Geometric mitosis and Newton–Okounkov polytopes

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In [K], a convex-geometric algorithm was introduced for building new analogs of Gelfand–Zetlin polytopes for arbitrary reductive groups. Conjecturally, these polytopes coincide with the Newton–Okounkov polytopes of flag varieties for a geometric valuation. I outline an algorithm (geometric mitosis) for finding collection of faces in these polytopes that represent a given Schubert cycle. For $GL_n$ and Gelfand–Zetlin polytopes, this algorithm reduces to a geometric version of Knutson–Miller mitosis introduced in [KST].

First, recall the mitosis on parallelepipeds from [KST, Section 6]. Let $\Pi(\mu, \nu) \subset \mathbb{R}^n$ be a parallelepiped given by inequalities $\mu_i \leq x_i \leq \nu_i$ for $i = 1, \ldots, n$. For every face $\Gamma \subset \Pi(\mu, \nu)$, we now define a collection of faces $M(\Gamma)$ called the mitosis of $\Gamma$. Let $k$ be the minimal number such that $\Gamma \subset \{x_i = \mu_i\}$ for all $i > k$ (in particular, $\Gamma \nsubseteq \{x_k = \mu_k\}$) and $\nu_i \neq \mu_i$ for at least one $i > k$. If no such $k$ exists then $M(\Gamma) = \emptyset$. Under the isomorphism $\mathbb{R}^n \cong \mathbb{R}^k \times \mathbb{R}^{n-k}$, $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_k) \times (x_{k+1}, \ldots, x_n)$ the face $\Gamma$ gets mapped to $\Gamma' \times v$ where $v = (\mu_{k+1}, \ldots, \mu_n)$ is a point and $\Gamma' \subset \mathbb{R}^k$ is a parallelepiped in $\mathbb{R}^k$. Let $E_i \subset \mathbb{R}^{n-k}$ for $i = k+1, \ldots, n$ be the segment with vertices $(\mu_{k+1}, \ldots, \mu_{i-1}, \mu_i, \nu_{i+1}, \ldots, \nu_n)$ and $(\mu_{k+1}, \ldots, \mu_{i-1}, \nu_i, \nu_{i+1}, \ldots, \nu_n)$ (that is, the union $\bigcup_{i=k+1}^n E_i$ is a broken line that connects points $(\mu_{k+1}, \ldots, \mu_n)$ and $(\nu_{k+1}, \ldots, \nu_n)$). Then $M(\Gamma)$ consists of all faces $\Gamma' \times E_i$ for $k+1 \leq i \leq n$ such that $E_i$ is not a single point (in particular, $\dim \Delta = \dim \Gamma + 1$ for any $\Delta \in M(\Gamma)$). Definition of $M(\Gamma)$ is motivated by the identity [KST, Proposition 6.8] for a Demazure-type operator applied to an exponential sum over $\Gamma$.

This geometric version of mitosis reduces easily to the combinatorial mitosis of [KnM] as follows. Every face of $\Pi(\mu, \nu)$ can be represented by a $2 \times n$ table $(a_{ij})_{i=1,2, j=1,\ldots,n}$ whose cells are either filled with $+$ or empty. Namely, the face satisfies the equality $x_i = \mu_i$ or $x_i = \nu_i$ if and only if $a_{1i} = +$ or $a_{2i} = +$, respectively (in particular, if $\mu_i = \nu_i$ then the $i$-th column has two $+$.). On the level of tables, operation $M$ coincides the mitosis of [KnM] after reflecting our tables in a vertical line.

**Example 1:** If $\Pi(\mu, \nu) \subset \mathbb{R}^4$, where $\mu = (1,1,1,1)$ and $\nu = (2,2,1,2)$ (that is, $\mu_3 = \nu_3$), then the vertex $\Gamma = \{x_1 = \nu_1, \ x_2 = \mu_2, \ x_4 = \mu_4\}$ is represented by the table

\[
\begin{array}{cccc}
+ & + & + \\
+ & + & + \\
\end{array}
\]

The set $M(\Gamma)$ consists of two edges represented by the tables

\[
\begin{array}{cccc}
+ & + & + \\
+ & + & + \\
\end{array} \quad \& \quad
\begin{array}{cccc}
+ & + & + \\
+ & + & + \\
\end{array}
\]

We now briefly recall a construction from [K, Section 3.3]. Let $G$ be a connected reductive group of semisimple rank $r$. Let $\alpha_1, \ldots, \alpha_r$ denote simple roots of $G,$
and $s_1, \ldots, s_r$ the corresponding simple reflections. Fix a reduced decomposition $w_0 = s_{i_1} s_{i_2} \cdots s_{i_d}$ where $w_0$ is the longest element of the Weyl group of $G$. Let $d_i$ be the number of $s_{i_j}$ in this decomposition such that $i_j = i$. Consider the space

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \cdots \oplus \mathbb{R}^{d_r}$$

and choose coordinates $x = (x_1, \ldots, x_1^{d_1}; \ldots; x_r^{d_r})$ with respect to this decomposition. Put $\sigma_i(x) = \sum_{j=1}^{d_i} x_j$. Define the projection $p$ of $\mathbb{R}^d$ to the real span $\mathbb{R}^r$ of the weight lattice of $G$ by the formula $p(x) = \sigma_1(x) \alpha_1 + \cdots + \sigma_r(x) \alpha_r$. Let $\lambda$ be a dominant weight of $G$. There is an elementary convex-geometric algorithm for constructing a polytope $P_{\lambda}(i_1, \ldots, i_d) \subset \mathbb{R}^d$ that yields the Weyl character $\chi(V_{\lambda})$ of the irreducible $G$-module $V_{\lambda}$, that is,

$$\chi(V_{\lambda}) = \sum_{x \in P_{\lambda} \cap \mathbb{Z}^d} e^{p(x)}$$

(see Theorem [K, Theorem 3.6] for more details). The polytope $P_{\lambda}$ can be used to extend the results of [KST] from $GL_n$ to $G$ since its polytope ring is isomorphic to the cohomology ring of the complete flag variety $G/B$ (with rational coefficients).

In particular, if $G = SL_n$ and $w_0 = (s_1)(s_2s_1)(s_3s_2s_1) \cdots (s_{n-1} \cdots s_1)$, then we get the classical Gelfand–Zetlin polytope [K, Theorem 3.4]. However, if $G = Sp_4$ the resulting polytopes seem to be different from string polytopes of Berenstein–Littelmann–Zelevinsky.

**Example 2:** Take $G = Sp(4)$ (that is, $d = 4$ and $r = 2$) and $w_0 = s_2 s_1 s_2 s_1$ (here $a_1$ denotes the shorter root and $a_2$ denotes the longer one). Let $\lambda = \lambda_1 \omega_1 + \lambda_2 \omega_2$ be a strictly dominant weight of $Sp_4$. Choose a point $a_\lambda = (a, b, c, d)$ such that $p(a_\lambda) = w_0 \lambda = -\lambda$. Label coordinates in $\mathbb{R}^4$ by $x := x_1^1 - a$, $y := x_2 - b$, $z := x_1^2 - c$ and $t := x_2^2 - d$. The polytope $P_{\lambda}(2, 1, 2, 1)$ is given by inequalities

$$0 \leq x \leq \lambda_1, \quad z \leq x + \lambda_2, \quad y \leq 2z, \quad y \leq z + \lambda_2, \quad 0 \leq t \leq \lambda_2, \quad t \leq \frac{y}{2}$$

(see [K, Example 3.4]). It has 11 vertices, hence, it is not combinatorially equivalent to string polytopes for $Sp_4$ defined in [L].

**Remark:** Let $X = Sp_4/B$ be the complete flag variety for $Sp_4$, and $L_\lambda$ the line bundle on $X$ corresponding to the weight $\lambda$. Recently, I checked that after a unimodular change of coordinates $P_{\lambda}(2, 1, 2, 1)$ coincides with the Newton–Okounkov polytope $\Delta_{\nu}(X, L_\lambda)$ for the lowest term valuation $\nu$ corresponding to the flag of translated Schubert varieties: $w_0 X_{id} \subset s_1 s_2 s_1 X_{ss} \subset s_1 s_2 X_{ss} \subset s_1 X_{ss} \subset X$ (cf. [Ka, Remark 2.3]).

By construction, the intersection of the polytope $P_{\lambda} := P_{\lambda}(i_1, \ldots, i_d)$ with $(c + R^{d_2})$ is either a parallelepiped $P(\mu(c), \nu(c))$ or is empty for any $c \in \mathbb{R}^d$. This property of $P_{\lambda}$ gives $r$ mitosis operations $M_1, \ldots, M_r$ corresponding to parallelepipeds $P_{\lambda} \cap (c + R^{d_1}), \ldots, P_{\lambda} \cap (c + R^{d_r})$, respectively. Mitosis on parallelepipeds allows us to produce collections of faces of $P_{\lambda}$ that represent a given Schubert cycle in $G/B$ (in the sense of [KST, Theorem 5.1]), that is, the exponential sum over
the union of these faces yields the Demazure characters. The algorithm is as follows. For an element \( w \in W \) of the Weyl group, denote by \([X_w] = [BwB/B]\) the Schubert cycle corresponding to \( w \). Let \( s_{j_1} \ldots s_{j_d} \) be a reduced decomposition of \( w_0 w_0^{-1} \) such that \((j_1, \ldots, j_d)\) is a subword of \((i_1, \ldots, i_d)\). Then \([X_w]\) is represented by the union of faces produced from a vertex of \( P_\lambda \) by applying successively the operations \( M_{j_1}, \ldots, M_{j_d} \). For \( G = SL_n \) and \( w_0 = (s_1)(s_2s_1)(s_3s_2s_1) \ldots (s_{n-1} \ldots s_1) \), this algorithm can be described combinatorially using mitosis of Knutson–Miller on pipe-dreams (see [KST]).

For other reductive groups, one can also describe the mitosis algorithm combinatorially using suitable analogs of pipe-dreams.

**Example 3:** We continue Example 2. The vertex \( a_\lambda \) is the intersection of 4 facets: \( 0 = x, y = 2z, 0 = t, t = \frac{y}{2} \). Let us encode faces that contain \( a_\lambda \) by tables using the following rules:

\[
\begin{array}{c}
+ \iff 0 = x \\
+ \iff t = \frac{y}{2} \\
+ \iff y = 2z
\end{array}
\]

Here are three examples:

\[
a_\lambda = \begin{array}{c}
+ \\
+ \\
+
\end{array} ; \{0 = y = t\} = \begin{array}{c}
+ \\
+
\end{array} ; \{y = 2z\} = \begin{array}{c}
+
\end{array} .
\]

Every face \( \Gamma \) defines two (possibly degenerate) rectangles \( \Pi_1(\Gamma) = \Gamma \cap \{ z = z_0, t = t_0 \} \) and \( \Pi_2(\Gamma) = \Gamma \cap \{ x = x_0, y = y_0 \} \) (we choose \( x_0, y_0, z_0 \) and \( t_0 \) so that the dimensions of \( \Pi_1(\Gamma) \) and \( \Pi_2(\Gamma) \) are maximal possible). For instance, the face \( \Gamma = \{ 0 = y = t \} \) defines two segments. Note that \( \Pi_i(\Gamma) \) is a face of the rectangle \( \Pi_i(\Gamma) \), and hence, there is a well-defined operation \( M_i \) of mitosis on parallellograms for \( i = 1, 2 \). It is not hard to check that in terms of tables, \( M_1 \) and \( M_2 \) do the following:

\[
a_\lambda \xrightarrow{M_1} \begin{array}{c}
+ \\
+ \\
+
\end{array} ; \{0 = y = t\} \xrightarrow{M_1} \begin{array}{c}
+ \\
+
\end{array} ; \{y = 2z\} \xrightarrow{M_1} \begin{array}{c}
+
\end{array} .
\]

**References**


