state that the "non-reservation price equilibria (if they exist) are too complicated for us to solve" and argue that these equilibria are "somewhat less appealing intuitively than the previous reservation price equilibria" (pp. 81). They mention that the reservation price property is "required in particular for demand functions to be downward sloping" (pp. 74). However, in order to make firms indifferent over the range of prices in a mixed strategy equilibrium, a firm's total demand must be downward sloping. In our model, it may be that the demand of an *individual consumer* is upward sloping in a non-reservation price equilibrium, but it need not be.

ture is not very satisfactory for a number of reasons. First, RPE implicitly assume certain out-of-equilibrium beliefs and this literature has not made it clear whether these out-of-equilibrium beliefs satisfy game theoretic refinement concepts commonly employed in asymmetric information games. Second, the literature shows that for certain parameter values RPE do not exist (no matter what the out-of-equilibrium beliefs are), especially when the search cost is relatively small and the uncertainty about costs is relatively large (cf., Dana (1994) and Jansen et al. (2011)). When RPE do not exist, it is an unanswered question what kind of equilibria do exist. Third, one would expect that when costs are uncertain consumers may want to search in equilibrium. When consumers observe a high price they are uncertain about whether this is due to a relatively high (common) production costs or whether this particular firm is charging a high margin. One may expect that this uncertainty leads to consumers searching more, but the RPE which are characterized in these homogeneous goods markets, have firms charging prices below the consumer reservation price and therefore all consumers buy at the first firm.

In response to these points, this paper argues that, first, RPE are very sensitive to how one specifies the out-of-equilibrium beliefs, and that these equilibria do not satisfy, for example, the logic of the D1 equilibrium refinement (hereafter the D1 logic) that is commonly employed in games with asymmetric information (cf., Cho and Sobel, 1990). Second, we prove that non-RPE exist for all parameter values, that there are parameter values for which multiple non-RPE exist, and we find that in any equilibrium satisfying the D1 logic, firms price in such a way that consumers find it indeed optimal to follow a non-reservation price strategy. Third, in all equilibria satisfying the D1 logic consumers actively search beyond the first firm. In particular, there is a region of "high" prices that are set with positive probability such that consumers are indifferent between buying and searching and consumers continue to search with strictly positive probability.

The non-RPE we characterize are not only interesting from a methodological perspective, but also provide a different perspective on the implications of cost uncertainty for consumer behaviour. First, the literature mentioned above shows that in RPE consumers are better off without cost uncertainty as the ex ante expected market price they have to pay is lower if they know the costs than if they do not. When cost uncertainty is large we find that consumers may benefit from cost uncertainty as they rationally search more than without cost uncertainty. As consumers search more and compare more prices, firms have less market power arising from search frictions and set lower prices on average. This "additional search" effect can be quantitatively significant.

Second, our results have implications for the empirical work on consumer search models, and in particular on the question of what search strategies consumers actually follow in real online or offline markets. In a recent paper De los Santos, Wildenbeest and Hortacsu (2012) show that actual search strategies by consumers buying books online do not follow the sequential search protocol. De los Santos et al. (2012) take two predictions from the sequential search protocol: (i) consumers buy from the last store visited unless all stores have been visited[7] and (ii) the decision whether to continue to search depends on the outcome of the previous search. They find that consumers go back and buy from shops they already have visited before they have visited all firms, and they do not find that consumers searching once are more likely to buy at relatively low prices compared to the first price observation of consumers searching twice. By making a distinction between consumers searching sequentially and reservation price strategies, our results show that both findings are not inconsistent with the sequential search protocol, although they are inconsistent with consumers following reservation price strategies. We show that when the cost uncertainty is large, non-RPE typically have a region of intermediate prices where the probability of a sale is lower when the price is low. This would imply that the consumers may well condition their search behaviour on current price observations, but that this does not imply that consumers are more likely continuing to search if they observe higher prices.[8] In particular, we show that for some parameter values

[7]De los Santos et al. (2013) interpret their observations in the context of a learning model, where consumers learn from prices about unobserved characteristics that drive the pricing decisions of firms.

[8]Even though we consider consumers having unit demand, interpreting the probability with

consumers accept higher prices in the first search round and reject lower ones. As we study a duopoly market in the main part of the paper, consumers cannot go back to previously sampled firms before they have sampled all firms. The first prediction from the sequential search protocol is thus trivially satisfied in our basic model. We show, however, that one type of equilibrium we derive can be analyzed for N firms and in that case, the optimal sequential search behaviour of consumers is consistent with consumers going back to previously sampled firms, before they have sampled all firms.[9]

Third, there is marketing oriented literature on reference price effects (see, e.g., Putlet (1992), Kalyanaram and Winer (1995) and Mazumdar, Raj and Sinha (2005)). This literature points to the fact that consumers have particular pricing points around which consumer demand is very sensitive to price changes. This may lead to situations where consumer demand drops significantly if firms price above this reference point, whereas at higher prices consumers are willing to buy again. Such "reference point" demand behaviour can occur in non-RPE when the cost uncertainty is large. After observing intermediate prices above the "reference price", consumers rationally infer that these prices are not chosen by high cost firms. Knowing costs are low, consumers find these prices too high to buy, however, and they continue to search for sure. This inference creates a gap in the equilibrium price distribution of the low cost firms. In all of the equilibria with a gap, consumer behaviour is such that at prices above the gap consumers buy again, but with a relatively low probability.

The rest of the paper is organized as follows. Section 2 describes the model and the equilibrium concept we use. Section 3 describes our analytical results. We first show that any RPE assumes specific out-of-equilibrium beliefs that, for example, do not satisfy the D1 logic. We then characterize non-RPE, where the determination of

which a consumer buys as the consumer's demand, non-reservation price equilibria always have price regions where individual demands are downward sloping, but there may also be regions (depending on the parameter values) where individual demands are upward sloping.

[9] In the discussion section at the end of the paper, we come back to the complications regarding studying non-reservation price equilibria with more than two firms.

the upper bound is independent of the specific assumption about out-of-equilibrium beliefs. We then show that the independence of specific out-of-equilibrium beliefs implies that the density function of the low cost firm at the highest price at which consumers buy with positive probability has to be equal to zero so that after observing this price, consumers infer that costs are high (and therefore are not inclined to buy at higher prices). Next, we characterize the equilibrium price distributions for high and low cost firms. The distributions are such that the cumulative distribution function of a high cost firm first-order stochastically dominates that of a low cost firm. We then show that in any equilibrium, there is active search by consumers. Using these characterizations we finally show that an equilibrium always exists. Section 4 shows by means of a numerical analysis, the effects of cost uncertainty on profits, expected prices and consumer welfare. It also performs a comparative statics analysis with respect to the model parameters. Section 5 briefly discusses a generalization of our duopoly model to N firms and shows that in equilibrium the optimal search rule of consumers may imply that consumers first continue searching at another firm, and then go back to a previously sampled firm before all firms are sampled. Section 6 concludes with a discussion, while proofs are given in two Appendices.

2 The Model

The sequential search model we analyze is based on Dana (1994) and Janssen et al. (2011). Essentially, the model incorporates cost uncertainty in Stahl (1989). Simplifying the analysis, in order to focus on non-RPE, we consider a duopoly model with inelastic demand. The two firms sell a homogenous good and face the same marginal production costs. Marginal cost can be either high c_H or low c_L, with probabilities α and $1-\alpha$, respectively. Without a loss of generality, we normalize fixed costs to zero. Firms know the cost realization, but consumers do not. After observing the realization of cost, firms simultaneously set prices and we denote the (symmetric) price distributions chosen by firms by $F_L(p)$ and $F_H(p)$ when cost is

Low or High, respectively. The highest price which will be charged by low and high cost firms is denoted by $\overline{p}_L$ and $\overline{p}_H$, respectively. Each firm's objective is to maximize profits, taking the prices charged by the other firm and consumers' search behavior as given.

On the demand side of the market we have a unit mass of risk-neutral consumers with identical preferences. Each consumer $j \in [0, 1]$ has a unit demand and has the same constant valuation $v > 0$ for the good. Observing a price below v, consumers will either buy one unit of the good or search for a lower price. In the latter case, they have to pay a search cost s to obtain one additional price quote, i.e. search is *sequential*. A fraction $\lambda \in (0, 1)$ of consumers, *shoppers*, have a zero search cost. These consumers sample all prices and buy at the lowest price. The remaining fraction of $1 - \lambda$ consumers – *non-shoppers* – have a positive search cost $s > 0$ and visit each of the two firms at their first search with equal probability. These consumers face a non-trivial problem when searching for low prices, as they have to trade off the search cost with the expected benefit from search. After observing their first price quote, non-shoppers update their beliefs about firms' underlying production costs. Consumers can always go back to previously visited firms incurring no additional cost.[10] We assume that v is large relative to c and s so that v is not binding. The probability that non-shoppers buy after observing price p is denoted by $\beta(p)$. With the remaining probability $1-\beta(p)$ these consumers continue to search. As consumers do not know the underlying production cost, $\beta(p)$ does not depend on the cost realization. Denote by ρ_i the consumers' reservation price if they were to infer that the firms' production cost equals c_i for sure. That is, if after observing a price p consumers infer that cost equals c_i for sure, then consumers are willing to buy at any price at or below ρ_i and prefer to continue to search if prices are larger.

It is by now a standard argument in the search literature that due to the presence of shoppers and non-shoppers there does not exist an equilibrium in pure strategies and that an equilibrium in mixed strategies does not have mass points at particular

[10] Janssen and Parakhonyak (2014) analyze the case where this assumption is replaced by costly recall.

prices.[11] As a consequence, if a firm sets a price equal to the upper bound $\overline{p}_i$, $i = H, L$, of the price distribution it will not sell to the shoppers and will sell at most to half of the non-shoppers. Profit $\pi(p|c_i)$ when setting price p and cost is $c_i, i = H, L$ can thus be written as

$$\begin{aligned}\pi(p|c_i) = \quad & \Bigg[\lambda(1 - F_i(p)) + \frac{1-\lambda}{2}\beta(p) + \frac{1-\lambda}{2}(1 - \beta(p))(1 - F_i(p)) + \\ & \qquad \frac{1-\lambda}{2}\int_p^{\overline{p}}(1 - \beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\Bigg](p - c_i).\end{aligned} \tag{1}$$

This expression can be understood as follows. First, a firm only attracts the shoppers if the other firm charges a higher price, which occurs with probability $1 - F_i(p)$. The number of non-shoppers buying from firm i gives a more complicated expression. There is a fraction $(1-\lambda)/2$ of non-shoppers that randomly first visits firm i and they buy immediately from that firm with probability $\beta(p)$.The remaining non-shoppers that randomly first visit firm i continue searching the other firm and come back to firm i if the other firm has a higher price. Finally, the non-shoppers that first visit the other firm and decide to continue to search buy from firm i if it has a lower price. As firm i does not know which price the other firm charges, this expression involves an expected number of consumers.

As this is a game with asymmetric information about production cost, the appropriate equilibrium concept is that of a Perfect Bayesian Equilibrium where the out-of-equilibrium beliefs satisfy some reasonable restrictions. To see how the specification of out-of-equilibrium beliefs plays a role, assume that consumers hold out-of-equilibrium beliefs such that, if a price above their reservation price is observed, they think that the lowest cost level has been realized with probability one and therefore continue to search. In such a case, in equilibrium no firm would set a price

[11] In our context, with asymmetric information between consumers and firms, the argument is more subtle. In principle, firms equilibrium pricing strategies could have a mass point, with prices in a left-neighborhood of that point not being in the support of the mixed strategy distribution. After observing an out-of-equilibrium price in this region, consumers could believe that these prices are set in case firms have low cost, giving them an incentive to continue searching (in which case it is not optimal to set these prices). One can show, however, that these out-of-equilibrium beliefs are inconsistent with the D1 logic.

above the reservation price and therefore such a price observation is clearly an out-of-equilibrium event. Thus, these optimistic beliefs about expecting low cost (and thus low prices on the next search) may support equilibria where the highest price charged by firms is the reservation price of the Stahl (1989) model where firms are known to have low cost. This out-of-equilibrium belief is, however, not reasonable, as we will argue.

Ideally, the characterization of the upper bound of the price distributions should not depend on arbitrary assumptions regarding out-of-equilibrium beliefs. As we will see in the next section, this can only be achieved if at the upper bound consumers believe that the underlying cost is high. If an out-of-equilibrium price above the upper bound is then observed, consumers will want to continue to search independent of their beliefs of the underlying cost. One way to achieve this is to require that a strong refinement holds and that equilibria satisfy the logic of the D1 criterion (Cho and Sobel, 1990). The D1 criterion was developed in the context of pure signaling games with one sender. The game we consider here is a two-sender game. The beliefs of the receivers (the non-shoppers) in our model are only based on the single price they have observed, and because of the assumption of common cost the firms have to be of the same type, the out-of-equilibrium belief of non-shoppers is simply a mapping from the observed price to the type distribution of cost, like in the one-sender game. Deviation by a sender (firm) to an out-of-equilibrium price generates a set of possible optimal actions of the receiver (non-shopper).

Consider a firm i that unilaterally deviates to a certain price p that lies outside the support of its equilibrium price distribution. Let $B_i(p)$ be the set of a firm i's total demand from shoppers and non-shoppers that can be generated by buying probabilities $\beta_i(p)$ of non-shoppers (at firm i at price p) that are best responses to some non-shoppers' belief. Each $q_i(p) \in B_i(p) \subset [0,1]$ is the demand of firm i at price p for some profile of non-shoppers' beliefs about firm i's type and optimal choices given these beliefs when the other firm plays according to its equilibrium strategy. In the spirit of the D1 criterion,[12] we compare the sets of demands for

[12]A similar treatment is given in Janssen and Roy (2010) for a more complicated inference

which it is gainful for different types of firm i to deviate to price p.

More precisely, consider any perfect Bayesian equilibrium where the equilibrium profit of firm i when it is of type τ is given by $\pi_{\tau}^{i*}, \tau = H, L$. Consider any p outside the support of the equilibrium price distribution. If for $\tau, \tau' \in \{H, L\}$, $\tau' \neq \tau$,

$$\{q_i \in B_i(p) : (p - c_\tau)q_i \geq \pi_\tau^{i*}\} \subset \{q_i \in B_i(p) : (p - c_{\tau'})q_i > \pi_{\tau'}^{i*}\}$$

where "$\subset$" stands for strict inclusion, then the D1 logic suggests that the out-of-equilibrium beliefs of buyers (upon observing a unilateral deviation by firm i to price p) should assign zero probability to the event that firm i is of type τ and thus (as there are only two types and firms have a common type), assign probability one to firm j being of type τ'.

Definition 1. *A symmetric perfect Bayesian equilibrium satisfying the D1 logic is characterized as follows:*

1) *each type $\tau = H, L$ of firm i uses a price strategy $F_L(p)$, $F_H(p)$ that maximizes its (expected) profit, given the competing firms' price strategies and the search behavior of consumers;*

2) *given the distribution of firms' prices, consumers' search strategy, characterized by $\beta(p)$, is optimal given their beliefs, and they update their beliefs about cost given the price they observe,* $\Pr(c = c_H|p)$, *by using Bayes' Rule if possible and formulate out-of-equilibrium beliefs that are consistent with the D1 logic whenever they observe an out-of-equilibrium price.*

In what follows, we concentrate on the characterization and existence of perfect Bayesian equilibria satisfying the D1 logic to ensure that at the upper bound of the price distribution consumers believe that cost is high so that independent of specific assumptions about out-of-equilibrium beliefs consumers prefer to continue to search if they observe a price that is larger than the upper bound.

problem where consumers observe all prices and there are N firms.

3 Characterisation and Existence of D1 Equilibria

In this section we provide a characterisation of the set of equilibria that satisfies the D1 logic as defined in the previous section. To this end, let $P = [0, \overline{p}]$, with $\overline{p} = \max\{\overline{p}_L, \overline{p}_H\}$ and $\beta(p) : P \to [0, 1]$. Define convex sets $P_{(0,1)} = \{p : p \in P, 0 < \beta(p) < 1\}$, $P_1 = \{p : p \in P, \beta(p) = 1\}$ and $P_0 = \{p : p \in P, \beta(p) = 0\}$. We consider equilibria where $\beta(p)$ is continuously differentiable in the interior of each of these sets and we show that such an equilibrium always exists.

We first show that RPE do *not* satisfy the D1 logic. To do so, we define RPE as an equilibrium where non-shoppers buy at prices at or below a certain reservation price ρ. Janssen et al. (2011), among others, have shown that in a RPE (i) both types of firms choose to set the reservation price with positive density and (ii) the expected price when cost is low, $E(p|c_L)$, is lower than the expected cost when cost is high, $E(p|c_H)$. Accordingly, the updated belief about cost after observing the reservation price, $\Pr(c_L|\rho)$, is larger than 0. The next Proposition shows that after observing a price $p = \rho + \varepsilon$, for some small ε, the D1 logic forces non-shoppers to believe that the cost is high and that therefore they prefer to buy at these prices. This defies, however, the property of a reservation price. Thus, any perfect Bayesian equilibrium satisfying the D1 logic must be a non-RPE.

Proposition 1. *Any reservation-price equilibrium does not satisfy the D1 logic.*

The D1 logic asks which type of firm (high or low cost) has the most incentive to deviate to prices above the reservation price. It turns out that high cost firms have more incentive to deviate. Thus, as (i) both high and low cost firms put positive density on charging a price equal to the reservation price, (ii) at the reservation price consumers are indifferent between buying and continuing to search and (iii) the expected price when cost is high is strictly larger than when cost is low, consumers strictly prefer to buy at prices just above the reservation price if they believe these out-of-equilibrium prices to be set by high cost firms. Given these beliefs, firms would then, however, have an incentive to deviate and set these higher prices defying the notion of equilibrium. Thus, RPE require that after observing prices above the reservation price, consumers infer that cost is low with sufficiently high probability, while the logic of the D1 criterion requires consumers to believe cost is high upon observing such high prices.

To characterize the price distributions of non-RPE, we first show that the upper bounds of the low and high cost price distributions have to be identical. If this were not the case, there would be a region of prices above the upper bound of, say, the low cost distribution that are only chosen by high cost firms, and this would imply that $\beta(p) = 1$. Low cost firms would then have an incentive, however, to deviate to such prices.

Lemma 1. *In any equilibrium,* $\overline{p}_L = \overline{p}_H \equiv \overline{p}$.

If firms set a price equal to the upper bound $\overline{p}$, their profits will be equal to $\frac{1-\lambda}{2}\beta(\overline{p})(\overline{p} - c_i)$. As in equilibrium, for any price in the support of the price distribution this expression has to be equal to (1), we have that

$$\begin{aligned} \lambda(1 - F_i(p)) + \frac{1-\lambda}{2}\beta(p) + \frac{1-\lambda}{2}(1 - \beta(p))(1 - F_i(p)) + \\ \frac{1-\lambda}{2}\int_p^{\overline{p}}(1 - \beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p} = \frac{1-\lambda}{2}\beta(\overline{p})\frac{\overline{p} - c_i}{p - c_i}. \end{aligned} \tag{2}$$

At intervals of prices in the support of the price distribution where $\beta(p) = 1$, or, $\beta(p) = 0$, this equation can be solved for $F_i(p)$ in a straightforward manner. If $0 < \beta(p) < 1$, (2) can be transformed into an exact differential equation that can be solved as shown in the proof of the following Proposition.

Proposition 2. *The equilibrium price distribution, which makes firms indifferent between all the prices, is given by:*

$$F_i(p) = \begin{cases} \dfrac{2\sqrt{1-(1-\lambda)\beta(\overline{p})} - \int_p^{\overline{p}} \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(\widetilde{p}-c_i)^2\sqrt{1-(1-\lambda)\beta(\widetilde{p})}}d\widetilde{p}}{2\sqrt{1-(1-\lambda)\beta(p)}} & \text{if } p \in P_{(0,1)} \\ 1 - \frac{1-\lambda}{2\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - 1 - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_1 \\ 1 - \frac{1-\lambda}{1+\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_0 \end{cases} \tag{3}$$

Using the characterization of the price distributions, we can now state that $F_H(p)$ first-order stochastically dominates the low cost distribution $F_L(p)$. Thus, as in RPE we continue to have the expected price when cost is low, $E(p|c_L)$, being lower than the expected price when cost is high, $E(p|c_H)$.

Corollary 1. *For all $p < \overline{p}$, $F_L(p) \geq F_H(p)$ and whenever $0 < F_H(p) < 1$, $F_L(p) > F_H(p)$.*

Using these characterizations of the distribution functions it is not too difficult to see that if we want that the upper bound of the distributions $\overline{p}$ is not determined by arbitrary out-of-equilibrium beliefs, it must be the case that after observing $\overline{p}$ consumers believe that firms have high cost for sure, and that given this inference, non-shoppers are indifferent between buying now and continuing to search. If this were not the case, and non-shoppers would have out-of-equilibrium beliefs such that $\Pr(c = c_H|p) = 1$ for prices $p > \overline{p}$, then they would prefer to buy at these prices, giving firms an incentive to deviate (see the proof of Proposition 1 for details). Thus, the upper bound of the price distributions has to be equal to the reservation price in case consumers know cost is high, i.e.,

$$\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp = s. \tag{4}$$

As $F_H(p)$ first-order stochastically dominates $F_L(p)$ this implies that if an out-of-equilibrium price larger than $\overline{p}$ is observed, consumers will always want to continue to search independent of their beliefs of the underlying cost.

The above also implies that in any equilibrium satisfying the D1 logic it must be the case that consumers actively search with strictly positive probability.

Proposition 3. *In any Perfect Bayesian Equilibrium that satisfies the D1 logic where $\beta(p)$ is continuously differentiable, it must be the case that consumers update their beliefs about the underlying cost in such a way that $\Pr(c_H|\overline{p}) = 1, \beta(\overline{p}) < 1$ and that*

$$\beta'(\overline{p}) = -\frac{\beta(\overline{p})}{\overline{p} - c_L}.$$

It follows that there is a region of prices that both types of firms charge with strictly positive probability where consumers continue to search with strictly positive probability.

Corollary 2. *The probability that in equilibrium non-shoppers actively search is positive.*

To fully characterize a D1 equilibrium of the model, we have to inquire into the non-shoppers' equilibrium strategy, $\beta(p)$, with $0 \leq \beta(p) \leq 1$. Optimal search behaviour implies that whenever $0 < \beta(p) < 1$ the non-shopper is indifferent between buying now and continuing to search, implying that

$$\frac{(1-\alpha)f_L(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}\Phi_L(p) + \frac{\alpha f_H(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}\Phi_H(p) = s, \tag{5}$$

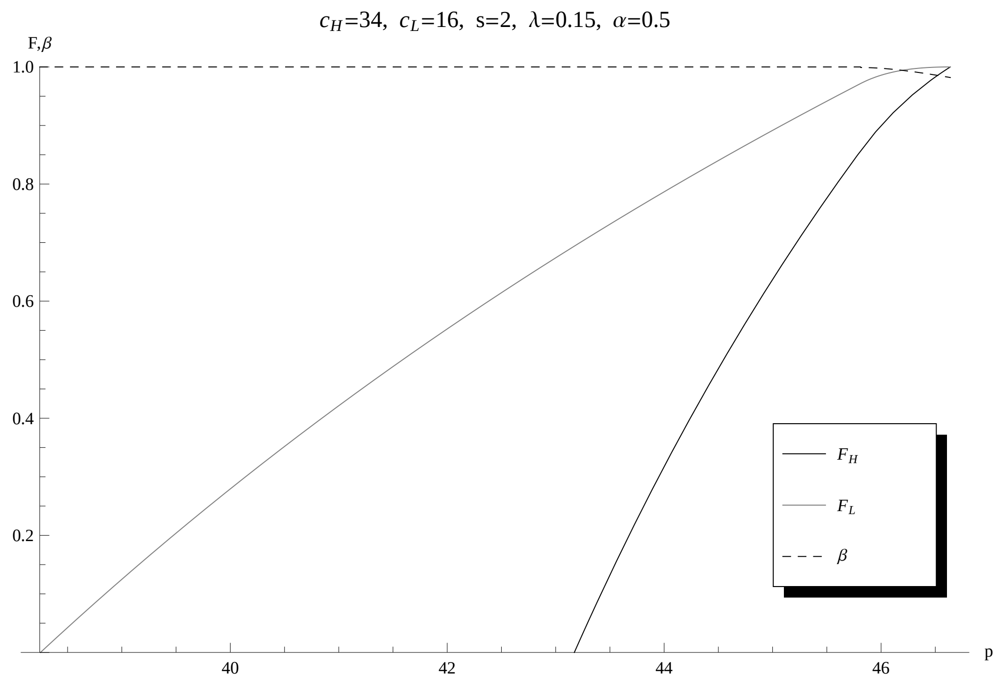
where $\Phi_i(p) = \int_0^p F_i(x)dx$. This equation says that after a non-shopper observes price p he will update his beliefs about the underlying cost of the firms and given these updated beliefs concludes that buying now yields the same expected pay-off as continuing to search. Optimal search behaviour also implies that the non-shoppers strictly prefer to buy ($\beta(p) = 1$) if, and only if, the LHS of (5) is strictly smaller than s and that the non-shoppers strictly prefer to search ($\beta(p) = 0$) if, and only if, the LHS of (5) is strictly larger than s. Together with (3) this behaviour characterizes an equilibrium.

As shown in the Appendix, equation (5) defines a differential equation which starting from initial conditions for $\overline{p}$ and $\beta(\overline{p})$ defines the function $\beta(p)$ going downward.[13] This function can continue to satisfy $0 < \beta(p) < 1$ or it may at some price point p reach the boundaries $\beta(p) = 1$ or $\beta(p) = 0$. If for some prices $\beta(p) = 1$, the following lemma shows that (5) implies that in any equilibrium $\beta'(p) = 0$ has to hold at the largest price point p where $\beta(p) = 1$.

Lemma 2. *Let p^* be such that $\beta(p^*) = 1$ and for any sufficiently small $\epsilon > 0$ $\beta(p^* + \epsilon) < 1$. If p^* is in the interior of the support of $F_i(p)$, $i = L, H$, then it must be that $\beta'(p^*) = 0$.*

[13]Note that (5) implies we should have $\beta'(\overline{p}) = -\beta(\overline{p})(\overline{p} - c_i)$ as derived in Proposition 3.

Figure 1: No gap equilibrium



We will now inquire into the existence question. The main question is whether for all parameter values $c_L, c_H, \lambda, \alpha$ and s we can find values of $\overline{p}$ and $\beta(\overline{p})$ such that equation (3) defines proper distribution functions that are upward sloping, and that the search strategy of non-shoppers satisfies the optimality condition (5). For relatively small cost differences $c_H - c_L$ it turns out that this question reduces to the question of whether we can find $\overline{p}$ and $\beta(\overline{p})$ such that equation (4) and $\beta'(p) = 0$ at the largest price p^* where $\beta(p^*) = 1$. In this solution the distribution functions defined in (3) are upward sloping , i.e., $f_L(p), f_H(p) \geq 0$, and for all prices smaller than p^* as defined in Lemma 2, $\beta(p) = 1$. We will call such an equilibrium a "*no gap equilibrium*" and an example is given in Figure 1. This Figure illustrates that at high prices $\beta(p) < 1$ and at lower prices $\beta(p) = 1$ and the price distributions do not have a gap. Figure 1 also illustrates that the demand of individual consumers is downward sloping.

For larger cost differences $c_H - c_L$ these requirements do not constitute an equilibrium, however, as ρ_L will be smaller than $\underline{p}_H$,where ρ_L is implicitly defined by

$$\int_{\underline{p}_L}^{\rho_L} F_L(p)dp = s. \tag{6}$$

If $\rho_L < \underline{p}_H$, then it cannot be the case that $\beta(\underline{p}_H) = 1$. The reason is that at prices smaller than $\underline{p}_H$ and larger than ρ_L non-shoppers infer that cost cannot be high and therefore prefer not to buy, but to continue to search, i.e., $\beta(p) = 0$ for all $\rho_L < p < \underline{p}_H$. On the other hand, non-shoppers will always buy immediately at prices smaller than ρ_L, i.e., $\beta(p) = 1$ for all $p < \rho_L$. This in turn implies that there will be a gap in the price distribution of low cost firms at prices just above ρ_L. There are two possibilities in this case. First, low cost firms do not charge prices in the interval $(\rho_L, \underline{p}_H)$ where $\beta(p) = 0$. In this case it is clear that $\beta(\underline{p}_H) = \beta(\rho_L) = 1$ would imply that low cost firms are not indifferent between charging $\underline{p}_H$ and ρ_L. Thus, we have $\beta(\underline{p}_H) < 1$. A second case that can arise is when low cost firms charge prices in the subset of the interval $(\rho_L, \underline{p}_H)$ where $\beta(p) = 0$. To make low cost firms indifferent between charging $\underline{p}_H$ and prices in the interval $(\rho_L, \underline{p}_H)$ it must be that $\beta(\underline{p}_H) = 0$. The fact that for large cost differences we should have $\beta(\underline{p}_H) < 1$ is the main reason why Dana (1994) and Janssen et al. (2011) find that (independent of out-of-equilibrium beliefs) RPE do not always exist (as in these equilibria $\beta(\underline{p}_H) = 1$).

Thus, for larger cost differences any equilibrium has a gap in the low cost price distribution where $\beta(p) = 1$ for all $p \leq \rho_L, \beta(p) = 0$ for all prices p in the interval $(\rho_L, \underline{p}_H)$ and $\beta(\underline{p}_H) < 1$. There are different types of these gap equilibria. One dimension along which these equilibria differ is whether or not low cost firms choose prices in the interval $(\rho_L, \underline{p}_H)$ where $\beta(p) = 0$. If they do, it is only consumers who have observed both prices who buy, and they buy at the lowest price in the market. We therefore denote such an equilibrium as a *competitive gap equilibrium.* If firms do not choose with positive probability prices in this interval, then we simply speak of a *regular gap* equilibrium. Another dimension along which these equilibria differ is whether or not there is a subset of prices in the interval $(\underline{p}_H, \overline{p})$ where non-shoppers buy for sure and $\beta(p) = 1$. If there is such an interval, we will denote such an equilibrium as a *monopolistic gap* equilibrium. Combining these different dimensions, we thus may have four different types of gap equilibria. Note that in all these equilibria, the gap is only in the low cost price distribution.

Figure 2 illustrates a regular gap equilibrium where $0 < \beta(p) < 1$ for all prices $p \in [\underline{p}_H, \overline{p}]$. In this case $\overline{p}$ and $\beta(\overline{p})$ are determined such that (4) and (6) hold. The cost difference is larger than that in Figure 1 and low cost firms either set low prices, or choose prices that are much larger, with a gap in between. In this equilibrium, there are four regions of prices where non-shoppers exhibit different behavior. At high prices (above $\overline{p}$), the consumers definitely continue to search. Consumers are indifferent between buying and continuing to search for all prices $p \in [\underline{p}_H, \overline{p}]$ as they update the beliefs about cost being low and the probability of finding lower prices if continuing to search. At prices below $\underline{p}_H$ (and above ρ_L) non-shoppers search for sure. Finally, at prices below ρ_L non-shoppers buy for sure. Although the parameter value of c_H is larger in Figure 2 than in Figure 1 (40 against 34, but average cost is the same) the support of the high cost price distribution has lower prices due to the fact that non-shoppers search more actively.

Figure 3 illustrates a monopolistic gap equilibrium. At prices close to $\overline{p}$, but also at prices close to $\underline{p}_H$ non-shoppers are indifferent between buying and continuing to search and $\beta(p) < 1$. At prices at and close to $\underline{p}_H$ $\beta(p) > 0$ and $\beta'(p) > 0$, the low cost distribution function is much steeper in this price region than the high cost distribution function. There is a relatively small gap in the low cost price distribution and $\beta(p) = 1$ for all $p \leq \rho_L$. At the lowest price p such that $\beta(p) = 1$, $\beta(p)$ is not continuously differentiable.[14] This equilibrium can co-exist with the equilibrium represented in Figure 2.

When the cost difference is large and the fraction of shoppers is large, the only way to satisfy condition (6) is to have a competitive gap equilibrium where $\beta(\underline{p}_H) = 0$. Figure 4 provides an example. In this case, low cost firms choose to set prices just below $\underline{p}_H$ with positive probability before a gap is created to ensure that (6) holds.

[14] Note that Lemma 2 only applies to the largest price p where $\beta(p) = 1$.

Figure 2: Regular gap equilibrium

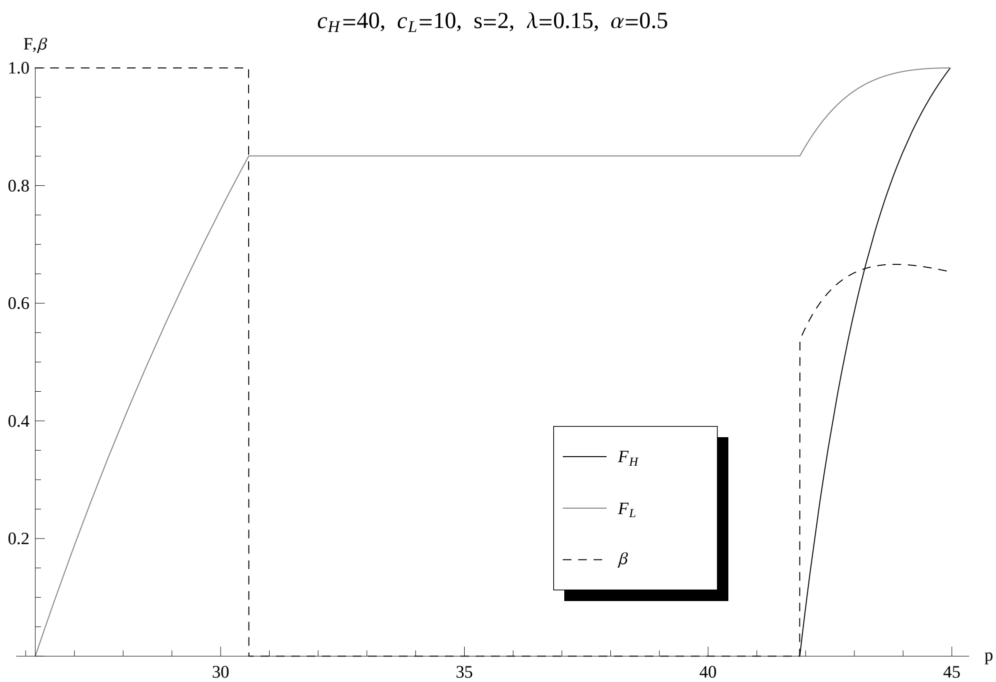


Figure 3: Monopolistic gap equilibrium

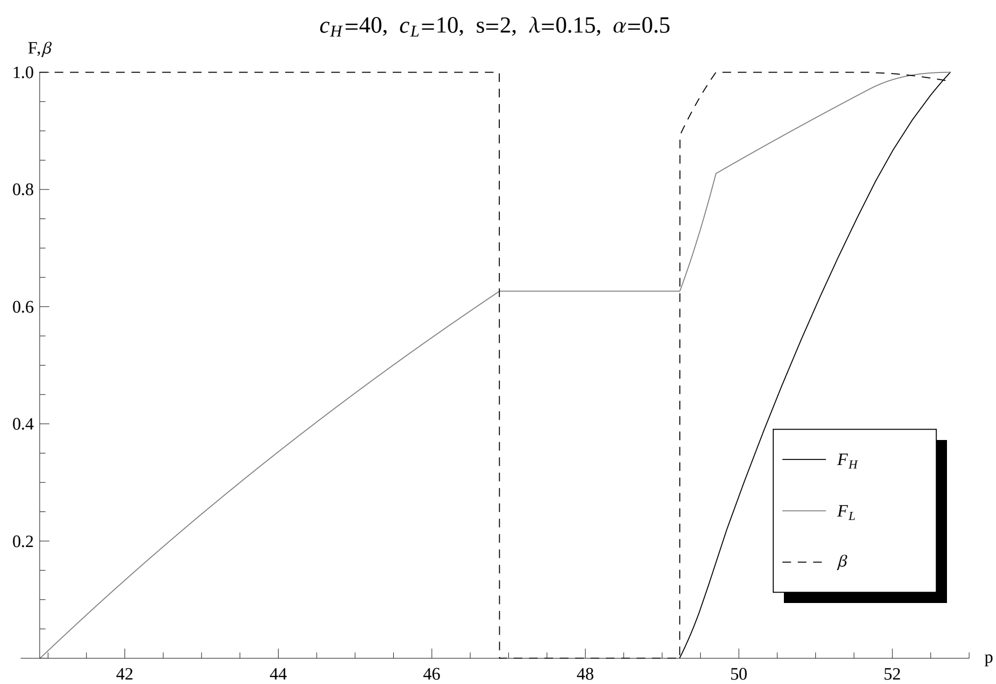
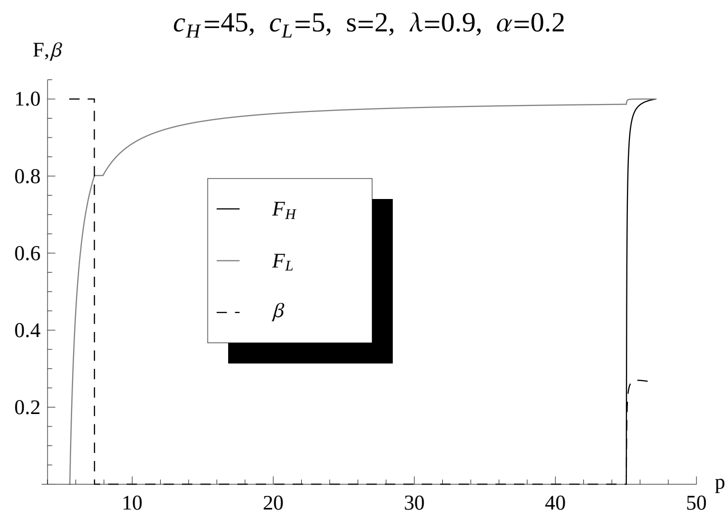


Figure 4: Competitive gap equilibrium



One can numerically compare for a given cost realization (i) the expected first price observation conditional on the price being accepted and (ii) the expected first price observation conditional on it not being accepted. De los Santos et al. (2012) observe that in their sample the first conditional expected price is larger than the second, and they rightly claim that this is inconsistent with RPE. For the parameter values used in Figures 2-4 one can compute and compare both conditional expected prices to conclude that for the high cost realization these non-RPE are consistent with the finding of De los Santos et al. (2012): for Figure 2 the numbers are 42.94 and 42.83, respectively, for Figure 3 the numbers are 50.81 and 49.83, respectively, while for Figure 4 they are 45.19 and 45.11, respectively.

In any gap equilibrium, we have some interval of prices just above ρ_L that are not charged with positive probability. In this interval it is natural to have out-of-equilibrium beliefs satisfying $\Pr(c_L|p) = 1$ for all $p \in (\rho_L, \underline{p}_H)$ implying that $\beta(p) = 0$. This out-of-equilibrium belief not only follows from the D1 logic, but also from the weaker notion of the Intuitive Criterion (Cho and Kreps, 1987). The reason is as follows: by setting a price equal to $\underline{p}_H$ a high cost firm already attracts all shoppers and all non-shoppers that first visited that firm. Of the remaining non-shoppers it will sell to all who continue to search after having visited the first firm.

By deviating to a lower price, a firm can never get a higher demand, and lowering the price, can only lower the profits. A low cost firm may have an incentive to deviate to prices $p \in (\rho_L, \underline{p}_H)$ if $\beta(p)$ is high enough. As the high cost type does not have an incentive to deviate and the low cost type may have an incentive (depending on the reaction of the non-shoppers), the Intuitive Criterion implies that $\beta(p) = 0$ for all $p \in (\rho_L, \underline{p}_H)$.

The following Theorem shows that an equilibrium satisfying the D1 logic exists for all values of the exogenous parameters.

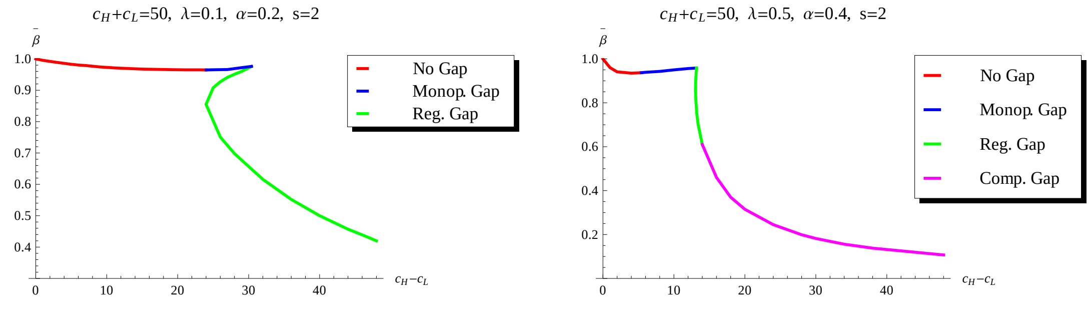
Theorem 1. *For any values of s, λ, c_L, c_H and α an equilibrium satisfying the D1 logic exists.*

The proof is constructive and shows that for any combination of parameter values one of the four types of equilibria defined above has to exist. The proof consists of several lemmas and is given in Appendix II. For a range of parameter values the equilibrium is not unique, while for other parameter values the equilibrium is unique. To better understand the equilibrium structure, Figure 5 shows for given values of s, λ and α how the equilibrium configuration may depend on the cost difference $c_H - c_L$.

For relatively small values of λ, Figure 5(a) shows there are three possible equilibrium configurations, depending on whether the cost difference is small, large or intermediate. If the cost difference is relatively small, there is a unique equilibrium without a gap in the low cost distribution. When c_H is close to c_L the value of $\beta(\overline{p})$ becomes closer to 1 and in the limit, when cost uncertainty disappears the Stahl equilibrium is the only possible equilibrium. If, on the other hand, the cost difference $c_H - c_L$ is relatively large, then there exists a unique regular gap equilibrium. The value of $\beta(\overline{p})$ has to be relatively low to satisfy the equilibrium conditions for such an equilibrium to exist. Finally, if the cost difference $c_H - c_L$ is at intermediate values, a monopolistic gap equilibrium co-exists together with two regular gap equilibria.[15] For larger values of λ, Figure 5(b) distinguishes four possible equilibrium configurations, while equilibrium is unique for each value of the cost difference

[15] This multiplicity of equilibria seems to be genuine and it is not easy to select among these

Figure 5: $\overline{\beta}$ as a function of cost difference.



(a) (b)

Figures 2, 3 and 4 show that non-RPE do not exhibit a simple monotone relationship between price and the probability of buying (or the probability of continuing to search). In a gap equilibrium, the $\beta(p)$ functions have an increasing segment, indicating that at higher prices the probability of consumers buying is higher (and thus the probability they continue searching is lower). Figure 4 shows an extreme case of this where there is a region of prices that are set by low cost firms such that non-shoppers continue to search for sure, while at higher prices the probability of continuing to search is lower. Thus, these Figures indicate that the optimal search behaviour may be highly nonmonotonic in price. De los Santos et al. (2012) empirically find that it is not the case that at higher prices, consumers are more likely to continue to search. Our analysis shows that this does not rule out that consumers search sequentially, although it does rule out that consumers follow reservation price strategies.

Equilibria where the low cost price distribution has non-compact support may be interpreted as a search theoretic foundation for the reference price principle that is discussed in marketing (see the references in the Introduction). In our model, reference prices endogenously arise from the fact that consumers rationally infer that a certain low price will only be set when cost is low, and if the common cost

equilibria. Fershtman and Fishman (1992) use a stability argument to argue that one of the equilibria in their search model is unstable. It is difficult to see how a stability argument can be invoked in our context as the behaviour of consumers is not characterized by a single parameter as in their model, but by the function $\beta(p)$.

is really low, then the chances of finding low prices are good so, it is rational to continue searching for better deals. Thus, it is better for firms not to set prices just above these reference prices. If higher prices are to be set, it is better to choose prices in a higher range where the probability of a sale is large enough.

4 Comparative Statics and A Comparison with Related Models

We are now in a position to compare the equilibrium outcomes of our model with two benchmark models, and to perform some numerical comparative statics analysis. On one hand, we use Stahl (1989) as a benchmark to show the implications of cost uncertainty. On the other hand, we use Dana (1994), or equivalently Janssen et al. (2011), as a benchmark for the outcome of RPE with cost uncertainty. As shown in Janssen et al. (2011) the expected price under RPE is larger than the weighted average of the expected price of the high and low cost equilibria as developed by Stahl (1989) and in that sense, consumers are worse off under cost uncertainty. In this Section we show that this result may well be reversed for non-RPE.

There are several effects that play a role when comparing the outcomes of non-reservation equilibria with those of RPE. First, for a given upper bound $\overline{p}$, lowering $\beta(\overline{p})$ from an initial value of 1 (which is the value in the case of RPE) implies that there are more consumers making price comparisons. This implies firms tend to lower their prices as a reaction to the increased competition. A second effect is a direct consequence: as for a given upper bound expected prices will be lower, therefore searching for lower prices becomes more beneficial (as the expected prices after a search are lower), lowering the upper bound (as it is equal to the high-cost reservation price). The third effect is that in a non-RPE non-shoppers believe that cost is high after observing the upper bound, while in a RPE as in Dana (1994) and Janssen et al. (2011) the upper bound equals the weighted average of the reservation prices when cost is certainly low or certainly high. Higher upper bounds of the price distribution tend to be associated with higher expected prices.

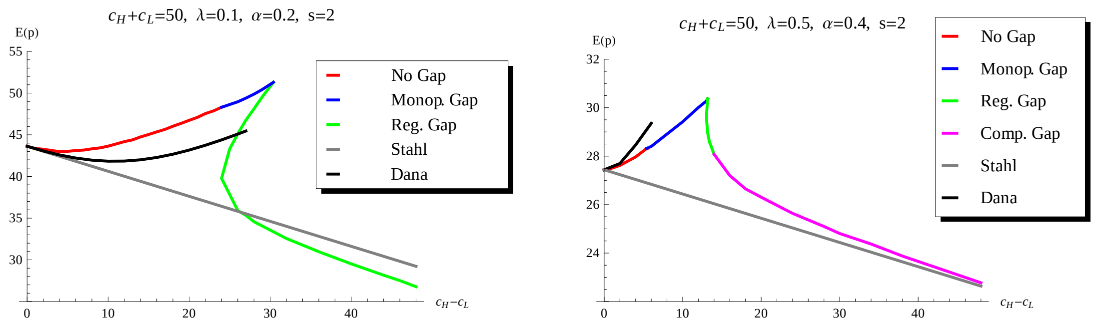
Figure 6 shows the typical effect on ex ante expected prices of these three effects. Expected price is a good measure of the surplus of the non-shoppers. When they continue to search non-shoppers pay the search cost, but they also get to buy at the lowest of two prices. As in equilibrium, when they search twice they are indifferent between buying and searching, the additional expected benefit of the possibility of buying at a lower price is exactly offset by the cost of the additional search. In both situations, the average cost is taken to be 25 and the cost difference $c_H - c_L$, measured on the horizontal axis, varies between 0 (implying the cost is known to be 25) and 50 (where $c_L = 0$ and $c_H = 50$). When the cost difference is 0 all models results in the same expected price. In the Stahl (1989) model where cost is known the expected price is a fixed number larger than the cost level, where the fixed number depends on λ and s, but not on c. The ex ante expected price reported here for the Stahl model is the weighted costs plus this fixed number. This expected price is thus decreasing in the cost difference $c_H - c_L$, if $\alpha < 0.5$ (as in Figure 6). The expected price in Dana (1994) or Janssen et al. (2011) is known to be higher than the ex ante weighted average of the expected prices in the Stahl model. The Figures also show that the RPE analyzed in these two papers does not exist for larger cost differences. Figures 6(a) and 6(a) show that for smaller cost differences expected prices are even larger than the ones reported in Janssen et al. (2011). The Figures also show, however, that for larger cost differences the expected price in a non-RPE becomes smaller and that it can even become smaller than the ex ante weighted average of expected prices in the Stahl model. Figure 6(a) shows that this difference can be in the order of 10%, which is non-negligible.

In the different panels of Figure 7, we perform a numerical comparative static analysis showing how expected price and the probability that non-shoppers search twice, which is given by

$$E(1-\beta(p)) = \alpha \int_{\underline{p}_H}^{\rho_H} (1-\beta(p)) f_H(p) dp + (1-\alpha) \int_{\underline{p}_L}^{\rho_H} (1-\beta(p)) f_L(p) dp$$

changes with the changes in the different exogenous parameters s, λ and a.

Figure 6: Expected prices as a function of cost difference.

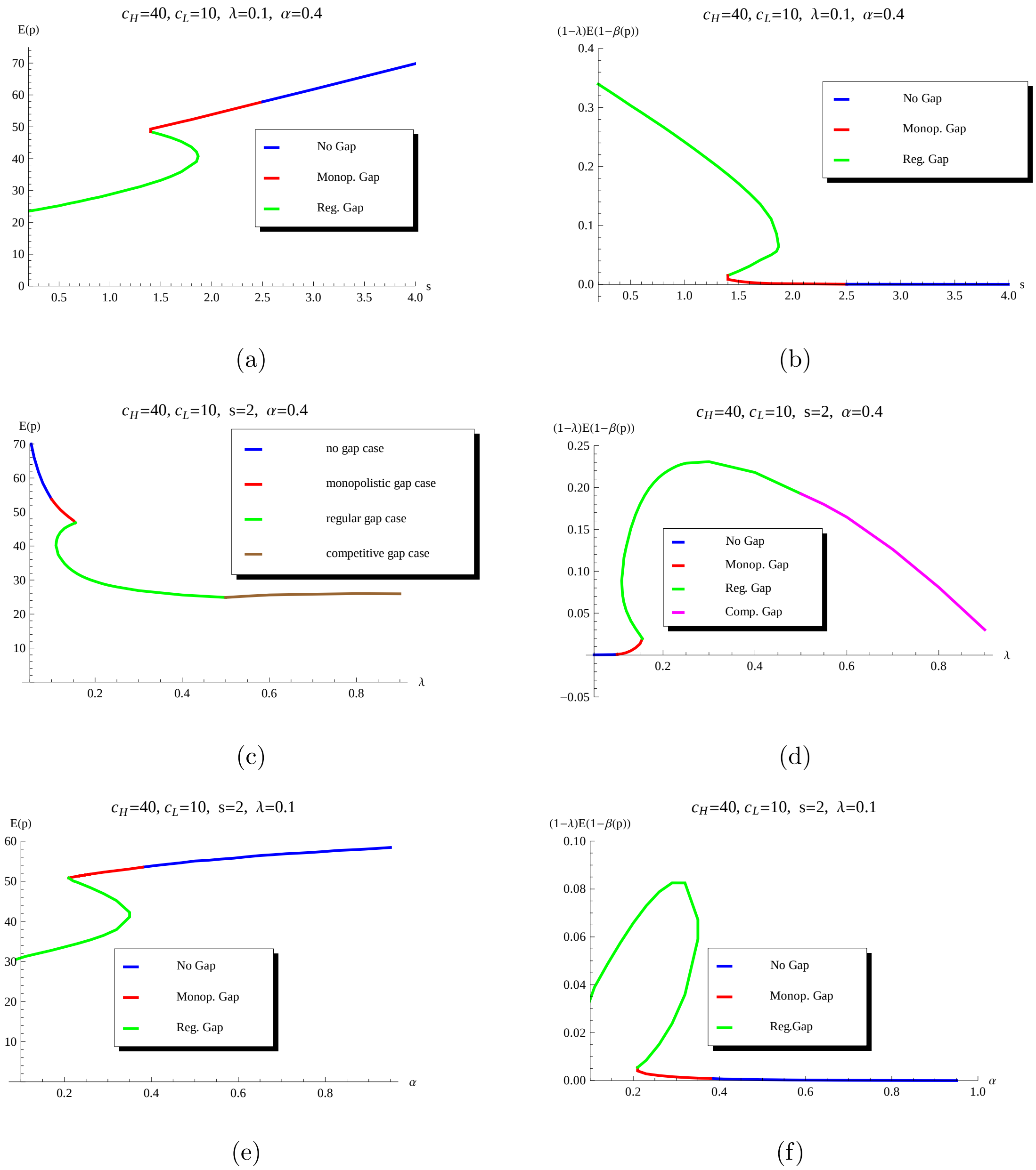


(a) (b)

The first two panels (7(a) and 7(b)) show the dependence on search cost. For small search cost, a large fraction of non-shoppers performs two searches and the expected price is close to the average marginal cost of 25. When the search cost increases from initially low levels, the expected price increases and the fraction of non-shoppers performing two searches decreases (giving firms more market power). At search cost levels close to 2, there are multiple gap equilibria, and it may be that the expected price is decreasing in search cost. When the search cost further increases a no gap equilibrium emerges and the probability of non-shoppers searching twice becomes very close to 0. Panel (7(b)) also shows that starting from an initially small search cost, non-shoppers will search less when the search cost increases. In this way, non-shoppers partially mitigate the increase in market power typically associated with higher search cost.

The middle two panels (7(c) and 7(d)) show the dependence on the fraction of shoppers. When λ is small, there are many non-shoppers and a no gap equilibrium exists. In such an equilibrium very few non-shoppers perform two searches and the expected price is high. When λ increases, the expected price decreases, but in the area where multiple equilibria exist the difference in the expected price can be quite large as the fraction of non-shoppers performing two searches differs greatly between the different equilibria. When λ increases further, we enter the area where only competitive gap equilibria exist. In this case increasing λ leads to a higher probability that low-cost firms price in the area where $\beta(p) = 0$ and the average

Figure 7: Comparative Statics.



price increases slightly.

The last two panels (7(e) and 7(f)) show the dependence on the probability that the cost is high. When this probability is high, there is a no-gap equilibrium and consumers search very little, since there is a low probability of obtaining a substantially lower price. In this region the higher the α, the higher the expected price. For lower values of α there is a monopolistic gap equilibrium with qualitatively similar properties. When α is sufficiently low, there are multiple gap equilibria and the incentives to search can be high, pushing the prices down. The expected price can be both increasing and decreasing in α depending on which of the regular gap equilibria is chosen.

5 Oligopoly Markets: an Extension

In general, it is difficult to analytically characterize non-RPE when there are more than two firms in the market due to the fact that depending on the prices observed, consumers may perform a different number of searches, creating complications for solving for the price distribution of firms. Nevertheless, the following result on the optimal search behaviour of consumers helps to considerably reduce the complexities in analyzing certain types of equilibria under an oligopoly. In this result we denote by p^t the price a non-shopper observes in search round t.

Proposition 4. *Suppose, the consumer was indifferent between continuing to search or buying after the first price observation $p^{(1)}$ and $f_H(p) > f_L(p)$ for all $p \in P_{(0,1)}$. Then if the consumer continued, she stops searching after the second price observation $p^{(2)}$ and buys at $\min\{p^{(1)}, p^{(2)}\}$.*

There are two interesting aspects about this Proposition. First, if a non-shopper observes two prices $p^{(1)}$ and $p^{(2)}$, with $p^{(1)} < p^{(2)}$, then the Proposition says the consumer will stop searching and go back to the first firm if the high cost density is larger than the low cost density. Thus, going back to previously sampled firms before all firms are searched may well be consistent with a sequential search. De los Santos et al. (2012) have observed that consumers do go back to previously sampled firms before having visited all firms. This is inconsistent with reservation price strategies, as they noted, but not necessarily with sequential search.

Second, if in a non-RPE we have that $f_H(p) > f_L(p)$ in the price region where $\beta(p) < 1$, then we know that non-shoppers will never search beyond the second firm, and the profit function under oligopoly can be written as

$$\begin{aligned}\pi(p|c_i) \;=\; & \left[\lambda(1-F_i(p))^{N-1} + \frac{1-\lambda}{N}\beta(p) + \frac{1-\lambda}{N}(1-\beta(p))(1-F_i(p)) + \right. \\ & \left. \frac{1-\lambda}{N-1}\frac{N-1}{N}\int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](p-c_i)\end{aligned}$$

so that the differential function which has to be solved to find the distribution functions reduces to

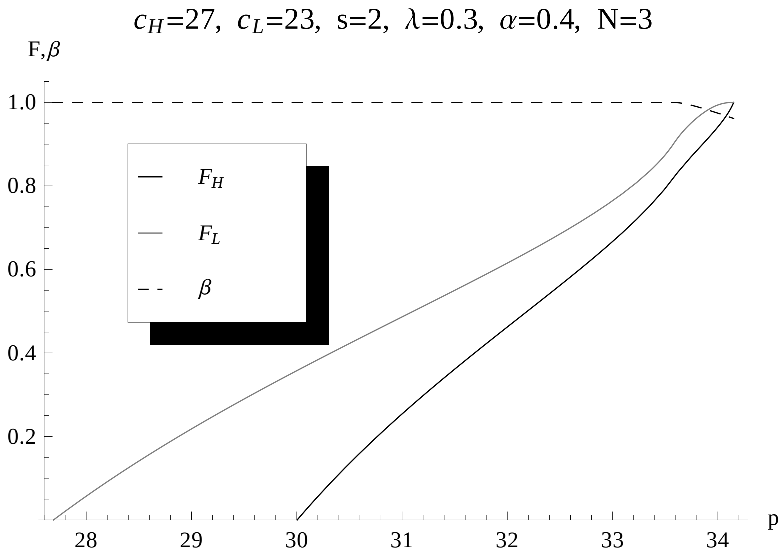
$$-2\left[1+\frac{\lambda N(N-1)}{2(1-\lambda)}(1-F_i)^{N-2}-\beta(p)\right]dF_i+\left[\beta'(p)F_i+\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]dp=0. \tag{7}$$

This differential equation can be solved numerically, and it can be checked whether $f_H(p) > f_L(p)$ indeed holds for all prices in the price region where $\beta(p) < 1$. In Figure 8 we illustrate the distribution functions that solve (7) for particular parameter values. It can be checked that the condition on the density functions is satisfied.

6 Discussion and Conclusion

In this paper we have considered search markets where consumers do not know the underlying common costs of firms. If consumers do not know the prices different firms charge, it is natural that they also do not know the underlying cost. We have argued that in this environment of cost uncertainty, the standard RPE considered in the consumer search literature suffer from some severe limitations. It has already been shown in the literature that RPE do not always exist in such an environment, but we add that RPE implicitly assume specific out-of-equilibrium beliefs that do not seem to always be reasonable. We characterize non-RPE that do not depend on

Figure 8: Equilibrium price distributions and stopping probability for $N = 3$



assumptions regarding out-of-equilibrium beliefs in defining the upper bound of the price distributions.

In non-RPE non-shoppers are indifferent between buying and continuing to search over a range of prices. As prices in this range are set with positive probability, these non-RPE have active search with positive probability in equilibrium. Thus, we extend the Rothschild (1974) finding by showing in a model with endogenous price setting that consumers do not choose reservation price strategies. An extreme example of a non-reservation price strategy which arises in our equilibrium analysis is when the cost difference is large, and the low cost distribution does not have connected support. In this case, over the whole range of prices that are charged when cost is high, consumers randomize between searching and buying and at an interval of prices below this price range, consumers rationally decide to continue to search as they expect cost to be low.

We show that non-RPE always exist. This result is important as it resolves the issue raised in the earlier literature as to what type of perfect Bayesian equilibria may exist in an asymmetric information game, if a RPE does not exist.

We also show that non-RPE have very different properties from RPE. One important result in this respect is that the expected market prices may well be lower under cost uncertainty than under cost certainty. The reason is that cost uncertainty makes it possible for consumers to rationally search more than under cost certainty and this additional search has a quantitatively important pro-competitive effect on prices.

Our results on non-RPE also have important consequences for the empirical literature on consumer search models that is recently taking off. We have shown that non-RPE may be consistent with the observations of De los Santos et al. (2012) as (i) consumers may rationally continue to search at lower prices, while they buy at higher prices and (ii) consumers may stop searching and buy at a previously visited store, before they have observed all prices in the market. Moreover, the price distributions of non-RPE are quite different from the regular price distributions found in RPE. It would be challenging to see whether these price distributions provide a good fit with data.

As a first inquiry into non-RPE, we have made some assumptions that restrict the immediate application of this paper to real world markets. We mainly consider duopoly markets and also consider the uncertainty that is characterized by two cost states only. There does not seem to be a particular reason why non-RPE cannot be characterized (or at least numerically calculated) for these different possible extensions, and some of these extensions are clearly non-trivial. One issue that needs to be addressed in the generalizations to oligopoly markets is how consumer inferences after observing two (or more) prices interact with the consumer search decisions. In the extension analyzed in this paper, we dealt with the easiest of different possible cases that can arise. In general, however, different possible search behaviours interact in a complicated way with the incentive of firms to choose different prices. This paper made a first step analyzing non-RPE. There are many theoretical and empirical challenges that lie ahead.

7 Appendix I: Proofs of Lemmas, Propositions and Corollaries

Proposition 1. *Any reservation-price equilibrium does not satisfy the D1 criterion.*

Proof. As only non-shoppers buy at the reservation price, in a reservation price equilibrium the profits of low and high cost firms are given by $\pi_L = \frac{1-\lambda}{2}(\rho - c_L)$ and $\pi_H = \frac{1-\lambda}{2}(\rho - c_H)$, respectively. If non-shoppers buy with probability $\beta(p)$ after observing an out-of-equilibrium price $p > \rho$, then the deviating firm makes a profit of $\pi_i = \frac{1-\lambda}{2}\beta(p)(p - c_i), i = 1, 2$. This is larger than the equilibrium profit if

$$\beta(p) > \frac{\rho - c_i}{p - c_i}.$$

As the RHS of this inequality is decreasing in c_i for all $p > \rho$, the high cost firms have a wider range of responses from the consumers for which it is profitable for them to deviate to prices $p > \rho$. The D1 refinement thus requires that the out-of-equilibrium belief $\Pr(c_L|p) = 0$ for all $p > \rho$.

As after observing the reservation price non-shoppers are indifferent between buying and continuing to search

$$\rho = s + \Pr(c_L|\rho)E(p|c_L) + \Pr(c_H|\rho)E(p|c_H).$$

As $\Pr(c_L|\rho) > 0$ and $E(p|c_L) < E(p|c_H)$, it follows that for some ε small enough and $p = \rho + \varepsilon$,

$$p < s + E(p|c_H).$$

Thus, given the D1 out-of equilibrium beliefs non-shoppers prefer to buy at prices just above ρ. Therefore it is optimal for both types of firms to deviate from the RPE and choose a price (just) above the reservation price. □

Lemma 1. *In any equilibrium, $\overline{p}_L = \overline{p}_H \equiv \overline{p}$.*

Proof. If the upper bounds are not equal it must be the case that $\overline{p}_H > \overline{p}_L$, or vice versa. As the argument in both cases is identical, we only consider the case where $\overline{p}_H > \overline{p}_L$, Due to the fact that the price distributions do not have mass points, it must be the case that in a left neighborhood of $\overline{p}_H$ high cost firms charge prices with strictly positive probability. For any small $\varepsilon > 0$ consider then the interval $(\overline{p}_H - \varepsilon, \overline{p}_H)$. If a low type firm would not charge prices in this interval, consumers would know that cost is high after observing prices in this interval. Given that consumers are (at least) indifferent between buying and not buying at $\overline{p}_H$ (as, if consumers prefer to continue to search after observing $\overline{p}_H$, no firm would ever charge $\overline{p}_H$), they strictly prefer to buy at prices the interval $(\overline{p}_H - \varepsilon, \overline{p}_H)$. But then low cost firms would prefer to set prices in this interval as well instead of charging $\overline{p}_L$. Thus, $\overline{p}_L = \overline{p}_H$.

□

Proposition 2. *The equilibrium price distribution, which makes firms indifferent between all the prices, is given by:*

$$F_i(p) = \begin{cases} \frac{2\sqrt{1-(1-\lambda)\beta(\overline{p})} - \int_p^{\overline{p}} \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(\widetilde{p}-c_i)^2\sqrt{1-(1-\lambda)\beta(\widetilde{p})}} d\widetilde{p}}{2\sqrt{1-(1-\lambda)\beta(p)}} & \text{if } p \in P_{(0,1)} \\ \\ 1 - \frac{1-\lambda}{2\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - 1 - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_1 \\ 1 - \frac{1-\lambda}{1+\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_i}{p-c_i} - \int_p^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right] & \text{if } p \in P_0 \end{cases} \quad (8)$$

Proof. Assuming the function $\beta(p)$ is differentiable, equation (2) can be rewritten as

$$-2\left[1-(1-\lambda)\beta(p)\right] f_i(p) + (1-\lambda)\beta'(p)F_i(p) = -(1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}$$

by taking the derivative of both sides of the equality sign. This equation can be explicitly written as a differential equation:

$$-2\left[1-(1-\lambda)\beta(p)\right] dF_i + \left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right] dp = 0. \quad (9)$$

As

$$-2\frac{\partial\left[1-(1-\lambda)\beta(p)\right]}{\partial p} \neq \frac{\partial\left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]}{\partial F_i}$$

this is an inexact linear differential equation. However, it can be made exact by dividing (9) by $\sqrt{1-(1-\lambda)\beta(p)}$:

$$-2\sqrt{1-(1-\lambda)\beta(p)}dF_i + \frac{\left[(1-\lambda)\beta'(p)F_i + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]}{\sqrt{1-(1-\lambda)\beta(p)}}dp = 0.$$

The solution to this exact differential function is a function $Z(F_i, p) = C_i$ (where C_i is an integration constant) with $\frac{\partial Z}{\partial p} = \frac{\left[(1-\lambda)\beta'(p)F_i+(1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}\right]}{\sqrt{1-(1-\lambda)\beta(p)}}$ and $\frac{\partial Z}{\partial F_i} = \sqrt{1-(1-\lambda)\beta(p)}$. It follows that the solution $Z(F_i, p)$ is given by

$$-2F_i\sqrt{1-(1-\lambda)\beta(p)} + \int \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2\sqrt{1-(1-\lambda)\beta(p)}}dp + C_i = 0.$$

This equation can be solved explicitly for $F_i(p)$, to yield (3), where the integration constant C_i is found by setting $F_i(\overline{p}) = 1$.

If, $\beta(p) = 1$ or $\beta(p) = 0$ in an interval of prices $(\widehat{p}, \widetilde{p})$, then the equilibrium price distribution can be simply directly calculated from (2). □

Corollary 1. *For all $p < \overline{p}$, $F_L(p) \geq F_H(p)$ and whenever $0 < F_H(p) < 1$, $F_L(p) > F_H(p)$.*

Proof. From the previous Proposition, it follows that $F_H(p) < F_L(p)$ if, and only if, $\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_H)}{(p-c_H)^2\sqrt{1-(1-\lambda)\beta(p)}} > \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_L)}{(p-c_L)^2\sqrt{1-(1-\lambda)\beta(p)}}$ for all p. This, is the case if

$$(p-c_H)^2(\overline{p}-c_L) < (p-c_L)^2(\overline{p}-c_H).$$

This can be rewritten as

$$(c_H-c_L)p^2 - ((c_H-c_L)\overline{p}p + c_Lc_H(c_L-c_H) < 0$$

or $p^2 - \overline{p}p - c_Lc_H < 0$, which is definitely the case. □

Proposition 3. *In any Perfect Bayesian Equilibrium that satisfies the D1 criterion where $\beta(p)$ is continuously differentiable, it must be the case that consumers update their beliefs about the underlying cost in such a way that* $\Pr(c_H|\overline{p}) = 1, \beta(\overline{p}) < 1$ *and that*

$$\beta'(\overline{p}) = -\frac{\beta(\overline{p})}{\overline{p}-c_L}$$

Proof. First, it is clear that $\beta(\overline{p}) > 0$ as otherwise no high cost firm would charge $\overline{p}$ with positive density. Following the same logic as in the proof of Proposition 1, but replacing ρ by $\overline{p}$ and adjusting the argument for $\beta(\overline{p})$, it is easy to see that the D1 logic requires that the out-of-equilibrium belief $\Pr(c_L|p) = 0$ for all $p > \overline{p}$. As again, non-shoppers have to be indifferent between buying and continuing to search after observing $\overline{p}$ we have that

$$\overline{p} = s + \Pr(c_L|\overline{p})E(p|c_L) + \Pr(c_H|\overline{p})E(p|c_H).$$

As it follows from Corollary 1 that $E(p|c_L) < E(p|c_H)$, we can follow the same logic as in the proof of Proposition 1 to show that if $\Pr(c_L|\overline{p}) > 0$ non-shoppers prefer to buy at prices just above $\overline{p}$ and then it is optimal for both types of firms to deviate. Thus, it must be that $\Pr(c_H|\overline{p}) = 1$.

From Lemma 1 it follows that in the interval $(\overline{p} - \varepsilon, \overline{p})$ both types of firms charge prices with strictly positive probability. As the low cost density at $\overline{p}$ should be 0 and at prices below $\overline{p}$ it is positive, it follows that the profits the low cost firm makes by selling only to non-shoppers reaches a maximum at $p = \overline{p}$. Maximizing $\frac{1-\lambda}{2}\beta(p)(p - c_L)$ and imposing the maximum gives

$$\beta'(\overline{p})(\overline{p} - c_L) + \beta(\overline{p}) = 0,$$

which can only be the case when $\beta(\overline{p}) < 1$. □

Corollary 2. *The probability that in equilibrium non-shoppers actively search is positive.*

Proof. Since $\beta(\overline{p}) > 0$ we have that $\beta'(\overline{p}) < 0$, and therefore, $\beta(\overline{p}) < 1$. Thus, for some ε small enough there exist an interval $(\overline{p} - \varepsilon, \overline{p})$ where $\beta(p) < 1$ and as these prices are charged with positive probability, there is a strictly positive probability that consumers search in any equilibrium satisfying the D1 criterion whenever $c_L < c_H$. □

Lemma 2. *Let p^* be such that $\beta(p^*) = 1$ and for any sufficiently small $\epsilon > 0$ $\beta(p^* + \epsilon) < 1$. If p^* is in the interior of the support of $F_i(p), i = L, H$, then it must be that $\beta'(p^*) = 0$.*

Proof. Suppose, $\beta'(p^*) < 0$. Denote $\Delta f_i = f_i(p^* - \varepsilon) - f_1(p^* + \varepsilon)$. Then, since

$$f_i(p^*) = \frac{(1-\lambda)\beta'(p)F_i(p) + (1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2}}{2\left[1-(1-\lambda)\beta(p^*)\right]},$$

and $F_L > F_H$ we have $\Delta f_l < \Delta f_h$ (both negative).

Denote

$$a_i = (1-\lambda)\beta(\overline{p})(\overline{p} - c_i)$$

Then

$$\frac{a_L}{(p-c_L)^2} < \frac{a_H}{(p-c_H)^2}$$

which implies that $f_L < f_H$ for prices higher than p^*. This gives $\frac{f_L(p^*-\varepsilon)}{(1-\alpha)f_L(p^*-\varepsilon)+\alpha f_H(p^*-\varepsilon)} = \frac{f_L(p^*+\varepsilon)+\Delta f_L}{(1-\alpha)f_L(p^*+\varepsilon)+\alpha f_H(p^*+\varepsilon)+(1-\alpha)\Delta f_L+\alpha\Delta f_H} > \frac{f_L(p^*+\varepsilon)}{(1-\alpha)f_L(p^*+\varepsilon)+\alpha f_H(p^*+\varepsilon)}$. Thus, if consumers are indifferent at $p^* + \varepsilon$, they must strictly prefer to continue searching at $p^* - \varepsilon$, which can not be the case. Therefore, $\beta'(p^*) = 0$ (since it cannot be grater than 0). □

Proposition 4. *Suppose, the consumer was indifferent between continuing to search or buying after the first price observation $p^{(1)}$ and $f_H(p) > f_L(p)$ for all $p \in P_{(0,1)}$. Then if the consumer continued, she stops searching after the second price observation $p^{(2)}$ and buys at $\min\{p^{(1)}, p^{(2)}\}$.*

Proof. Consider a consumer who has observed two prices $p^{(1)}$ and $p^{(2)}$. Given that the consumer was indifferent after observing $p^{(1)}$, the optimal stopping rule for the first round gives

$$w_1(p^{(1)})(\Phi_L(p^{(1)}) - s) + (1 - w_1(p^{(1)}))(\Phi_H(p^{(1)}) - s) = 0,$$

where

$$w_1(p^{(1)}) = \frac{\alpha f_L(p^{(1)})}{\alpha f_L(p^{(1)}) + (1-\alpha)f_H(p^{(1)})}.$$

After observing $p^{(2)}$ the decision of the consumer is determined by the sign of

$$w_2(p^{(1)}, p^{(2)})(\Phi_L(p^{(1)}) - s) + (1 - w_1(p^{(1)}, p^{(2)}))(\Phi_H(p^{(1)}) - s),$$

where

$$w_2(p^{(1)}, p^{(2)}) = \frac{\alpha f_L(p^{(1)}) f_L(p^{(2)})}{\alpha f_L(p^{(1)}) f_L(p^{(2)}) + (1-\alpha) f_H(p^{(1)}) f_H(p^{(2)})}.$$

Note, that if $w_2(p^{(1)}, p^{(2)}) < w_1(p^{(1)})$ this sign is always negative and the consumer prefers to stop. This is the case if

$$\frac{\alpha f_L(p^{(1)})}{\alpha f_L(p^{(1)}) + (1-\alpha) f_H(p^{(1)})} > \frac{\alpha f_L(p^{(1)}) f_L(p^{(2)})}{\alpha f_L(p^{(1)}) f_L(p^{(2)}) + (1-\alpha) f_H(p^{(1)}) f_H(p^{(2)})},$$

which can be rewritten as

$$\alpha^2 f_L(p^{(1)}) f_L(p^{(2)}) + \alpha(1-\alpha) f_L(p^{(1)}) f_H(p^{(1)}) f_H(p^{(2)}) >$$
$$\alpha^2 f_L(p^{(1)}) f_L(p^{(2)}) + \alpha(1-\alpha) f_L(p^{(1)}) f_H(p^{(1)}) f_L(p^{(2)}),$$

and reduces to

$$f_H(p^{(2)}) > f_L(p^{(2)}).$$

□

8 Appendix II: Proof of Theorem 1

Here, we prove the existence of equilibrium (Theorem 1) in several lemmas. In general, and as explained in the main text, we need to prove that the two functional equations (3) characterizing the distribution functions and the optimality condition (5) for the search rule of non-shoppers has a solution such that the distribution functions are well-defined, i.e. the densities are positive. When $\beta(p) = 0$ or $\beta(p) = 1$, these conditions are trivially satisfied. When $0 < \beta(p) < 1$ two boundary conditions need to be satisfied and we have two parameters to satisfy them: $\overline{p}$ and $\beta(\overline{p})$. First, we need that $f_L(\overline{p}) = 0$, which implies that $\int_0^{\overline{p}} F_H(x)dx = s$. The second boundary condition is different for different parameter values. For the purpose of formulating this second boundary condition, implicitly define $p^* \leq \underline{p}_H$ as

$$\pi_L(p^*) = \pi_L(\underline{p}_H),$$

or

$$\left[\lambda(1-F_L(p^*))+\frac{1-\lambda}{2}+\frac{1-\lambda}{2}\int_{p^*}^{\underline{p}_H} f_i(\widetilde{p})d\widetilde{p}+\right.$$

$$\left.\frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](p^*-c_L)=$$

$$\left[\lambda(1-F_L(\underline{p}_H))+\frac{1-\lambda}{2}-\frac{1-\lambda}{2}(1-\beta(\underline{p}_H))F_L(\underline{p}_H)+\right.$$

$$\left.\frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_L(\widetilde{p})d\widetilde{p}\right](\underline{p}_H-c_L).$$

That is, p^* is the largest price smaller than $\underline{p}_H$ that makes low cost firms indifferent between (i) setting this price and having uninformed consumers immediately buy at this price and not buying for sure at any price in the interval $(p^*, \underline{p}_H)$ and (ii) choosing $\underline{p}_H$ and have uninformed consumers buying with probability $\beta(\underline{p}_H)$. To see that p^* is uniquely defined consider the following two cases. If low quality firms do not charge prices in the interval $(p^*, \underline{p}_H)$ with positive probability, then the demand at p^* is independent of p^* and thus the profit expression is increasing in p^*. In that case, if $\beta(\underline{p}_H) = 1$, then $p^* = \underline{p}_H$, while if $\beta(\underline{p}_H) < 1$, then $p^* < \underline{p}_H$. If, on the other hand, low cost firms do charge prices in the interval $(p^*, \underline{p}_H)$ with positive probability, then the profit at p^* can be written as

$$\pi_L(p^*)=\left[\frac{1+\lambda}{2}(1-F_L(p^*))+\frac{1-\lambda}{2}F_L(\underline{p}_H)+\frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](p^*-c_L),$$

which using (3) for $\beta(p^*) = 1$, can be written as

$$\frac{1+\lambda}{2}\left(\frac{1-\lambda}{2\lambda}\left[\beta(\overline{p})\frac{\overline{p}-c_L}{p^*-c_i}-1-\int_{p^*}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right]\right)(p^*-c_L)$$
$$+\left(\frac{1-\lambda}{2}F_L(\underline{p}_H)+\frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)(p^*-c_L),$$

or

$$\frac{1-\lambda^2}{4\lambda}\left[\beta(\overline{p})\left(\overline{p}-c_L\right)-\left(1+\int_{p^*}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)\left(p^*-c_L\right)\right]$$
$$+\left(\frac{1-\lambda}{2}F_L(\underline{p}_H)+\frac{1-\lambda}{2}\int_{\underline{p}_H}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)(p^*-c_L),$$

which is clearly increasing in p^*.

The second boundary condition can then be stated as follows.

(i) If $\rho_L \geq \underline{p}_H$, then $\beta'(p)=0$ when p is such that $\beta(p)=1$ (Lemma 2);

(ii) If $\beta(p)<1$ for all $p\in[\underline{p}_H,\overline{p}]$, then $p^*=\rho_L$;

(iii) If $\rho_L<\underline{p}_H$ and there is an interval $[x,y]$ of prices p such that $\beta(p)=1$ for all $p\in[x,y]$, then $\lim_{p\downarrow y}\beta'(p)=0$, and $p^*=\rho_L$.

To simplify notation, we rewrite the distribution functions as

$$F_i(p)=\frac{2g(\overline{p})-\int_p^{\overline{p}}\frac{a_i}{(x-c_i)^2g(x)}dx}{2g(p)} \qquad i=L,H, \tag{10}$$

where $g(p)=\sqrt{1-(1-\lambda)\beta(p)}$ and $a_i=(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)$, and proceed as follows. We first note that (5) and (3) only need to hold in an interval of prices where $\beta(p)<1$ and that this is a subset of $[\rho_L,\overline{p}]$. Lemma A.1 shows that this implies that $f_L(p)$ and $f_H(p)$ are either both positive or both negative over the relevant interval. We next show that $f_H(\overline{p})>0$. Together with Lemma A.1, this shows that if the indifference equation for consumers has a solution, then the price distribution functions are well-defined, increasing functions. We then rewrite the system into five proper differential equations and invoke the Pickard-Lindelof theorem of differential equations to show that the system has indeed a (mathematical) solution that it is locally unique. Finally, we show that we can satisfy the boundary conditions given above.

Lemma A.1. *For any* $p\in P_{(0,1)}$, $f_L(p)\cdot f_H(p)\geq 0$.

Proof. As $\Phi_L(\rho_L)=\int_0^{\rho_L}F_i(x)dx=\Phi_H(\rho_H)=\int_0^{\rho_H}F_i(x)dx=s$, and $\Phi_i(p)$ are increasing functions it follows that $\Phi_L(p)>s$ and $\Phi_H(p)<s$ for all $\rho_L<p<\rho_H$.

As (5) can be rewritten as

$$\frac{(1-\alpha)f_L(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}(\Phi_L(p)-s)+\frac{\alpha f_H(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}(\Phi_H(p)-s)=0$$

it follows that both the weights $\frac{(1-\alpha)f_L(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}$ and $\frac{\alpha f_H(p)}{(1-\alpha)f_L(p)+\alpha f_H(p)}$ have to be positive, which can only be the case if $f_L(p)$ and $f_H(p)$ have the same sign. $\square$

Lemma A.2. *For all* $p \in [\max(\rho_L, \underline{p}_H), \overline{p}] \cap P_{(0,1)}$, $f_i(p) > 0$, $i = L, H$.

Proof. As the function $\beta(p)$ is differentiable, equation (2) can be rewritten as

$$-2\left[1-(1-\lambda)\beta(p)\right]f_i(p)+(1-\lambda)\beta'(p)F_i(p)=-(1-\lambda)\beta(\overline{p})\frac{\overline{p}-c_i}{(p-c_i)^2},$$

which at $\overline{p}$ reduces to

$$-2\left[1-(1-\lambda)\beta(\overline{p})\right]f_i(\overline{p})=-(1-\lambda)\left[\frac{\beta(\overline{p})}{\overline{p}-c_i}+\beta'(\overline{p})\right].$$

As the RHS of this expression equals 0 for $c_i = c_L$, the RHS is clearly negative for $c_i = c_H$ for any choice of $0 < \beta(\overline{p}) < 1$. Thus, $f_H(\overline{p}) > 0$. By continuity there exists $\varepsilon > 0$ such that for all $p \in [\overline{p}-\varepsilon, \overline{p}]$ $f_H(p) > 0$. Then, by Lemma A.1 $f_L(\cdot)$ is also positive in the interior of this interval. Moreover, Lemma A.1 implies that if $f_L(\cdot)$ and $f_H(\cdot)$ change sign it must happen at the same price, which we denote as $p_0 \in [\max(\rho_L, \underline{p}_H), \overline{p}]$. By differentiating (3) and taking the ratio of the derivatives we obtain

$$\frac{(\overline{p}-c_H)(p_0-c_L)^2}{(p_0-c_H)^2(\underline{p}-c_L)}=\frac{F_H(p_0)}{F_L(p_0)}$$

Note, that the left-hand side of this expression is larger than 1 (since $p_0 < \overline{p}$), while by Corollary 1 the right-hand side is smaller than 1. Therefore, there is no such p_0 and both densities must be positive. $\square$

It thus directly follows from Lemma A.1 and A.2 that both density functions have to be positive for all For all $p \in [\max(\rho_L, \underline{p}_H), \overline{p}] \cap P_{(0,1)}$. As for all other prices $\beta(p) = 0$ or $\beta(p) = 1$, the density functions are positive for all prices.

Lemma A.3. *The solution to the uninformed indifference equation* (5) *can be written as*

$$g'(p) = \frac{(1-\alpha)\frac{a_L}{2(p-c_L)^2 g(p)}(\Phi_L(p)-s) + \alpha\frac{a_H}{2(p-c_H)^2 g(p)}(\Phi_H(p)-s)}{(1-\alpha)F_L(p)(\Phi_L(p)-s) + \alpha F_H(p)(\Phi_H(p)-s)} \tag{11}$$

Proof. Taking the derivative of (10) gives

$$f_i(p) = \frac{1}{g(p)}\left(\frac{a_i}{2(p-c_i)^2 g(p)} - F_i(p)g'(p)\right).$$

Then the optimal stopping rule can be rewritten as

$$\begin{aligned} 0 \;=\; & \frac{(1-\alpha)\left(\frac{a_L}{2(p-c_L)^2 g(p)} - F_L(p)g'(p)\right)}{(1-\alpha)\left(\frac{a_L}{2(p-c_L)^2 g(p)} - F_L(p)g'(p)\right) + \alpha\left(\frac{a_H}{2(p-c_H)^2 g(p)} - F_H(p)g'(p)\right)}(\Phi_L(p)-s) + \\ & \frac{\alpha\left(\frac{a_H}{2(p-c_H)^2 g(p)} - F_H(p)g'(p)\right)}{(1-\alpha)\left(\frac{a_L}{2(p-c_L)^2 g(p)} - F_L(p)g'(p)\right) + \alpha\left(\frac{a_H}{2(p-c_H)^2 g(p)} - F_H(p)g'(p)\right)}(\Phi_H(p)-s), \end{aligned}$$

which can easily be rewritten as the equation in the statement of the Lemma. □

We proceed with some facts about the function $g'(p)$. Define

$$A(p) \equiv (1-\alpha)\frac{a_L}{2(p-c_L)^2 g(p)}(\Phi_L(p)-s) + \alpha\frac{a_H}{2(p-c_H)^2 g(p)}(\Phi_H(p)-s) \tag{12}$$

and

$$B(p) \equiv (1-\alpha)F_L(p)(\Phi_L(p)-s) + \alpha F_H(p)(\Phi_H(p)-s). \tag{13}$$

We can then write $g'(p) = \frac{A(p)}{B(p)}$.

Lemma A.4. *Equation $A(p) = 0$ has at most one root on $[\underline{p}_H, \overline{p}]$.*

Proof. From the definition of $A(p)$ it follows that $A(p) = 0$ if, and only if,

$$\frac{\alpha(s-\Phi_H(p))}{(1-\alpha)(\Phi_L(p)-s)} = \frac{\frac{a_L}{(p-c_L)^2}}{\frac{a_H}{(p-c_H)^2}} = \frac{(p-c_H)^2(\overline{p}-c_L)}{(p-c_L)^2(\overline{p}-c_H)}.$$

It is clear that the LHS of this expression is decreasing in p (since both numerator and denominator are positive for $p \in [\underline{p}_H, \overline{p}]$), while the RHS is increasing in p. Thus, there is at most one p where $A(p) = 0$. □

Lemma A.5. *For all $p \in [\underline{p}_H, \overline{p}]$ such that $A(p) > 0$, it must be the case that $B(p) > 0$.*

Proof. $A(p) > 0$ if, and only if,

$$\frac{(1-\alpha)(\Phi_L(p) - s)}{\alpha(s - \Phi_H(p))} > \frac{\frac{a_H}{(p-c_H)^2}}{\frac{a_L}{(p-c_L)^2}} = \frac{(p-c_L)^2(\overline{p} - c_H)}{(p-c_H)^2(\overline{p} - c_L)}. \tag{14}$$

Similarly, $B(p) > 0$ if, and only if,

$$\frac{(1-\alpha)(\Phi_L(p) - s)}{\alpha(s - \Phi_H(p))} > \frac{F_H(p)}{F_L(p)}. \tag{15}$$

The LHS of (14) and (15) are identical; as the RHS of (14) is larger than 1, while by Corollary 1 the RHS of (15) is smaller than 1, the statement follows. □

Now note, that if $\rho_L \geq \underline{p}_H$ then due to Lemma 2 for all $p = [\rho_L, \overline{p}]$ it must be the case that $B(p) > 0$ (this immediately follows from Lemmas A.4 and A.5). The next lemma establishes the same result for the case $\rho_L < \underline{p}_H$.

Lemma A.6. *If $\rho_L < \underline{p}_H$,then $B(p) > 0$ for all $\underline{p}_H \leq p \leq \overline{p}$.*

Proof. It is clear that $B(\overline{p}), B(\underline{p}_H) > 0$ and that $B(p)$ is continuously differentiable on $\underline{p}_H < p < \overline{p}$. We will show that $B(p)$ cannot be equal to 0. Suppose it is and that there is a x such that $B(x) = 0$. We show that this implies that $B'(x) > 0$, which is inconsistent with the fact that $B(\overline{p}), B(\underline{p}_H) > 0$.

It is clear that $B'(p) = \alpha f_L(p)(\Phi_L(p) - s) + (1-\alpha) f_H(p)(\Phi_H(p) - s) + \alpha F_L^2(p) + (1-\alpha)F_H^2(p)$. Using the fact that $\alpha(\Phi_L(x) - s) = -(1-\alpha)F_H(x)(\Phi_H(x) - s)/F_L(x)$, we can write

$$B'(x) = (1-\alpha)\left[f_H(x) - f_L(x)\frac{F_H(x)}{F_L(x)}\right](\Phi_H(x) - s) + \alpha F_L^2(x) + (1-\alpha)F_H^2(x).$$

As $f_i(p) = \frac{1}{g(p)}\left(\frac{a_i}{2(p-c_i)^2 g(p)} - F_i(p)g'(p)\right)$ it follows that

$$B'(x) = \frac{(1-\alpha)}{2g^2(p)}\left[\frac{a_H}{(p-c_H)^2} - \frac{a_L}{(p-c_L)^2}\frac{F_H(x)}{F_L(x)}\right](\Phi_H(x) - s) + \alpha F_L^2(x) + (1-\alpha)F_H^2(x).$$

As $\frac{F_H(p)}{F_L(p)} < 1$ and $\frac{a_H}{(p-c_H)^2} > \frac{a_L}{(p-c_L)^2}$ it follows that all terms are positive and $B'(x) > 0$. □

For the existence proof we also need that consumers prefer to buy as long as $\beta(p) = 1$. The proof is a simple adaptation of a proof given by Dana (1994) that in a reservation price equilibrium uninformed consumers strictly prefer to buy at all prices in the support of the price distribution of the high cost firm that are strictly smaller than the reservation price.

Lemma A.7. *If $\beta(p) = 1$ on a certain interval $[x, y]$ and uninformed consumers weakly prefer buying to continuing searching at $p = y$, then these consumers strictly prefer buying to continuing searching at $p \in [x, y)$.*

Proof. If $\beta(p) = 1$, then

$$\frac{f_H(p)}{f_L(p)} = \frac{(p - c_H)^2(\overline{p} - c_L)}{(p - c_L)^2(\overline{p} - c_H)}.$$

This expression is decreasing in p. Thus, after observing a larger price, updating beliefs results in uninformed consumers believing it is more likely that cost is high. The expected pay-off of continuing to search is thus larger at larger prices. At the same time, the pay-off of buying at a higher price decreases. Thus, if a consumer is indifferent between the two options at $p = x$, then he must strictly prefer buying at $p < x$. $\square$

We also need that along the equilibrium path we construct consumers prefer to continue searching when $\beta(p) = 0$. This is, however, trivial, as $\beta(p) = 0$ only occurs along the equilibrium path when $\rho_L < p < \underline{p}_H$, but in that case consumers infer that it is only low cost firms that charge such prices, and non-shoppers prefer to search on as these prices are above ρ_L.

In the proof of the Theorem, we use the fact that the system of differential equations (5) and (3) has a unique solution. To this end, we prove in the next Lemma that this is the case by applying the Pickard-Lindelof theorem.

Lemma A.8. *The system of differential equations given by (3) and (5) with boundary values $\Phi_i(p_0), F_i(p_0), \beta(p_0)$, $i = L, H$ such that*

$$F_L(p_0)(\Phi_L(p_0) - s) + (1 - \alpha)F_H(p_0)(\Phi_H(p_0) - s) > 0$$

$$\beta(p_0) < 1$$

has a unique solution in a neighbourhood of p_0.

Proof. To apply the Pickard-Lindelof theorem, we need to rewrite our system in the form where the derivatives of certain functions are expressed as functions of these functions themselves. We do this in the following way: we define a function $\Phi_i(p) = \int_{\underline{p}}^{\overline{p}} F_i(p)dp$, so that $\Phi_i'(p) = F_i(p)$, and a function $z_i(p) = \int_0^p \frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(x-c_i)^2 g(x)}dx$, so that $z_i'(p) = -\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2 g(p)}$. Using these transformations, we can rewrite our system as :

$$z_i'(p) = -\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2 g(p)}, i = L, H;$$

$$\Phi_i'(p) = \frac{2g(\overline{p}) - z_i(p)}{2g(p)}, i = L, H;$$

and

$$\begin{aligned} g'(p) &= \frac{\alpha\frac{a_L}{2(p-c_L)^2 g(p)}(\Phi_L(p)-s) + (1-\alpha)\frac{a_H}{2(p-c_H)^2 g(p)}(\Phi_H(p)-s)}{\alpha\left(1-\frac{z_L(p)}{2g(p)}\right)(\Phi_L(p)-s) + (1-\alpha)\left(1-\frac{z_H(p)}{2g(p)}\right)(\Phi_H(p)-s)} = \\ &\quad -\frac{\alpha\frac{a_L}{(p-c_L)^2}(\Phi_L(p)-s) + (1-\alpha)\frac{a_H}{(p-c_H)^2}(\Phi_H(p)-s)}{\alpha\left(z_L(p)-2g(p)\right)(\Phi_L(p)-s) + (1-\alpha)\left(z_H(p)-2g(p)\right)(\Phi_H(p)-s)}, \end{aligned}$$

whenever $g(p) > \sqrt{\lambda}$ ($\beta(p) < 1$) and $g'(p) = 0$ if $g(p) = \sqrt{\lambda}$.

To apply the Pickard-Lindelof theorem, we need that the RHS of this system of differential equations is Lipschitz-continuous with respect to $(g, z_i, \Phi_i), i = L, H$.

Denoting $b_i = -\frac{(1-\lambda)\beta(\overline{p})(\overline{p}-c_i)}{(p-c_i)^2}, i = L, H$, the derivatives of the vector-function representing the RHS of the system of five differential equations for $z_L', z_H', \Phi_L', \Phi_H', g'$ with respect to g, z_i, Φ_i is summarized in the matrix

$$\nabla = \begin{pmatrix} \frac{b_L}{g^2} & 0 & 0 & 0 & 0 \\ \frac{b_H}{g^2} & 0 & 0 & 0 & 0 \\ -\frac{2g(\overline{p})-z_L}{2g^2} & -\frac{1}{2g} & 0 & 0 & 0 \\ -\frac{2g(\overline{p})-z_H}{2g^2} & 0 & -\frac{1}{2g} & 0 & 0 \\ D_1 & D_2 & D_3 & D_4 & D_5 \\ & & & & \end{pmatrix},$$

where

$$D_1 = \frac{2(\alpha b_L(\Phi_L - s) + (1-\alpha)b_H(\Phi_H - s))(\alpha(\Phi_L - s) + (1-\alpha)(\Phi_H - s))}{[\alpha z_L(\Phi_L - s) + (1-\alpha)z_H(\Phi_H - s) - 2g(\alpha(\Phi_L - s) + (1-\alpha)(\Phi_H - s))]^2}$$

$$D_2 = -\frac{\alpha(\Phi_L - s)(\alpha b_L(\Phi_L - s) + (1-\alpha)b_H(\Phi_H - s))}{[\alpha z_L(\Phi_L - s) + (1-\alpha)z_H(\Phi_H - s) - 2g(\alpha(\Phi_L - s) + (1-\alpha)(\Phi_H - s))]^2}$$

$$D_3 = -\frac{(1-\alpha)(\Phi_H - s)(\alpha b_L(\Phi_L - s) + (1-\alpha)b_H(\Phi_H - s))}{[\alpha z_L(\Phi_L - s) + (1-\alpha)z_H(\Phi_H - s) - 2g(\alpha(\Phi_L - s) + (1-\alpha)(\Phi_H - s))]^2}$$

$$D_4 = \frac{(2b_L g - 2b_H g + b_H z_L - b_L z_H)\alpha(1-\alpha)(\Phi_1 - s)}{[\alpha z_L(\Phi_L - s) + (1-\alpha)z_H(\Phi_H - s) - 2g(\alpha(\Phi_L - s) + (1-\alpha)(\Phi_H - s))]^2}$$

$$D_5 = -\frac{(2b_L g - 2b_H g + b_H z_L - b_L z_H)\alpha(1-\alpha)(\Phi_1 - s)}{[\alpha z_L(\Phi_L - s) + (1-\alpha)z_H(\Phi_H - s) - 2g(\alpha(\Phi_L - s) + (1-\alpha)(\Phi_H - s))]^2}$$

Due to our condition $F_L(p_0)(\Phi_L(p_0) - s) + (1-\alpha)F_H(p_0)(\Phi_H(p_0) - s) > 0$ all D_i's are bounded and our vector-function is continuously differentiable. It is known that if a function is continuously differentiable on $[\underline{p}_H, \overline{p}]$, then it is Lipschitz-continuous on this interval.[16] The statement of the Lemma then is an application of the Pickard-Lindelof theorem. $\square$

We are now in the position to prove the existence theorem.

Theorem 1. *An equilibrium satisfying the logic of D1 exists.*

Proof. Fix some $\overline{p} > \max(\rho_L^{NU}, c_H + s)$, where ρ_L^{NU} is the standard Stahl reservation price in case there is no ex ante cost uncertainty and cost is known to be low. We show that for any $\overline{p}$ all equilibrium conditions except $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp = s$ can be satisfied. Then we show, that this last condition always can be satisfied by the appropriate choice of $\overline{p}$.

As $\overline{p} > \rho_L^{NU}$ consumers can be indifferent after observing $\overline{p}$ and therefore there is a $\overline{\beta} \equiv \beta(\overline{p}) < 1$ with $\beta'(\overline{p}) = -\overline{\beta}(\overline{p} - c_L)$. For any such $\overline{\beta} < 1$ let $\beta(p), F_L(p)$ and $F_H(p)$ be defined in the usual way with $F_L(\overline{p}) = F_H(\overline{p}) = 1$. That is, for any

[16] Proof of this statement can be found on this web-page: http://unapologetic.wordpress.com/2011/05/04/continuously-differentiable-functions-are-locally-lipschitz/

p in the support of the mixed strategy distribution, firms make the same profits and consumers are indifferent between buying and continuing to search whenever $0 < \beta(p) < 1$. Each $\overline{\beta}$ defines its own function $\beta(p)$. We use the notation $\beta(p; \overline{\beta})$ when we want to make explicit that the function $\beta(p)$ depends on the choice of $\overline{\beta}$. From Lemmas A.4, A.5 and A.6 it follows that the conditions of Lemma A.8 are satisfied for any p at which a consumer is indifferent. Thus, for any $\overline{\beta}_1 \neq \overline{\beta}_2$ it must be the case that $\beta(p, \overline{\beta}_1) \neq \beta(p, \overline{\beta}_2)$. For some choices of $\overline{\beta}$ it may well be the case that there is $p' \in (\underline{p}_H, \overline{p})$ such that $\beta(p', \overline{\beta}) = 1$. Denote $\tilde{p} = \sup p'$.

Claim 1. We claim that there is such a $\overline{\beta}_0$ that F_L and F_H are solutions to the equal profit condition for firms and either (i) $\tilde{p}$ exists and $\beta'(\tilde{p}, \overline{\beta}_0) = 0$ or (ii) $\beta'(\underline{p}_H, \overline{\beta}_0) < 0$ with $\beta(\underline{p}_H, \overline{\beta}_0) = 1$, where $\beta(p, \overline{\beta}_0)$ is the solution to the indifference condition of consumers.

We prove this claim in two steps.

Claim 2. There is a $\overline{\beta}_1$ (small enough) such that the equal profit conditions are satisfied and the solution to the indifference condition of consumers is such that $\beta(p, \overline{\beta}_1) < 1$ for all $p \in (\underline{p}_H, \overline{p}]$.

Suppose, that this is not the case, and for any choice of $\overline{\beta}_1$ there is such $p_0 \in (\underline{p}_H, \overline{p})$ that $\beta(p_0, \overline{\beta}_1) = 1$. Then, $\pi_L(p_0) \geq \frac{1-\lambda}{2}(p_0 - c_L) > \frac{1-\lambda}{2}(c_H - c_L)$. On the other hand $\pi_L(\overline{p}) = \frac{1-\lambda}{2}\overline{\beta}_1(\overline{p} - c_L)$. Therefore, there is such $\overline{\beta}_1$ that $\pi_L(p_0) > \pi_L(\overline{p})$ which cannot be the case in equilibrium. Thus the claim is correct.

Claim 3. There is $\overline{\beta}_2$ (large enough) such that equal profit conditions are satisfied and the solution of the indifference condition of consumers is such that $\tilde{p}$ exists.

Indeed, since $\beta(\overline{p}, 1) = 1$, $\beta'(\overline{p}, 1) < 1$ (see Proposition 3) and β is continuously differentiable in both arguments, $\overline{\beta}_2$ always exists.

Now, we show that claim 1 follows from claims 2 and 3. Let $\overline{\beta}_0$ be the smallest $\overline{\beta}$ such that a $p_0 \in [\underline{p}_H, \overline{p}]$ exists with $\beta(p_0, \overline{\beta}_0)$=1. If $p_0 = \underline{p}_H$ then (ii) holds. If not, then $\tilde{p} = \sup p_0$ and since $\beta(\cdot, \cdot)$ is continuously differentiable in both arguments we obtain $\beta'(\tilde{p}, \overline{\beta}_0) = 0$.

Now we analyze these two cases sequentially. First, suppose (i) holds. Note, that the condition imposed by Lemma 2 is automatically satisfied for this $\overline{\beta}_0$.

Suppose it is the case that if we define $\beta(p, \overline{\beta}_0) = 1 \ \forall p < \tilde{p}$ the reservation price for the low cost difference implicitly defined by equation (6) yields as a solution that $\rho_L \geq \underline{p}_H$. In this case all the equilibrium conditions are satisfied.

Now, suppose that if $\beta(p, \overline{\beta}_0) = 1 \ \forall p < \tilde{p}$ we get that $\rho_L < \underline{p}_H$. This $\overline{\beta}_0$ does not satisfy the equilibrium requirements, since it needs to be the case that $\beta(p) = 0$ for all $p \in (\rho_L, \underline{p}_H)$. In this case we construct a solution with a gap in the low-cost distribution $F_L(p)$. In order to do so we choose some $\hat{p} \in [\underline{p}_H, \tilde{p}]$, and construct a solution with $\beta(p) < 1, \forall p \in [\underline{p}_H, \hat{p})$ and $\beta(p) = 1, \forall p \in [\hat{p}, \tilde{p}]$ using $F_L(\hat{p}), F_H(\hat{p})$ obtained as solutions on $p \in [\hat{p}, \overline{p}]$ as boundary values (which we can always do due to Lemma A.8). Define $p^* \leq \underline{p}_H$ as in the beginning of the Appendix.

Note, that $\lim_{\hat{p} \downarrow \underline{p}_H} p^* = \underline{p}_H$ implies there exists a $\hat{p}$ such that $p^* > \rho_L$. From Lemmas A.4 and A.5 it follows that for any $p \in [\underline{p}_H, \hat{p})$ $\beta'(p) > 0$. Together with Lemma A.8 this implies that $\beta(\underline{p}_H)$ is decreasing in $\hat{p}$. Thus, either (a) we can find a $\hat{p} \in [\underline{p}_H, \tilde{p}]$ such that $p^* = \rho_L$ and all the equilibrium conditions are satisfied, or (b) $p^* > \rho_L$ for all $\hat{p} \in [\underline{p}_H, \tilde{p}]$, or (c) $\beta(p) = 0$ for some $p > \underline{p}_H$.

First deal with case (b) and consider the set $\overline{\beta} < \overline{\beta}_0$. Note that for any such $\overline{\beta}$ we have $\beta(p, \overline{\beta}) < 1, \forall p \in [\underline{p}_H, \overline{p}]$ and that $\pi_L(\overline{p}) = \frac{1-\lambda}{2}\overline{\beta}(\overline{p} - c_L)$. Thus, $\lim_{\overline{\beta} \to 0} \pi_L(\overline{p}) = 0 \Rightarrow \lim_{\overline{\beta} \to 0} p^* = c_L$. However, $\rho_L \geq c_L + s$. Since $\beta(\cdot, \cdot)$ is continuous in both arguments, and for $\overline{\beta}_0$ $p^* > \rho_L$ there must be $\overline{\beta}_1$ such that $p^* = \rho_L$. This is the equilibrium if $\beta(p) > 0$ for all $p > \underline{p}_H$.

Now, suppose that we have case (c), possibly in combination with case (b), or even (a) and $\overline{\beta}$ is such that $\beta(p, \overline{\beta}) = 0$ for some $p \in [\underline{p}_H, \overline{p}]$. Denote $\overline{\beta}_0$ as the upper bound such that $\beta(\underline{p}_H, \overline{\beta}_0) = 0$. We show that it is possible to construct an equilibrium such that $\beta(p, \overline{\beta}_0) = 0$ for all $p \in (\rho_L, \underline{p}_H]$ and $p^* = \rho_L$. We construct an equilibrium where low-cost firms still choose prices in a left region of p_H, $[\underline{p}', \underline{p}_H]$. In this region consumers search with probability one, and the profit function is defined by

$$\pi_L(p) = \left(\frac{1+\lambda}{2}(1 - F_L(p) + \frac{1-\lambda}{2} \int_p^{\overline{p}} (1 - \beta(\tilde{p})) f(\tilde{p}) d\tilde{p} \right) (p - c_L)$$

Note, that for any choice of $\underline{p}'$ it must be the case that $p^* < \underline{p}'$. However, by choosing $\underline{p}'$ sufficiently low, $F_L(\underline{p}')$ can be chosen arbitrary close of zero, which implies that ρ_L is arbitrary large since $\int_{\underline{p}}^{\rho_L} F_L(p)dp = s$. Therefore, there is such a $\underline{p}'$ that $p^* = \rho_L$, which completes the proof of case (i) for fixed $\overline{p}$.

Consider then briefly case (ii) where $\beta'(\underline{p}_H, \overline{\beta}_0) < 0$ with $\beta(\underline{p}_H, \overline{\beta}_0) = 1$. If the reservation price for the low cost distribution implicitly defined by equation (7) yields as a solution that $\rho_L \geq \underline{p}_H$ all the equilibrium conditions are satisfied. If not, then we can proceed as in the two cases (b) and (c) analyed above, and show along the same lines that an equilibrium exists.

We have now proved that for any fixed $\overline{p} > \max(c_H + s, \rho_L^{NU})$ we can satisfy all equilibrium conditions apart from the fact that in equilibrium we should have $\int_{\underline{p}_H}^{\overline{p}} F_H(p) = s$. We now prove that we can always choose $\overline{p}$ such that this indifference condition is also satisfied. To do so, we first realize that if $c_H > c_L$ we have

$$\lim_{\overline{p} \downarrow \rho_L^{NU}} \int_{\underline{p}_H}^{\overline{p}} F_H(p)dp < s.$$

We next show that for $\overline{p}$ large enough, the other equilibrium conditions can only be satisfied if

$$\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp > s.$$

As $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp$ is continuous in $\overline{p}$ it follows then that there must be a $\overline{p}$ such that $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp = s$.Thus, the only thing to be proved is that for $\overline{p}$ large enough, $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp > s$. To this end, it follows from

$$\pi(\overline{p}|c_H) = \frac{1-\lambda}{2}(1 - \beta(\overline{p}))(\overline{p} - c_H) < \frac{1-\lambda}{2}(\overline{p} - c_H)$$

and

$$\pi(\underline{p}_H|c_H) > \frac{1+\lambda}{2}(\underline{p}_H - c_H)$$

and the fact that a firm has to be indifferent between charging the upper and lower bound of the price distribution that

$$\overline{p} - \underline{p}_H > \frac{2\lambda}{1+\lambda}(\overline{p} - c_H). \tag{16}$$

Thus, the support of the mixed strategy distribution grows without bound when $\overline{p}$ becomes larger. Suppose then that $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp < s$ even for large $\overline{p}$. This would imply that for all $\epsilon > 0$ there exist a large $\overline{p}$ such that $F_H(\frac{\overline{p}+\underline{p}_H}{2}) < \epsilon$. Let us then consider the profit a firm makes when setting prices $\underline{p}_H$ and $[\overline{p}+\underline{p}_H]/2$:

$$\pi(\underline{p}_H|c_H) = \left[\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\left(\int_{\underline{p}_H}^{\frac{\overline{p}+\underline{p}_H}{2}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p} + \int_{\frac{\overline{p}+\underline{p}_H}{2}}^{\overline{p}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right)\right](\underline{p}_H - c_H),$$

and

$$\pi\left(\frac{\overline{p}+\underline{p}_H}{2}|c_H\right) = \left[\frac{1+\lambda}{2} - \left[\lambda + \frac{1-\lambda}{2}(1-\beta(\frac{\overline{p}+\underline{p}_H}{2}))\right] F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right) + \frac{1-\lambda}{2}\int_{\frac{\overline{p}+\underline{p}_H}{2}}^{\overline{p}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right]\left(\frac{\overline{p}+\underline{p}_H}{2} - c_H\right).$$

As by choosing $\overline{p}$ we can make $F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right)$ arbitrarily small and as $1-\beta(\widetilde{p}) < 1$, it is clear that □

$$\pi(\underline{p}_H|c_H) < \left[\frac{1+\lambda}{2} + \frac{1-\lambda}{2}F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right) + \frac{1-\lambda}{2}\int_{\frac{\overline{p}+\underline{p}_H}{2}}^{\overline{p}} (1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p}\right](\underline{p}_H - c_H),$$

so that

$$\pi\left(\frac{\overline{p}+\underline{p}_H}{2}|c_H\right) - \pi(\underline{p}_H|c_H) > \left[\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\int_{\overline{p}/2}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p} - \frac{1-\lambda}{2}F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right)\right]\frac{\overline{p}-\underline{p}_H}{2} - \lambda F_H\left(\frac{\overline{p}+\underline{p}_H}{2}\right)\left(\frac{\overline{p}+\underline{p}_H}{2} - c_H\right).$$

using (16) it follows that

$$\pi\left(\frac{\overline{p}+\underline{p}_H}{2}|c_H\right) - \pi(\underline{p}_H|c_H) > \left\{\left[\frac{1+\lambda}{2} + \frac{1-\lambda}{2}\int_{\overline{p}/2}^{\overline{p}}(1-\beta(\widetilde{p}))f_i(\widetilde{p})d\widetilde{p} - \frac{1-\lambda}{2}F_H(\frac{\overline{p}+\underline{p}_H}{2})\right]\frac{\lambda}{1+\lambda} - \lambda F_H(\frac{\overline{p}+\underline{p}_H}{2})\right\}(\overline{p} - c_H),$$

which is clearly positive for large $\overline{p}$. This implies that for large $\overline{p}$ a high cost firm cannot be indifferent over the whole support of the price distribution if $\int_{\underline{p}_H}^{\overline{p}} F_H(p)dp < s$.

Finally, Lemmas A.1 and A.2 guarantee that the distribution functions are well-defined. This completes the proof.

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