Rationality of Voting and Voting Systems: Lecture I

Can a Reasonable Person Have Intransitive, Incomplete and Discontinuous Preferences?

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The recognized aims of scholarly work

- make sense of things
  1. explanation
  2. understanding
- predict things
- influence things
- design things

Explanations come in many forms:

In terms of substance:

- causal
- functional
- teleological
- genetic

In terms of form:

- deductive-nomological
- inductive-statistical

The ultimate goal: theory

What is it? Views differ. E.g.

- a set of interrelated laws (like in mechanics)
- a basic interpretation of phenomena of interest (corpuscular theory of light)
- a set of statements with a hierarchical structure conjoined with rules of derivation
- a study of principles characterizing certain field of interest (like in game theory)
- a basic way of describing objects of interest (e.g. systems theory, cybernetics)
- a basic principle characterizing objects of interest (e.g. prospect theory)
- the study of some field of interest from a given perspective (multi-polar systems theory of international relations)

Our focus

- to outline the standard theory regarding rational behavior
- to review some of the challenges faced by this theory
- to suggest that – intuitively speaking – rationality may violate all basic principles associated with it in the standard theory
- some alternatives to the received will be discussed in the course the lectures

Old time religion

Research strategy:

- to predict or understand behavior (e.g. manage conflicts) one needs to know the goals and beliefs of the parties involved
- the goals are preferred states of the world
- given the goals, the beliefs restrict the action possibilities to those believed to lead to those goals
- assuming that goals are many and resources limited, the principle of rationality calls for acts that lead to goal achievement in a rational manner (e.g. with minimum associated costs)
- prediction: the actors will resort to those acts that will lead to their goals in a rational way
- design principle: look for those mechanisms that result in desired outcomes as game-theoretic equilibria

Pascal's wager

state act	exists s_1	doesn't exist s_2
believe a_1 do not believe a_2	eternal life hell	pious life in vain life without faith

tila teko	s_1	s_2
a_1	∞	$-c$
a_2	$-d$	e

Choice criteria

- dominance
- max-min (min-max)
- expected utility

$$EU(a_1) = pU(s_1) + (1-p)U(s_2) = p(\infty) + (1-p)(-c)$$

$$EU(a_2) = p(U(s_1) + (1-p)U(s_2) = p(-d) + (1-p)(e)$$

Another example

state outfit	sunny	rain
sunny outfit.	0	5
light rain outf.	1	3
heavy rain outf.	3	2

Other principles of choice

- maximax (highest value rule)
- satisficing
- Hurwicz's rule (weighted sum of max and min)

Representing goals and rationality

Definition

Rationality. A decision maker is rational if – when confronted with the choice between state a and state b – he/she will choose state a iff he/she prefers a to b.

Theorem

(Harsanyi 1977). Suppose that a preference relation is complete and transitive over the outcomes and that for each alternative state, the inferior and superior states constitute closed sets. Then the preference can be represented by a utility function.

Risk modality

Definition

Risky prospect: $C = (A_1, p_1; A_2, p_2; \ldots; A_n, p_n)$.

Definition

Expected utility (EU) property. A utility U function has the EU property iff

$$U(C) = U(A_1, p_1; A_2, p_2; \ldots; A_n, p_n)$$
$$= p_1 U(A_1) + p_2 U(A_2) + \ldots + p_n U(A_n)$$

Definition

Inferior set. Inferior probability set $I*$ for alternative B wrt alternatives A and C is the the set of probability numbers p satisfying

$$B \succeq (A, p; C, 1-p)$$

Definition

Superior set. Superior probability set $S*$ for alternative B wrt alternatives A and C is the the set of probability numbers p satisfying

$$B \preceq (A, p; C, 1-p).$$

Axioms:

1. weak preference relation over risky prospects is complete and transitive
2. monotonicity in prizes: If $A \succ B$ and $p > 0$, then

$$(A, p; C, 1-p) \succ (B, p; C, 1-p)$$

   and conversely.
3. continuity: For any alternative B wrt any pair of alternatives A and C, both the superior probability set $S*(B; A, C)$ and the inferior set $I*(B; A, C)$ are closed sets.

Theorem

Suppose that a decision maker's preferences among risky prospects satisfy completeness, transitivity, continuity and monotonicity in prizes. Then there exists a utility function $U = U(A)$ representing his preferences and having the EU property.

Analogous representation theorem has been proven for uncertain prospects.

Spatial representation

The individuals are supposed to be endowed with complete and transitive preference relations $\succeq$ over all point pairs in the space W. These relations are, moreover, assumed to be representable by utility functions in the usual way, that is

$$x \succeq y \Leftrightarrow u(x) \geq u(y), \forall x, y \in W$$

In strong spatial models the individual i's evaluations of alternatives are assumed to be related to a distance measure d_i defined over the space. Moreover, each individual i is assumed to have an ideal point x_i in the space so that

$$x \succeq y \Leftrightarrow d_i(x, x_i) \leq d_i(y, x_i), \forall x, y \in W$$

Spatial theory of voting: some classic results

- Black: single-peaked preferences and voting equilibrium
- Downs: median convergence in two-party competition
- Kramer: single-peakedness unlikely in multidimensional setting
- Plott equilibrium
- McKelvey's theorems
- Banks, Saari, Schofield: core conditions in multidimensional policy spaces

Empirical studies on voting

Strategy:

- estimate the actors' location in a many-dimensional space
- predict the voting or coalition formation on the basis of proximity of actors in the space

Allais paradox

$$\begin{aligned} r_1 &= (1,000,000, 1.0) \\ r_2 &= (5,000,000, 0.10; 1,000,000, 0.89; 0, 0.01) \end{aligned}$$

$$\begin{aligned} r_3 &= (5,000,000, 0.10; 0, 0.90) \\ r_4 &= (1,000,000, 0.11; 0, 0.89) \end{aligned}$$

In both pairs of choices, Allais's subjects chose the former (i.e. r_1 and r_3) alternative against the theoretical prediction.

Allais, cont'd

Yet, this choice behavior can be shown to be inconsistent with EU maximization. To see this, let us compute the expected utilities of the four risky prospects. If the decision maker is a EU maximizer, her choice behavior should reflect this.

$$
\begin{aligned}
EU(r_1) &= 1 \times u(1,000,000) \\
EU(r_2) &= 0.10 \times u(5,000,000) + 0.89 \times u(1,000,000) + 0.01 \times u(0) \\
EU(r_3) &= 0.10 \times u(5,000,000) + 0.90 \times u(0) \\
EU(r_4) &= 0.11 \times u(1,000,000) + 0.89 \times u(0)
\end{aligned}
$$

Allais, cont'd

Now, if one prefers r_1 to r_2 and is an EU maximizer, this means that

$$u(1,000,000) > 0.10 \times u(5,000,000) + 0.89 \times u(1,000,000) + 0.01 \times u(0)$$

Solving for $u(1,000,000)$ yields:

$$u(1,000,000) > \frac{0.10 \times u(5,000,000) + 0.01 \times u(0)}{0.11} \qquad (1)$$

If, on the other hand, r_3 is preferred to r_4, as we assumed, we get:

$$0.10 \times u(5,000,000) + 0.90 \times u(0) > 0.11 \times u(1,000,000) + 0.89 \times u(0)$$

and thus

$$u(1,000,000) < \frac{0.10 \times u(5,000,000) + 0.01 \times u(0)}{0.11} \qquad (2)$$

Ellsberg's paradox

	colour (and number) of balls		
	red	white or blue (60)	
options	(30)	white	blue
1	$100	$0	$0
2	$0	$0	$100
3	$100	$100	$0
4	$0	$100	$100

Ellsberg's observation

"Many people would choose 1 over 2, but 4 over 3. The latter choice behaviour is clearly inconsistent with EU theory". Let the probability of blue balls be q and that of the white ones $2/3 - q$. For a EU maximizer the preference for option 1 over option 2 means:

$$1/3 \cdot U(\$100) > q \cdot U(\$100).$$

On the other hand, the preference for option 4 over option 3 means:

$$2/3 \cdot U(\$100) > 1/3 \cdot U(\$100) + (2/3 - q) \cdot U(\$100),$$

whereupone:

$$q \cdot U(\$100) > 1/3 \cdot U(\$100),$$

which contradicts the first inequality.

Intransitivity of preferences over risky prospects

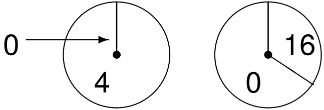
There is also experimental evidence that points to difficulty in forming transitive preference relation over risky prospects. Consider the following list of risky prospects:

1. $(\$5.00, 7/24; \$0, 17/24)$
2. $(\$4.75, 8/24; \$0, 16/24)$
3. $(\$4.50, 9/24; \$0, 15/24)$
4. $(\$4.25, 10/24; \$0, 14/24)$
5. $(\$4.00, 11/24; \$0, 13/24)$

The expected values of payoffs increase from top to bottom (from value \$1.46 to \$1.83). Tversky (1969) found in his experiments that a sizable subgroup of his experimental subjects exhibited behavior whereby in adjacent pairwise choices, they preferred the prospect with higher maximum value (and smaller expected payoff), but in the comparison between the extreme prospects preferred the one with the higher winning probability (and expected value).

Preference reversal phenomenon

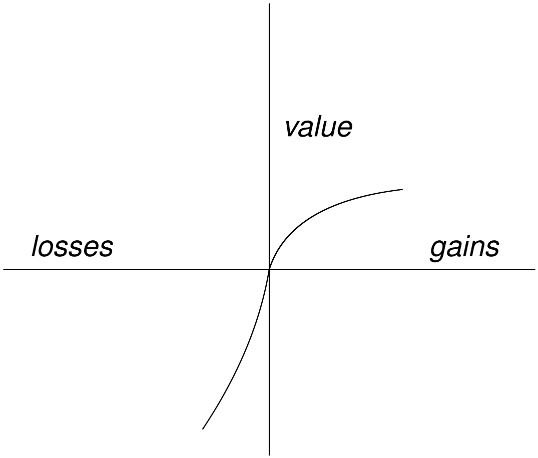
Figure: The preference reversal experiment



Framing

In one of their experiments Kahneman and Tversky (1979) confronted half of their experimental subjects with the choice between options (i) and (ii) and the other half with the choice between (iii) and (iv). (i) gives each subject first 1000 and then gives her a ticket to a lottery which gives an additional payoff of 1000 with probability $1/2$ and nothing with probability $1/2$. (ii) gives also the subject first 1000, but then gives an additional 500 with certainty. (iii) gives the subject first 2000 and then assigns her the lottery with payoffs -1000 and 0, each with probability $1/2$. (iv) similarly gives the subject first 2000, but withdraws -500 from this with certainty. An overwhelming majority of Kahneman and Tversky's experimental subjects preferred (ii) to (i) and somewhat smaller majority preferred ((iii) to (iv). Yet, it can be seen that (i) is in fact identical with (iii) and (ii) is identical with (iv). Yet, the preference of the majority of subjects seems to reverse depending on the way the options are framed.

Figure: Valuation of gains and losses according to prospect theory



Compromise effect

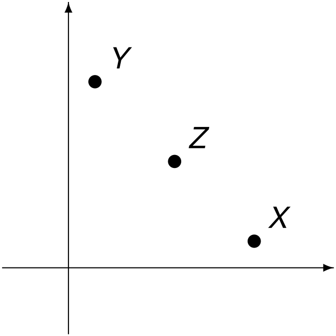


Figure: Compromise Effect

Asymmetric domination effect

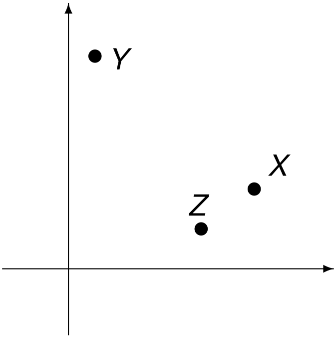


Figure: Asymmetric Domination

What has been learned?

Grether and Plott: “The fact that preference theory and related theories of optimization are subject to exception does not mean that that they should be discarded. No alternative theory currently available appears to be capable of covering the same extremely broad range of phenomena.” Some twenty-five years later this conclusion still seems correct. All alternatives to the EU and SEU theory seem either more limited in scope or poorer in informative content.

Cyclic preferences may make sense

Example

Three universities A, B and C are being compared along three criteria: (i) research output (scholarly publications), (ii) teaching output (degrees), (iii) external impact (expert assignments, media visibility, R& D projects, etc.)

crit. (i)	crit. (ii)	crit. (iii)
A	B	C
B	C	A
C	A	B

Cycle: $A \succ B \succ C \succ A \succ \ldots$.

issue	*theory*	*methods*	*social relevance*	*row choice*
plan	A	A	B	A
past record	A	B	A	A
community	B	B	B	B
column choice	A	B	B	*overall choice* ?

Table: Ostrogorski's paradox in MCDM setting

Simpson – again

Haunsperger and Saari, reinterpreted

Three candidates A, B, and C for presidency are having two televised debates which you watch intermittently, e.g. fives minutes here and there. On the basis of performances of the candidates over these periods you score the candidates on a 2-7 interval scale (the higher score, the better).

First debate:

	A	B	C
first period	2.69	2.63	2.62
second period	2.74	2.71	3.00

Second debate:

	A	B	C
first period y	2.89	2.81	2.80
second period	2.98	2.90	5.99

In terms of ordinal measurements the above tables look like this:

First debate

	A	B	C
first p.	4	5	6
second p.	2	3	1

Second debate

	A	B	C
first p.	4	5	6
second p.	2	3	1

Summing up the ordinal values gives in both debates the ranking: $A \succ C \succ B$. Hence, A seems to be the best candidate.

However, had you combined the observations in the four time instants, you would have had:

A	B	C
3	4	1
5	6	2
8	9	7
10	11	12

Now the order of candidates becomes: $C \succ A \succ B$, i.e. C is the best.

Simpson's paradox before Simpson

Cohen and Nagel (1934):

Example

death rate per 100.000	*New York*	*Richmond*
sub-population 1	179	162
sub-population 2	560	332
total death rate	187	226

Simpson's paradox, cont'd

Distribution of cards in a fair deck:

Example

	court cards	*plain cards*
red cards	6/52	20/52
black cards	6/52	20/52

Example, cont'd

The deck under scrutiny:

Example

	dirty		clean	
	court	plain	court	plain
red	4/52	8/52	2/52	12/52
black	3/52	5/52	3/52	15/52

The ratio plain cards/court cards among red dirty ones: $8/4 = 2$. The same ratio among black and dirty ones: $5/3 = 1.67$, i.e. smaller. In the subset of clean cards, these ratios are: $12/2 = 6$ among red cards and $15/3 = 5$ among the black ones. I.e. in both subsets (dirty, clean) there is positive association between plainness and redness: the red cards tend to be more often plain than the black cards.

Another interpretation

Example

	men		women	
	no treatment	treatment	no treatment	treatment
alive	4/52	8/52	2/52	12/52
dead	3/52	5/52	3/52	15/52

Same ratios as in the card example, but one would now perhaps be inclined to suggest that a causal connection exists.

Decomposition of conditional probabilities

The crux of this construction is the decomposition of $P(A|B)$ and $P(A|\neg(B))$:

$$\begin{aligned} P(A|B) &= [P(C|B)]P(A|BC) && (3)\\ &\quad +[P(\neg(C)|B)]P(A|B\neg(C)) && (4)\end{aligned}$$

$$\begin{aligned} P(A|\neg(B)) &= [P(C|\neg(B))]P(A|\neg(B)C) && (5)\\ &\quad +[P(\neg(C)|\neg(B))]P(A|\neg(B)\neg(C)) && (6)\end{aligned}$$

$P(A|B)$ is clearly a weighted average of $P(A|BC)$ and $P(A|B\neg(C))$. Similarly for $P(A|\neg(B))$. If the weights $P(C|B)$ and $P(\neg(C)|B)$ are identical, the paradox doesn't occur.

Baigent's theorem

Theorem

Anonymity and respect for unanimity cannot be reconciled with proximity preservation: choices made in profiles more close to each other ought to be closer to each other than those made in profiles less close to each other (Baigent 1987).

I.e. if a small group of voters changes its mind about preference ranking, the change in outcomes can be larger than had a large group of voters changed its mind. That is, smaller groups can, under any reasonable voting rule, have larger impact on outcomes than larger groups.

Stylized Greek Bailout Example

P_1		P_2		P_3		P_4	
1	2	1	2	1	2	1	2
bailout	bailout	default	default	default	bailout	bailout	default
default	default	bailout	bailout	bailout	default	default	bailout

We denote the voters' rankings in various profiles by P_{mi} where m denotes the profile and i denotes the voter. k and j are particular values of m such that $k \neq j$. N is the set of voters. We consider two types of metrics: d_r is defined on pairs of rankings and d_P refers to profiles. The former is denoted by d_r and the latter by d_P. The two metrics are related as follows:

$$d_P(P_k, P_j) = \sum_{i \in N} d_r(P_{ki}, P_{ji}).$$

Example, cont'd

In other words, the distance between two profiles P_k and P_j is the sum of distances between the pairs of rankings of the first, second, etc. voters. No further assumptions on the metric have been made. Take now two profiles, P_1 and P_3, from the above table and express their distance using metric d_P as follows:

$$d_P(P1, P3) = d_r(P_{11}, P_{31}) + d_r(P_{12}, P_{32}).$$

Since, $P_{12} = P_{32} = bailout > default$, and hence the latter summand equals zero, $d_P(P_1, P_3)$ reduces to: $d_P(P_1, P_3) = d_r(P_{11}, P_{31}) = d_r((bailout > default), (default > bailout))$.
Taking now the distance between P3 and P4, we get:

$$d_P(P_3, P_4) = d_r(P_{31}, P_{41}) + d_r(P_{32}, P_{42}).$$

Example, cont'd

Both summands are equal since by definition:
$d_r((default > bailout), (bailout > default)) = d_r((bailout > default), (default > bailout))$. Thus,
$d_P(P_3, P_4) = 2 \times d_r((bailout > default), (default > bailout))$, i.e., in terms of d_P, then, P_3 is closer to P_1 than to P_4. This makes sense intuitively.
The proximity of the social choices emerging out of various profiles depends on the applied choice procedure g. Let us make two very mild restrictions on choice procedures, viz. that they are anonymous and respect unanimity.

Example, cont'd

The former states that the choices are not dependent on the labelling of the voters. The latter, in turn, means that if all voters agree on a preference ranking, then that ranking is chosen. In our example, anonymity requires that whatever is the choice in P_3 is also the choice in P_4 since these two profiles can be reduced to each other by relabelling the voters. Unanimity, in turn, requires that $g(P_1) = bailout$, while $g(P_2) = default$. Therefore, either $g(P_3) \neq g(P_1)$ or $g(P_3) \neq g(P_2)$. Let's assume the former. It then follows that $d_r(g(P_3), g(P_1)) > 0$. Recalling the implication of anonymity, we now have:

$$d_r(g(P_3), g(P_{1)}) > 0 = d_r(g(P_3, g(P_4)).$$

Example, cont'd

In other words, even though P_3 is closer to P_1 than to P_4, the choice made in P_3 is closer to – indeed identical with – that made in P_4. This argument rests on the assumption that $g(P_3) \neq g(P_1)$. Similar argument can be made for the alternative assumption, viz. that $g(P_3) \neq g(P_2)$. The example thus shows that anonymity and respect for unanimity cannot be reconciled with a property called proximity preservation [Baigent 1987; Baigent and Klamler 2004]: choices made in profiles more close to each other ought to be closer to each other than those made in profiles less close to each other.

Example, cont'd

The example demonstrates that small mistakes or errors made by voters are not necessarily accompanied by small changes in voting outcomes. Indeed, if the true preferences of voters are those of P_3, then voter 1's mistaken report of his preferences leads to profile P_1, while both voters' making a mistake leads to P_4. Yet, the outcome ensuing from P_1 is further away from the outcome resulting from P_3 than the outcome that would have resulted had more, i.e., both, voters made a mistake (whereupon P_4 would have emerged).

Example, cont'd

The above example illustrates that the mistakes of voters can make a substantial difference. It should be emphasized that the violation of proximity preservation occurs in a wide variety of voting systems, viz. those that satisfy anonymity and unanimity. This result is not dependent on any particular metric with respect to which the distances between profiles and outcomes are measured. Therefore we can conclude that in nearly all reasonable voting systems it is possible that a smaller group of voters has a greater impact on voting outcomes than a larger group. Thus, we have case of a violation of local monotonicity (LM).

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