Regime-dependent robust risk measures with application in portfolio selection

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Outline

- Introduction
- Regime-dependent robust risk measures
- Application to portfolio selection problem
- Empirical illustrations
- Conclusions
Traditional risk measure

- The traditional risk measure can be regarded as an aggregation function \( \rho : L_p(\mathcal{F}) \rightarrow R \) with respect to the probability \( P \), here \( 1 \leq p < \infty \).
- The famous risk measure VaR can be described as follows:

\[
VaR(x) = \min \gamma \quad \text{s.t.} \quad \text{Prob}\{\gamma \leq x\} \leq \epsilon,
\]

\( \epsilon \in (0, 1] \) is a given loss tolerant probability (say, 5%).
- The computation of risk measure relies on the underlying distribution \( P \) of \( x \).
Unknown distribution

- Traditional distribution assumptions, such as normal or student’s t, does not fit the financial data well.
- Fully distributional information is hardly known in practice.

Deal with the unknown distribution

- **Sample average approximation** (Shapiro et al. [2009])
  Generate samples to represent the original distribution.

- **Worst-case estimation** (Bertsimas et al. [2011])
  Make decisions with the worst sample.

- **Distributional robust** (El Ghaoui et al. [2003])
  Finding a worst estimation among all possible known distributions.
We can estimate $\rho$ by assuming $P$ belongs to an uncertainty set $\mathcal{P}$. This gives us the following worst-case risk measure:

**Definition 1**

For given risk measure $\rho$, the worst-case risk measure with respect to $\mathcal{P}$ is defined as $w\rho(x) \triangleq \sup_{P \in \mathcal{P}} \rho(x)$.

- By constructing different uncertainty sets $\mathcal{P}$, we can derive different versions of worst-case risk measures.
- Typical uncertainty sets proposed in the literature include the box uncertainty, the ellipsoidal uncertainty, and the mixture distribution uncertainty.
Worst-case variance

- Lobo and Boyd [1999] proposed a worst-case analysis with respect to uncertain variance, and demonstrated the worst-case variance problem is a semidefinite program.

Worst-case VaR

- El Ghaoui et al. [2003] considered the worst-case value-at-risk (VaR) with uncertain first and second order moments, and showed the worst-case VaR constraint is equivalent to a second order cone constraint.
Worst-case CVaR

- Zhu and Fukushima [2009] considered the portfolio selection models with worst-case CVaR constraints, and solved it with discrete samples.
- Chen et al. [2011] considered the worst-case lower partial moments and worst-case conditional value-at-risk (CVaR) with respect to the first two order moments, and derived a tight bound for these two problems.
Regime-dependent robust risk measures

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New risk measure

Application

Empirical illustrations

Regime-switching environment

Regime switching

- We consider the uncertainty set which is regime-dependent.
- Regime switching describes the trend of macro economy and it can reflect dynamic correlations of return rates in different economic cycles.
- We assume there are $K$ regimes possibly appearing.
- We assume the regime switching is Markovian with the following transition probability matrix:

$$P_s = \begin{pmatrix}
  P_{s^1s^1} & P_{s^1s^2} & \cdots & P_{s^1s^K} \\
  P_{s^2s^1} & P_{s^2s^2} & \cdots & P_{s^2s^K} \\
  \vdots & \vdots & \ddots & \vdots \\
  P_{s^Ks^1} & P_{s^Ks^2} & \cdots & P_{s^Ks^K}
\end{pmatrix},$$

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Regime-dependent risk measure

- We assume that uncertainty sets $\mathcal{P}(s)$ are associated with the possible regime $s \in S$, $S$ is the regime set.
- Given a particular regime $s$, the regime-dependent worst-case risk measure can be defined as

$$w\rho^s(x) \triangleq \sup_{P \in \mathcal{P}(s)} \rho(x).$$

Combine the sub risks into one

- One takes the greatest risk measure value among all the possible regimes
- The other mixes the sub risks together with respect to their occurring probabilities

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We define them as follows:

**Worst regime risk measure**

For the risk measure $\rho$, the worst regime risk measure with respect to the regime set $S$ is defined as

$$\text{wr}\rho(x) \triangleq \sup_{s \in S} w\rho^s(x) = \sup_{s \in S} \sup_{P \in \mathcal{P}(s)} \rho(x).$$

**Mixed worst-case risk measure**

For the risk measure $\rho$, the mixed worst-case risk measure with respect to the regime set $S$ is defined as

$$\text{mw}\rho(x) \triangleq E_s[w\rho^s(x)] = E_s[\sup_{P \in \mathcal{P}(s)} \rho(x)] = \sum_{s} P^s_{s_0,s} (\sup_{P \in \mathcal{P}(s)} \rho(x)).$$
Properties of the regime-dependent risk measures

Proposition 1

Worst regime risk measure is equivalent to worst-case risk measure with respect to the uncertainty set \( \mathcal{P} = \bigcup_{s \in S} \mathcal{P}(s) \)

\[
wr \rho(x) = \sup_{P \in \mathcal{P}} \rho(x).
\]

Theorem of coherency

If \( \rho(\cdot) \) is a coherent risk measure, then \( wr \rho(\cdot) \) and \( mw \rho(\cdot) \) associated with any regime set \( S \) and uncertainty sets \( \mathcal{P}(s), s \in S \), are both coherent risk measures.
Market setting

- There are \( n \) risky assets in the security market
- \( w = [w_1, w_2, ..., w_n]^T \): The proportion vector of the wealth invested in \( n \) assets
- \( r = [r_1, r_2, ..., r_n]^T \): The random return rates of \( n \) assets
- \( x = -r^T w \): The loss function
- Regime-dependent uncertainty set:

\[
\mathcal{P}(s) = \{ P \mid E_P[\xi] = \mu(s), \sigma^2_P(\xi) = \Gamma(s) \}, \ s \in S, \quad (1)
\]

Such uncertainty set contains all possible distribution with given mean and variance under particular regime.
Mean-wrVaR model

The objective function:

$$\max_w E(x) - \lambda \cdot \text{wrVaR}(x) = \max_w E(x) - \lambda \cdot \sup_{s \in S} \sup_{P \in \mathcal{P}(s)} \text{VaR}_P(x).$$  \hspace{1cm} (2)$$

Constraints on portfolio:

$$e^T w = 1, \hspace{1cm} (3)$$

$$\underline{w} \leq w_i \leq \overline{w}, \hspace{0.2cm} i = 1, \ldots, n, \hspace{1cm} (4)$$
By introducing an auxiliary variable $y$, Mean-wrVaR model (2-4) can be expressed as:

$$\max \ E(-r^T w) - \lambda y$$

s.t. \quad \sup_{P \in \mathcal{P}(s)} \text{VaR}_P(-r^T w) \leq y, \ s \in S,$$

$$e^T w = 1,$$

$$w \leq w_i \leq \bar{w}, \ i = 1, \ldots, n.$$
By transforming the worst-case VaR constraint (6) with respect to the uncertainty set (1), (5-8) is equivalent to

$$\max \quad E_s(\mu(s))^T w - \lambda y$$

**(SOCP1)**: s.t. $$\kappa(\epsilon)||\Gamma^{1/2}(s)w||_2 - \mu^T(s)w \leq y, \ s \in S,$$

$$e^T w = 1,$$

$$w \leq w_i \leq \bar{w}, \ i = 1, ..., n,$$

which is a second order cone programming.
Replacing $w_r$VaR in (2) by $mw$VaR, we have the mean-$w_r$VaR portfolio selection model

$$\max E(-r^Tw) - \lambda \cdot E_s[\sup_{P \in \mathcal{P}(s)} \text{VaR}_P(-r^Tw)],$$

(9)

s.t. \quad e^Tw = 1,

(10)

$$w_s \leq w_i \leq w_b, \quad i = 1, \ldots, n.$$  

(11)
Again, problem (9)-(11) can be transformed into the following second order cone program:

\[
\text{(SOCP2) : } \begin{align*}
\max & \quad E_s(\mu(s) - y(s)) \\
\text{s.t.} & \quad \kappa(\epsilon)\|\Gamma^{1/2}(s)w\|_2 - \mu^T(s)w \leq y(s), \ s \in S, \\
& \quad e^T w = 1, \\
& \quad w \leq w_i \leq \bar{w}, \ i = 1, \ldots, n.
\end{align*}
\]

SOCP can be efficiently solved by some commercial optimization softwares, such as MOSEK.
Empirical illustrations

- We choose 10 stocks from different industries in American stock markets
- We use adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates from February 14, 1977 to January 30, 2012
- We divide the market into three regimes: the bull regime; the consolidation regime and the bear regime
- Assume the regime transition probability is stationary:

\[
P = \begin{bmatrix}
0.9475 & 0.0336 & 0.0189 \\
0.3333 & 0.3148 & 0.3519 \\
0.0471 & 0.0634 & 0.8895
\end{bmatrix}.
\]
Statistics information under three regimes

Table: expected return rates (%) under different regimes and the total sample

<table>
<thead>
<tr>
<th></th>
<th>DIS</th>
<th>DOW</th>
<th>ED</th>
<th>GE</th>
<th>IBM</th>
<th>MRK</th>
<th>MRO</th>
<th>MSI</th>
<th>PEP</th>
<th>JNJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu(s^1)$</td>
<td>0.099</td>
<td>0.068</td>
<td>0.112</td>
<td>0.095</td>
<td>0.078</td>
<td>0.101</td>
<td>0.116</td>
<td>0.076</td>
<td>0.115</td>
<td>0.110</td>
</tr>
<tr>
<td>$\mu(s^2)$</td>
<td>0.026</td>
<td>-0.079</td>
<td>0.134</td>
<td>-0.042</td>
<td>-0.142</td>
<td>0.108</td>
<td>0.172</td>
<td>0.070</td>
<td>0.170</td>
<td>0.1235</td>
</tr>
<tr>
<td>$\mu(s^3)$</td>
<td>-0.205</td>
<td>-0.185</td>
<td>0.094</td>
<td>-0.161</td>
<td>0.010</td>
<td>-0.076</td>
<td>-0.037</td>
<td>-0.261</td>
<td>-0.029</td>
<td>0.031</td>
</tr>
<tr>
<td>$\mu$</td>
<td>0.099</td>
<td>0.068</td>
<td>0.112</td>
<td>0.095</td>
<td>0.078</td>
<td>0.101</td>
<td>0.116</td>
<td>0.076</td>
<td>0.115</td>
<td>0.1105</td>
</tr>
</tbody>
</table>

Both first and second order moments have significant difference among different regimes. (The estimated covariance matrices are omitted.)
Then we can find the optimal portfolios of mean-wrVaR, mean-mwVaR models by solving (SOCP1), (SOCP2).

Besides, we show the optimal portfolio of mean-wVaR model in el Ghaoui et al. [2003] as a comparison.

Table: optimal portfolios under wrVaR, mvVaR and wVaR

<table>
<thead>
<tr>
<th>Portfolio Type</th>
<th>DIS</th>
<th>DOW</th>
<th>ED</th>
<th>GE</th>
<th>IBM</th>
<th>MRK</th>
<th>MRO</th>
<th>MSI</th>
<th>PEP</th>
<th>JNJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_{wVaR}(s_0) )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.00</td>
<td>0.260</td>
<td>0.140</td>
<td></td>
</tr>
<tr>
<td>( w_{wrVaR}(s_0) )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.00</td>
<td>0.154</td>
<td>0.245</td>
<td></td>
</tr>
<tr>
<td>( w_{mwVaR}(s^1) )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.087</td>
<td>0.000</td>
<td>0.300</td>
<td>0.012</td>
</tr>
<tr>
<td>( w_{mwVaR}(s^2) )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.000</td>
<td>0.000</td>
<td>0.300</td>
<td>0.100</td>
</tr>
<tr>
<td>( w_{mwVaR}(s^3) )</td>
<td>0.00</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.30</td>
<td>0.00</td>
<td>0.000</td>
<td>0.000</td>
<td>0.246</td>
<td>0.153</td>
</tr>
</tbody>
</table>

The confidence level is \( \epsilon = 0.05 \), the trade-off parameter is \( \lambda = 1 \), the lower bound and upper bound of portfolio weights are 0 and 0.3, respectively.
Both the optimal portfolios of mean-wVaR model and mean-wrVaR model do not rely on the current regime.

The mean-mwVaR model provides us with three optimal portfolios under three different regimes.

That is because the estimation of mwVaR relies on the regime appearing probability in the future.

The strategy derived under such mixed robust models reveals more information about market regimes than the traditional worst-case risk measures.
We compare the performance of three robust models by computing the expected return rate, the variance of the optimal portfolio, and the optimal estimation of the robust VaR.

Table: robust VaR value, expected return and variance under different robust portfolio selection models

<table>
<thead>
<tr>
<th></th>
<th>wVaR($s_0$)</th>
<th>wrVaR($s_0$)</th>
<th>mwVaR($s^1$)</th>
<th>mwVaR($s^2$)</th>
<th>mwVaR($s^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>robust VaR value</td>
<td>0.0891</td>
<td>0.1078</td>
<td>0.0809</td>
<td>0.0923</td>
<td>0.1053</td>
</tr>
<tr>
<td>expected return (%)</td>
<td>0.1028</td>
<td>0.1023</td>
<td>0.1035</td>
<td>0.1030</td>
<td>0.1027</td>
</tr>
</tbody>
</table>
The largest estimation on the robust VaR is attained under the mean-wrVaR model. And it leads to the most conservative strategy.

The estimation of robust VaR has significant difference with respect to three regimes under the mean-mwVaR model.

The expected return of mean-mwVaR model under the bull or consolidation regime is significantly larger than that of either the mean-wVaR model or the mean-wrVaR model, while the corresponding variance does not increase too much with regard to its magnitude.

wrVaR is suitable for conservative investors; mwVaR is more suitable for investors who focus on the market trend.
We propose in this paper two classes of robust risk measures, which are time-dependent and coherent.

We apply VaR to construct robust portfolio selection models and show that they can be transformed into second order cone programs.

Empirical illustrations show that our new models can flexibly reflect the influence of different market regimes on the investment return and risk.

We can also adopt other risk measures, such as CVaR, as the basic risk measure $\rho(\cdot)$ in our new risk measure definitions to construct other robust risk measures.

Another interesting topic is to extend the results in this paper to the multi-period situation.
Thank You Very Much for Your Attention!