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- **Context:** network of interacting individuals that contribute to a total productive value of the network
- **Problem:** how to share the value generated by the network
- Jackson & Wolinsky (1996) introduced **network games** (the value generated depends directly on the network structure) and showed that the Myerson value has a direct extension from communication games to network games
- **Crucial feature most often ignored:** the network is not static, it evolves over time. So far, only a few exceptions, like Jackson (2005), Navarro (2013), Caulier et al. (2013)
$N = \{1, \ldots, n\}$ set of players (fixed), connected in a network
Networks (1/2)

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- $\ell_i(g) = |L_i(g)|$ degree of player $i$ in $g$
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Given any $S \subset N$, let $g^S$ be the complete network among the players in $S$. 
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Networks (2/2)

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- The components of a network are the distinct maximal connected subgraphs.
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The components of a network are the distinct maximal connected subgraphs.

Let $C(g)$ be the set of components of $g$, $g = \bigcup_{g' \in C(g)} g'$
A network game is a pair \((N, \nu)\), where \(\nu : G \to \mathbb{R}\) is a value function.

\[ V = \text{set of all possible value functions} \]
A network game is a pair \((N, v)\), where \(v : G \rightarrow \mathbb{R}\) is a value function

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The unanimity game (basic value function) (form a basis of \(\mathcal{V}\))

\[ v_g(g') = \begin{cases} 1, & \text{if } g' \supseteq g \\ 0, & \text{otherwise} \end{cases} \]

Note that any \(v\) can be written as a linear combination of basic value functions \(v_g\) in a unique way (form a basis of \(\mathcal{V}\)).
A network $g$ is \textit{efficient} relative to $\nu$ if it maximizes $\nu$, i.e., 
$\nu(g) \geq \nu(g')$ for all $g' \in G$. 
A network $g$ is **efficient** relative to $v$ if it maximizes $v$, i.e.,

$$v(g) \geq v(g') \text{ for all } g' \in G.$$  

**Allocation rule** for $(N, v)$: how the value generated by $g$ is distributed among players.  

An allocation rule is a function $Y : G \times V \to \mathbb{R}^n$ such that

$$\sum_i Y_i(g, v) = v(g) \text{ for all } v \in V \text{ and } g \in G$$
An allocation rule $Y$ is *component balanced* if for any component additive $\nu$, $g \in G$ and $g' \in C(g)$

$$\sum_{i \in N(g')} Y_i(g, \nu) = \nu(g')$$

Component balance requires that if a value function is component additive, then the value generated by any component be allocated to the players among that component.
Component balance and equal bargaining power

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- An allocation rule satisfies \textit{equal bargaining power} if for any component additive $v$ and $g \in G$,

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Y_i(g, v) - Y_i(g - ij, v) = Y_j(g, v) - Y_j(g - ij, v)
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- Equal bargaining power does not require that players split the marginal value of a link. It just requires that they equally benefit or suffer from its addition. It is possible that

\[ Y_i(g, v) - Y_i(g - ij, v) + Y_j(g, v) - Y_j(g - ij, v) \neq v(g) - v(g - ij) \]
Myerson (1977) developed a variation of the Shapley value for communication games. The Myerson value also has a corresponding allocation rule in the context of network games:

$$Y_i^{MV}(g, \nu) = \sum_{S \subseteq \mathcal{N} \backslash i} (\nu(g|_{S \cup i}) - \nu(g|_S)) \left( \frac{|S|!(n - |S| - 1)!}{n!} \right)$$
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- Characterization of \( Y^{MV}(g, v) \) (Jackson & Wolinsky, 1996): \( Y \) satisfies component balance and equal bargaining power if and only if \( Y(g, v) = Y_{i}^{MV}(g, v) \) for all \( g \in G \) and any component additive \( v \).
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- Insensitivity of the Myerson value to alternative networks: 3-person society, two different value functions, \( v(12) = v(23) = v(12, 23) = 1, v(g) = 0 \) for all other networks, \( v'(g) = 1 \) for all \( g \neq g^\emptyset \). While player 2’s criticality is quite different under the two value functions, we have \( Y^{MV}(\{12, 23\}, v) = Y^{MV}(\{12, 23\}, v') = (\frac{1}{6}, \frac{2}{3}, \frac{1}{6}) \).
The egalitarian allocation rule $Y^e$ is defined by

$$Y^e_i(g, v) = \frac{v(g)}{n}$$
Egalitarian allocation rules

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- The component-wise egalitarian allocation rule $Y^{ce}$ is defined as follows for component additive $v$ and any $g$:
  $$Y^{ce}_i(g, v) = \begin{cases} \frac{v(h)}{|N(h)|}, & \text{if there exists } h \in C(g) \text{ such that } i \in h \\ 0, & \text{otherwise} \end{cases}$$
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- For a value function $v$ that is not component additive, $Y^{ce}(g, v) = Y^e(g, v)$ for all $g$. 
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  \end{cases}
  \]

- For a value function \( v \) that is not component additive, \( Y^{ce}(g, v) = Y^e(g, v) \) for all \( g \).

- \( Y^{ce} \) splits the value of a component network equally among all members of that component, but makes no transfers across components. \( Y^{ce} \) respects component balance.
Flexible-network rules (Jackson, 2005)

- The idea that the allocation of value is taking place with the perspective that the network is something that can be varied and inefficient networks should not be formed.
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An allocation rule $Y$ is a *flexible network rule* if $Y(g, v) = Y(g^N, \hat{v})$ for all $v$ and efficient $g$ (relative to $v$).

The *player-based flexible network allocation rule*

$$Y^P_{B\!F\!N}_i(g, v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{S \subset N \setminus i} (\hat{v}(g^{S \cup i}) - \hat{v}(g)) \left( \frac{|S|!(n - |S| - 1)!}{n!} \right)$$
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- The link-based flexible network allocation rule

\[
Y_{i}^{LBFN}(g, v) = \frac{v(g)}{\hat{v}(g^N)} \sum_{j \neq i} \sum_{g' \subseteq g^N - ij} \left[ \frac{1}{2} (\hat{v}(g' + ij) - \hat{v}(g')) \left( \frac{\ell(g')!(\ell(g^N) - \ell(g') - 1)!}{\ell(g^N)!} \right) \right]
\]

where \( \ell(g) = \frac{1}{2} \sum_{i} \ell_i(g) \), and \( \ell_i(g) \) is the degree of \( i \) in \( g \)