Chapter 1

Geometric optics

The main ideas of Hamiltonian mechanics originate in geometric optics. Therefore, we first discuss fundamental principles of geometric optics, and then formulate similar principles of Hamiltonian mechanics.

1.1 The least time principle of Fermat

We regard the propagation of light as a motion of some special kind of particles — photons. What is the trajectory of a photon? It turns out that, in most cases, the trajectory can be found using the following principle stated by Fermat in 1650: *the light chooses the trajectory between two given points so that the time it takes to traverse it is the minimal possible.*

The Fermat principle was preceded by experimental as well as theoretical research on propagation of light. Ancient Greeks did a lot. Ptolemy broke the classical Greek tradition of studying natural phenomena abstractly and deductively and conducted a number of experiments. In particular, he composed numerical tables showing the dependence of the angle of on refraction on the angle of incidence. Hero of Alexandria formulated the least distance principle, which is applicable to homogeneous isotropic media. According to Hero's principle, in such media, the light goes along straight lines. However, this principle did not explain the phenomenon of refraction. Fermat's principle provides an explanation for the refraction, and also for many other optical effects.

For example, to construct lenses, it suffices to use Fermat's principle alone. At a dawn, when we see the Sun near the horizon, it is in fact behind the horizon, i.e., the line connecting us (the observer) and the Sun intersects the surface of the Earth. The fact that we still see the Sun is due to the bending of light rays. Rays bend in order to optimize their path through the denser layers of the atmosphere, in which the speed of light is lower. The ray bending is also responsible for the fact that the visible shape of the Sun at the dawn is not round but rather oval.

One can sometimes see a mirage over a hot asphalt or a hot sand. The rays seem to reflect from the hot air. In reality, this effect can be explained using Fermat's principle: it is optimal for the rays to bend and traverse a significant distance through the hot air near the surface of the Earth, since the speed of light is higher in the hot air than in the cold air.

We now try to formalize Fermat's principle. Suppose that the speed of light at a given point depends only on the point and on the direction, in which the light goes (and not, say, on the position of the source of light and not on its intensity) Thus the speed of light is completely determined by the physical properties of the medium. The medium is said to be *isotropic* if the speed of light depends only on the point, not on the direction (in order to emphasize that we only talk about optical properties of the medium, we will sometimes say "optically isotropic medium"). We now assume for simplicity that the medium is isotropic. Then the speed of light is represented by a function of a point.

In real life, light propagates in the three-dimensional space, but we will assume that it propagates in the plane. Let $v(x, y)$ be the speed of light in a point with coordinates x and y. Then the light trajectory minimizes the integral

$$\int_{(x_0,y_0)}^{(x_1,y_1)} \frac{ds}{v(x,y)}$$

This integral is computed along a curve connecting the points (x_0, y_0) and (x_1, y_1) (these points are fixed) and is called *the optical length of the curve.* The parameter s is the arc-length parameter on the curve (i.e., a parameter, whose increment on any arc of the curve is equal to the length of the arc). The integral can be thought of as the one-variable integral

$$\int_0^L \frac{ds}{v(x(s), y(s))},$$

where L is the total length of the curve, and $x = x(s)$, $y = y(s)$ is the representation of the curve through the arc-length parameter s. Thus the optical length is a function of the curve, and this function is defined on all sufficiently smooth curves with fixed endpoints. Fermat's principle states that a light ray is a curve, on which this function attains its minimum.

Fermat's principle needs some corrections. For example, we need to consider not only the paths, on which the traversing time is minimal but also the paths representing critical points of the function of optical length.[1] This modification is similar to considering, instead of the minima of a one-variable function, the points, at which the derivative of the function vanishes. The points of the minimum satisfy this condition, but not only them.

Consider a homogeneous and isotropic medium, i.e., a medium, in which the speed of light depends neither on the point, nor on the direction. Then the optical length of a curve is proportional to the usual Euclidean length. Thus to minimize the time is the same as to minimize the length. We know which curves in a Euclidean space minimize the length — these are straight lines. Thus, in a homogeneous and isotropic medium, the light propagates along straight line rays.

However, these rays may reflect in surfaces, which does not follow directly from Fermat's principle, if the latter is understood verbatim. Indeed, if a light ray chooses between going along a straight line and reflecting, say, in the surface of a lake, then, according to Fermat's principle, it must choose the former. In real life, both variants are chosen, although the latter is chosen, that is to say, with a smaller weight depending on the reflecting properties of the surface. Why this is happening is a difficult question, which we will not discuss here. We would need to employ some quantum mechanics for that.

Suppose that the ray has to reflect in a smooth surface. A particular way of doing that is described by Fermat's principle. For simplicity, we consider a two-dimensional problem. A light ray emanates from a point A and reflects in a line l. An observer located at a point B sees the reflected light. What is the shape of the ray between the point A and the point B?

This is a mathematical problem:

1.1. Find the curve of the smallest length connecting the points A and B having at least one point in common with l. We assume that A and B are on the same side of the line l (otherwise the problem is obvious). See Figure 1.1.

This problem admits an elegant solution using the *method of reflections*, which is sometimes studied in high-school geometry classes. However, this

[1] If paths are regarded as points of a functional space, then one can talk about *critical points* of a functional defined on this space — points, at which the differential of the functional equals zero. Values of the functional at its critical points are called *critical values*. In the classical calculus of variations and theoretical mechanics, critical values are called *stationary values*. We will also sometimes use this terminology.

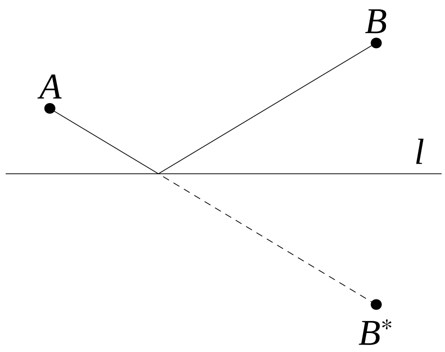

Figure 1.1: The curve of the smallest length connecting two given points and having a point in common with a given line.

problem can also be solved by "brute force". To this end, we observe that the optimal curve (if such curve exists) must consist of two straight line segments, one of which connects the point A with a point X on the line l, and the other connects the point X with the point B. Thus, our problem reduces to an optimization problem with one parameter. As a parameter, we can take the coordinate of the point X on the line l. Let u denote this coordinate. We will write $X(u)$ instead of X, meaning the point on the line l, whose coordinate is u. We of course assume that the coordinate u is chosen so that the distance between the points $X(u)$ and $X(u')$ equals $|u - u'|$.

1.2. Compute the derivative with respect to u of the distance between the points A and $X(u)$ (express the answer through the angle between the line segment $[A, X(u)]$ and the line l).

Let α be the angle between the line segment $[A, X(u)]$ and the line l. The answer to the problem posed above is: $\pm\cos\alpha$. See figure 1.1, in which some clarifications are provided. The sign depends on which of the two angles is taken as α. Now, in order to find the shortest reflected ray, we need to find the minimum of the following single variable function:

$$f(u) = |AX(u)| + |X(u)B|,$$

where $|AX|$ denotes the Euclidean distance between the points A and X. Equating the derivative of the function f to zero, we obtain that the angle of incidence must be equal to the angle of reflection.

When passing from one medium to another, a ray is *refracted.* This phenomenon is related with the fact that the speed of light changes on the

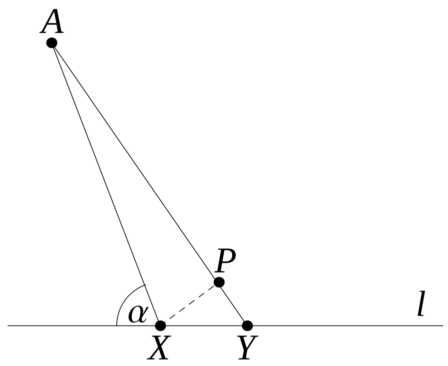

Figure 1.2: The derivative of the distance $|AX(u)|$ with respect to u equals $\cos\alpha$. Points $X = X(u)$ and $Y = X(u + \Delta u)$ are shown in the figure. The difference $|AY| - |AX|$ is approximately equal to $|PY|$, i.e., is approximately equal to $|XY| \cos\alpha = (\Delta u) \cos\alpha$.

boundary between the two media. Suppose that one medium occupies the upper half-plane, and the speed of light in this medium equals c_1, and the second medium occupies the lower half-plane, and the speed of light in this medium equals c_2. For example, we can think of having air above and water below. Let l be the horizontal line separating the two media. As before, we let $X(u)$ stand for the point in the line l, whose coordinate is equal to u.

Suppose that a light ray is attempting to reach a point B from a point A, the latter point being in the air and the former point being in the water. It is clear that the ray will first traverse the straight line segment connecting A with some point of the line l, and then traverse some straight line segment connecting this point with B. We arrive at the minimization problem for the following function:

$$f(u) = \frac{|AX(u)|}{c_1} + \frac{|X(u)B|}{c_2}.$$

Equating the derivative of the function f to zero, we obtain the following equation:

$$\frac{\cos\alpha_1}{c_1} = \frac{\cos\alpha_2}{c_2}.$$

Let α_1 be the angle between the line segment $[A, X(u)]$ and the line l, and let α_2 be the angle between the line l and the line segment $[X(u), B]$. The angles are chosen so that either both are acute or both are obtuse. We will assume that the both angles are acute, see Fig. 1.1.

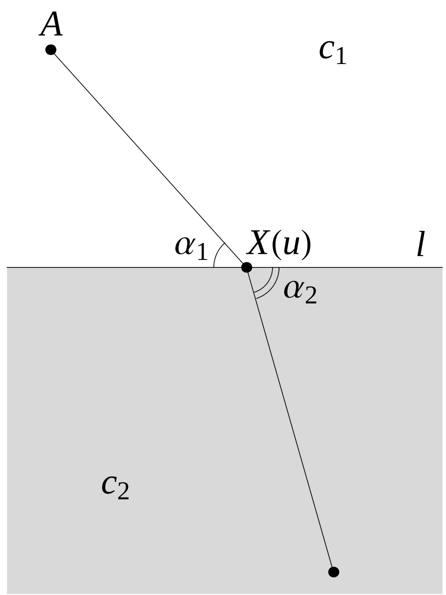

Figure 1.3: Refracting light ray passing from a medium, in which the speed of light equals c_1, to the medium, in which the speed of light equals c_2.

The thus obtained relation between $\cos \alpha_1$ and $\cos \alpha_2$ is called *Snell's law*. This law was obtained by Snell (Latin version: Snellius) in 1621.

The following heuristic argument allows to generalize Snell's law to the case of an isotropic medium, in which the speed of light (as a scalar-valued function of a point) depends only on the y-coordinate. Let us first imagine a medium consisting of a big number of horizontal layers. Each layer is bounded by two parallel horizontal planes (more precisely, if we think about a planar medium, by two horizontal lines). Suppose that each layer is homogeneous. Let c_i denote the speed of light in the i-th layer. Then Snell's law implies that the value

$$\frac{\cos \alpha_i}{c_i}$$

is constant, i.e., does not depend on i. We now let the layers become thinner and thinner. Every isotropic medium, in which the speed of light is a smooth function $v(y)$ of the coordinate y only, can be approximated by media consisting of finite but large number of thin horizontal layers. Passing to the limit (here we allow a "physical" level of rigor and not justify the limit), we obtain that

$$\frac{\cos\alpha(t)}{v(y(t))} = const$$

along every light ray (meaning that the value in the left-hand side does not depend on t). Here t is any parameter on the light ray (for example, we can impose that $x(t)$ and $y(t)$ are the coordinates of the photon at time t). The angle $\alpha(t)$ is the angle between the light ray and the horizontal direction at the point $(x(t), y(t))$. The obtained law is called *generalized Snell's law.*

Generalized Snell's law can be rewritten as a differential equation on the trajectory. Set

$$\kappa = \frac{\cos\alpha(t)}{v(y(t))}.$$

This value does not depend on t by generalized Snell's law. Suppose that a light ray is the graph of some function $y = y(x)$. Then we can take x instead of t on the given light ray. The angle $\alpha(x)$, i.e., the slope of the light ray (relative to the horizontal direction) is related to the derivative of the function $y(x)$. Namely, as we know, the derivative is equal to the tangent of the slope. The tangent and the cosine are related by the following formula:

$$1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}.$$

Thus we obtain the following differential equation on the light rays:

$$\frac{dy}{dx} = \frac{\sqrt{1-\kappa^2 v^2(y)}}{\kappa v(y)}.$$

The right-hand side of this equation depends only on y. Therefore, the obtained equation is separable.

1.1.1 Willebrord Snellius (1580–1626)

Dutch mathematician, physicist and astronomer, professor of Leiden university. He developed a triangulation method for geodesic measurements, in particular, for the computing the radius of the Earth. In 1621, he discovered the law of refraction of light. However, as it turned out, this law had been previously known to the Persian mathematician and physicist Ibn Sahl (the law of refraction appears in his treatise on the optical lenses dated 984).

1.1.2 Pierre de Fermat (1601–1665)

A councillor of the Toulouse parliament and an amateur mathematician. He is most famous for his mathematical inventions presented mainly in a private correspondence with mathematicians. The author of fundamental notions and methods of analytic geometry, number theory, analysis and probability. Fermat stated the least time principle and deduced Snell's law from it (1657). However, Fermat thought that the light propagates with infinite speed, which resulted in a somewhat misleading formulation of the least time principle.

1.1.3 Brachistochrone problem

One of the most interesting applications of generalized Snell's law is to the brachistochrone problem. Suppose that two points A and B are given, and the point B is located lower than A but not on the same vertical line. Consider various curves connecting A with B. Imagine a particle sliding along these curves without friction. A *brachistochrone* is a curve such that the sliding time — the time it takes a particle to slide along this curve — is the minimal possible. A sliding particle always starts in A and terminates in B; and the initial velocity is always zero (we do not push the sliding particles).

The problem of finding a brachistochrone can be formally reduced to a problem in geometric optics. Indeed, the sliding time along a curve is equal to the integral

$$\int \frac{ds}{v},$$

computed along this curve. If it happens so that the speed v of a sliding particle depends only on the position of the particle and not on the curve, then the problem will reduce to geometric optics.

The speed v can be found from the conservation of energy:

$$\frac{mv^2}{2} - mgy = 0.$$

(We let the y-axis be oriented downwards, so that the potential energy decreases with y, and we have the minus sign. We also assume that the point A is at height $y = 0$, hence the potential energy equals mgy. At the initial moment, the sliding particle is at height 0, and its speed vanishes, therefore, the total energy is equal to zero.) We obtain that

$$v = \sqrt{2gy}.$$

We see that, indeed, v depends only on a position but not on the curve. Moreover, v depends only on the y-coordinate. Therefore, we can use generalized Snell's law.

We have already established that Snell's law can be rewritten as a separable ODE on the light rays. Light rays can be found in the form of function graphs $y = y(x)$. In our case, the ODE takes the form

$$\frac{dy}{dx} = \frac{\sqrt{1 - 2g\kappa^2 y}}{\kappa\sqrt{2gy}}.$$

Separating the variables and integrating, we obtain:

$$\int \frac{\sqrt{2g\kappa^2 y}dy}{\sqrt{1 - 2g\kappa^2 y}} = \int dx.$$

In order to compute the integral in the left-hand side, it is convenient to make the following change of variables:

$$2g\kappa^2 y = \sin^2 \phi$$

(the old variable is y, the new variable is ϕ; these variables are related by the equation displayed above) Performing the integration, we obtain that

$$\frac{\phi}{2} - \frac{\sin 2\phi}{4} = g\kappa^2 x + C.$$

Replace 2ϕ with ϕ (this is just a reparameterization of a curve), x with $4g\kappa^2 x$ and y with $4g\kappa^2 y$ (this is a dilation, i.e., the change of the unit of length). We obtain:

$$x = \phi - \sin\phi + const, \quad y = 1 - \cos\phi.$$

It is easy to see that this equation describes a *cycloid*, i.e., the trajectory of a point attached to the edge of a wheel that is rolling along a horizontal line without sliding. To be more precise, we obtained the reflected (in the horizontal line) trajectory.

A *cycloid* is a periodic curve with countably many cusps. A brachistochrone is a part of a cycloid extending from a cusp to a point strictly before the next cusp. Note that, at the point A, the tangent line to the brachistochrone is vertical. This is in accordance to intuitive understanding: at the initial moment, a sliding particle needs to accelerate as fast as possible. On the other hand, this property of a brachistochrone makes it useless for practical purposes (such as finding the optimal shape of a mountain tunnel etc.).

1.1.4 Problems

1.3. Deduce from Fermat's principle that, under reflection in any smooth curve in the plane (located in a homogeneous and isotropic medium), the angle of reflection equals the angle of incidence.

1.4. All walls in a room are reflecting. A source of light is located at a single point of the room. Is it possible to choose the shape of the room and the location of the source so that to keep some places dark (not illuminated by the light of the source, not even multiply reflected)?

1.5. Consider an isotropic medium occupying the upper half-plane $y > 0$, in which the speed of light is given by the formula $v(x, y) = y$. Find the shape of light rays in this medium.

1.6. Consider an isotropic medium occupying the interior of the unit disk, in which the speed of light is given by the formula $v(x, y) = 1 - x^2 - y^2$. Find the shape of light rays in this medium.

1.7. The plane is filled with homogeneous and isotropic medium. There is a mirror in the plane, whose shape is the graph of a function f. It is known that the vertical flow of light reaching the mirror from above is reflected so that all reflected rays meet at a common point (called the focus of the mirror). Find all sufficiently smooth functions f with this property.

1.8. There are two homogeneous and isotropic media in the plane. The common boundary of the media has the shape of the graph of a function f. It is known that the vertical flow of light propagating downwards is refracted so that the refracted rays meet at a common point. Rewrite this condition as a differential equation on the function f. (The speed of light in the upper medium equals c_1, and the speed of light in the lower medium equals c_2). Assuming that $f'(0) = 0$, find a parabola that gives the best approximation of f near 0.

1.2 The Huygens principle

So far, we studied trajectories of individual photons. However, it is also interesting to have a description of for the propagation of a light stop, consisting of many photons. Usually, there are too many of them. So that it is impossible to keep track of individual particles. However, we can keep track of the shape of the light spot.

Suppose that the photons start emanating from a point source in all possible directions. What will be the shape of the light spot after time t? Let $X(\mathbf{x}_0, t)$ denote the light spot (thought of as a subset of the plane), where the point $\mathbf{x}_0$ is the location of the source. More precisely, $X(\mathbf{x}_0, t)$ is the set of all points, which light can reach in time $\leqslant t$. We do not assume now that the medium is isotropic. It can be non-homogeneous and unisotropic.

The *Huygens principle* describes the evolution law of the shape of $X(\mathbf{x}_0, t)$. Mathematically, this principle can be stated as follows:

Theorem 1.1 (Huygens principle)**.** *For every splitting $t = t_1 + t_2$ of the time interval t into a sum of two nonnegative time intervals,*

$$X(\mathbf{x}_0, t) = \bigcup_{\mathbf{x}_1 \in X(\mathbf{x}_0, t_1)} X(\mathbf{x}_1, t_2)$$

We now try to translate the formula into plain English. Consider the light spot in time t_1. Into each point of this spot, we place an imaginary source of light, and let all these sources illuminate the space for time t_2. In this way, we obtain many *secondary light spots*. To obtain the light spot in time t, we need to take the union of all the secondary light spots. We will prove the Huygens principle assuming that Fermat's principle holds verbatim, i.e., the light chooses only those trajectories which correspond to the minimal time.

Observe that the right-hand side of the equality expressing the Huygens principle includes the set $X(\mathbf{x}_0, t_1)$. Indeed, $\mathbf{x}_1$ must necessarily lie in the light spot $X(\mathbf{x}_1, t_2)$.

Proof of the Huygens principle. Let $\mathbf{x}$ be taken from the left-hand side of the Huygens principle, i.e., $\mathbf{x} \in X(\mathbf{x}_0, t)$. The light emanated from $\mathbf{x}_0$ can reach the point $\mathbf{x}$ in time $s \leqslant t$. If $s \leqslant t_1$, then the point $\mathbf{x}$ belongs to the set $X(\mathbf{x}_0, t_1)$, which is, as we have seen, a subset of the right-hand side. On the other hand, if $s > t_1$, then $s = t_1 + s_2$, where $s_2 \leqslant t_2$. Consider the trajectory of a photon emanated from $\mathbf{x}_0$ and reaching $\mathbf{x}$. In time t_1, the photon reaches some point $\mathbf{x}_1 \in X(\mathbf{x}_0, t_1)$, and then in time $s_2 \leqslant t_2$, it reaches the point $\mathbf{x}$, from which we deduce that $\mathbf{x} \in X(\mathbf{x}_1, t_2)$. Therefore, $\mathbf{x}$ lies in the right-hand side.

We now suppose that $\mathbf{x}$ lies in the right-hand side of the formula expressing the Huygens principle. This means that the point $\mathbf{x}$ can be reached by a photon going first from $\mathbf{x}_0$ to some point $\mathbf{x}_1 \in X(\mathbf{x}_0, t_1)$, and then from $\mathbf{x}_1$ to $\mathbf{x} \in X(\mathbf{x}_1, t_2)$. We know that the time it takes to traverse the first trajectory does not exceed t_1, and the time it takes to traverse the second trajectory does not exceed t_2. The union of the two trajectories may fail to be a genuine trajectory of a photon. However, this union is a curve such that it takes time t or less to traverse it. This implies that a photon emanated from $\mathbf{x}_0$ reaches the point $\mathbf{x}$ in time $\leqslant t$ (by Fermat's principle). □

Example. Consider a homogeneous but unisotropic medium. In such a medium all light spots obtained in time t differ by parallel translations only. Therefore, there is a figure $X(t)$ depending only on t such that

$$X(\mathbf{x}_0, t) = \mathbf{x}_0 + X(t)$$

for every point $\mathbf{x}_0$. Here $\mathbf{x}_0 + X$ stands for the parallel translate of X by the vector $\mathbf{x}_0$, i.e., the set of points $\mathbf{x}_0 + \mathbf{x}$, $\mathbf{x} \in X$. The *Minkowski sum* of two sets A, $B \subset \mathbb{R}^n$ is defined as follows:

$$A + B = \{\mathbf{a} + \mathbf{b} \mid \mathbf{a} \in A,\ \mathbf{b} \in B\}.$$

For example, if B is the ball of radius $\varepsilon > 0$ centered at the origin, then $A + B$ is the ε-neighborhood of the set A.

1.9. Let A be the union of two straight line segments in the plane

$$[(0,0),(1,0)] \cup [(0,0),(0,1)].$$

Draw the sets $A + A$, $A + A + A$, $A + A + A + A$ and compare them with $2A$, $3A$, $4A$.

Let us resume our discussion of a homogeneous unisotroic medium. We found out that such a medium is described by one function $X(t)$, taking values in subsets of $\mathbb{R}^n$. The Huygens principle can be restated as the following functional equation:

$$X(t_1 + t_2) = X(t_1) + X(t_2)$$

for all t_1 and t_2. It is interesting to find all solutions of this equation under sufficiently general assumptions on the function $X(t)$. It is natural to impose a kind of continuous dependence of $X(t)$ on t. Observe that the function $t \mapsto tA$ is a solution of our equation only if the set A is convex (prove!).

Let us go back to the case of a general (i.e., non-homogeneous and unisotropic) medium. Following Hamilton, define the function

$$W(\mathbf{x}_0, \mathbf{x}) = \inf\{t \mid \mathbf{x} \in X(\mathbf{x}_0, t)\}.$$

If the point $\mathbf{x}$ has coordinates (x_0, y_0), and the point $\mathbf{x}$ has coordinates (x, y), then the function $W(\mathbf{x}_0, \mathbf{x})$ can be written as a function of four variables: $W(x_0, y_0, x, y)$. In other words, $W(\mathbf{x}_0, \mathbf{x})$ is the minimal time it takes light to go from $\mathbf{x}_0$ to $\mathbf{x}$. If Fermat's principle holds verbatim, then this is simply the time, in which the light goes from $\mathbf{x}_0$ to $\mathbf{x}$, and this time is automatically minimal. In real life, this assertion is generally true only if the points $\mathbf{x}_0$ and $\mathbf{x}$ are sufficiently close (and even in this case, it is not always true). Note that the set $X(\mathbf{x}_0, t)$ can be given by the inequality

$$W(\mathbf{x}_0, \mathbf{x}) \leqslant t.$$

The surface $\Gamma_t(\mathbf{x}_0)$ given by the equation $W(\mathbf{x}_0, \mathbf{x}) = t$ for a fixed t is called a *light front*. The Huygens principle is often stated for light fronts rather than for light spots. In this case, one has to speak about the envelope of light fronts.

We now give this formulation for the plane. Suppose we are given a family of curves C_λ in the plane depending on one parameter λ. For example, such a family can be defined parametrically, if the curve C_λ is given by $x = x(t, \lambda)$, $y = y(t, \lambda)$ in terms of a parameter t. Here t is a parameter on the curve, and λ is a parameter indicating the choice of the curve C_λ among all curves of our one-parameter family (thus, if we fix the curve, then the parameter λ is also fixed). A curve C is called an *envelope* of the family C_λ if it is tangent to all curves of the family, and every point of C is the tangency point with some C_λ. We can now state the Huygens principle for light fronts.

Theorem 1.2. *Suppose that all light fronts in the plane filled with a given medium are smooth and convex. Suppose also that any pair of points is connected by a light ray. Fix a splitting $t = t_1 + t_2$ of a time interval t into a sum of two nonnegative time intervals. Then the light front $\Gamma_t(\mathbf{x_0})$ is the envelope of the "secondary" light fronts $\Gamma_{t_2}(\mathbf{x}_1)$ emanating from points $\mathbf{x}_1$ of the "primary" light front $\Gamma_{t_1}(\mathbf{x}_0)$.*

Proof. Under the assumptions we have made, the light spot $X(\mathbf{x}_0, t)$ is always the convex hull of $\Gamma_t(\mathbf{x}_0)$. Let $\mathbf{x}_1$ be any point of the light front $\Gamma_{t_1}(\mathbf{x}_0)$. Note that $\Gamma_{t_2}(\mathbf{x}_1)$ has a point in common with $\Gamma_t(\mathbf{x_0})$, namely, the point, which the ray emanated from $\mathbf{x}_0$ and passing through $\mathbf{x}_1$ reaches in time

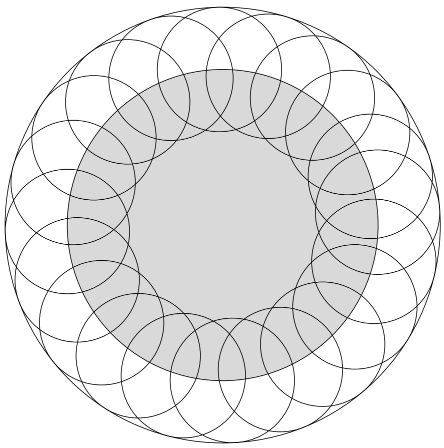

Figure 1.4: This picture illustrates Theorem 1.2. The interior of the primary light front is colored gray. The big circle in the picture is the envelope of all secondary fronts.

t. It suffices now to use the following fact: if one smooth curve lies in the convex hull of another smooth curve, and the two curves have a point in common, then they are tangent at this point. □

The *indicatrix* $I(\mathbf{x}_0)$ (of a given medium) at a point $\mathbf{x}_0$ is defined as the surface formed by the tips of all vectors $\mathbf{v}$ that are velocity vectors of photons passing through $\mathbf{x}_0$ in all possible directions. In each direction, there goes one such vector. For example, if the medium is isotropic, then the indicatrix is a ball. In general, we will assume that the indicatrices are smooth surfaces. An indicatrix can be thought of as a shape of infinitesimal light front. Namely, under some natural assumptions,

$$I(\mathbf{x}_0) = \lim_{t \to 0} \frac{1}{t} \left(\Gamma_t(\mathbf{x}_0) - \mathbf{x}_0 \right).$$

The set $\Gamma_t(\mathbf{x}_0) - \mathbf{x}_0$ is the light front translated so that the center coincides with the origin. What the limit of a family of sets means should of course be specified. However, we will give neither the precise statement of the result nor the precise assumptions, under which it holds. This is sufficiently cumbersome, and we are anyway discussing informal motivations only at this point. We confine ourselves with the remark that every point in the left-hand side can be obtained as the limit of some points from the right-hand side. Indeed, every element of the indicatrix is the velocity vector $\mathbf{v}$ of some ray. In time t, this ray reaches a certain point $\mathbf{x}(t) \in \Gamma_t(\mathbf{x}_0)$. By the definition of the velocity, we have

$$\mathbf{v} = \lim_{t \to 0} \frac{\mathbf{x}(t) - \mathbf{x}_0}{t}.$$

Theorem 1.3. *Consider a ray emanating from the point* $\mathbf{x}_0$ *and passing through the point* $\mathbf{x}$*. Consider also the front* $\Gamma = \{\mathbf{y} \in \mathbb{R}^2 \mid W(\mathbf{x}_0, \mathbf{y}) = t\}$*, containing the point* $\mathbf{x}$*. Let* $\mathbf{v}$ *be the velocity vector of light at the point* $\mathbf{x}$ *tangent to the given ray. Then the tangent line of the indicatrix* $I(\mathbf{x})$ *at the point* $\mathbf{v} \in I(\mathbf{x})$ *is parallel to that of the front at the point* $\mathbf{x}$*.*

We will only give a heuristic argument convincing that this must be true for the majority of good (in the optical sense) media. Consider a very small time interval Δt. The front $\Gamma_{\Delta t}(\mathbf{x})$ is approximated by $(\Delta t)I(\mathbf{x}) + \mathbf{x}$. We know from Theorem 1.2 that the tangent line of the front $\Gamma_{\Delta t}(\mathbf{x})$ at a point of the ray passing through $\mathbf{x}$ with the velocity $\mathbf{v}$ coincides with the tangent line to $\Gamma_{t+\Delta t}(\mathbf{x}_0)$ at the same point (recall that the point $\mathbf{x}_0$ is the source of the front $\Gamma = \Gamma_t(\mathbf{x}_0)$). As $\Delta t \to 0$, the first tangent line converges to the tangent line of the indicatrix $I(\mathbf{x})$ at the point $\mathbf{v}$, and the second tangent line converges to the tangent line of Γ at the point $\mathbf{x}$. We will not give a formal proof of the theorem (to this end, we would first need to make the statement more precise), however, the theorem should be intuitively clear (if it is not clear yet, try to draw pictures).

Theorem 1.4. *Suppose that the plane is filled with an isotropic medium, in which the speed of light is given by a function* $v(x, y)$*. Then the velocity of a light particle emanated from a point* (x_0, y_0) *at a point* (x, y) *equals*

$$\mathbf{v} = v(x, y)^2 \left(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y} \right),$$

where $W = W(x_0, y_0, x, y)$ *(the derivatives are taken for fixed* x_0 *and* y_0*).*

For example, if $v(x, y) = 1$ at all points (x, y), then $W(x_0, y_0, x, y)$ equals to the length of the line interval connecting the points (x_0, y_0) and (x, y), i.e.,

$$W(x_0, y_0, x, y) = \sqrt{(x - x_0)^2 + (y - y_0)^2}.$$

Set $f(x, y) = W(0, 0, x, y)$. It is clear that the gradient of the function f is directed radially and has length 1. This corresponds to the fact that the light emanated from the origin propagates along straight lines with constant speed.

Proof of Theorem 1.4. Observe that the vector with coordinates $\frac{\partial W}{\partial x}(x_0, y_0, x, y)$ and $\frac{\partial W}{\partial y}(x_0, y_0, x, y)$ is perpendicular to the light front at the point (x, y) (we mean the front emanated from the point (x_0, y_0) and containing the point (x, y)). This is a consequence of the following general statement: the vector $(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y})$ is perpendicular to a level curve of f.

Let $\mathbf{v}$ be the velocity vector of light at the point (x, y), directed along the ray emanating from (x_0, y_0). Since the medium is isotropic, all its indicatrices are round disks. In particular, the tangent line of the indicatrix at a point $\mathbf{v}$ is perpendicular to the vector $\mathbf{v}$. By Theorem 1.3, the tangent line of the front at the point (x, y) is parallel to the tangent line to the indicatrix at the point $\mathbf{v}$, i.e., is perpendicular to the vector $\mathbf{v}$. Thus the vector $\mathbf{v}$ is perpendicular to the light front. As we have already seen, the vector with coordinates $\frac{\partial W}{\partial x}(x_0, y_0, x, y)$ and $\frac{\partial W}{\partial y}(x_0, y_0, x, y)$ is also perpendicular to the light front. Therefore, these two vectors are proportional.

It remains to find the coefficient of proportionality. To this end, it suffices to compute the length of the vector with coordinates $\frac{\partial W}{\partial x}(x_0, y_0, x, y)$ $\frac{\partial W}{\partial y}(x_0, y_0, x, y)$. This is the gradient of the function W, equal to the time it takes light to reach the point (x, y) from the point (x_0, y_0). Thus the length of the gradient is approximated by $\Delta t/\Delta s$, where Δs is the length of a small arc of a ray, and Δt is the time it takes the photon to traverse this arc (we already know that the gradient of the function W is directed along along the ray emanating from the point (x, y) — we use this fact here). On the other hand, the speed of light at the point (x, y) is approximated by $\Delta s/\Delta t$. We can deduce from here that the length of the vector $(\frac{\partial W}{\partial x}, \frac{\partial W}{\partial y})$ equals $1/v(x, y)$, where $v(x, y)$ is the speed of light at the point (x, y).

The length of the vector $\mathbf{v}$ equals $v(x, y)$. Thus the coefficient of proportionality equals $v(x, y)^2$. □

It follows from the proof given above that the function W satisfies the following partial differential equation (called the *eikonal equation*):

$$\left(\frac{\partial W}{\partial x}\right)^2 + \left(\frac{\partial W}{\partial y}\right)^2 = \frac{1}{v(x, y)^2}.$$

Indeed, in the left-hand side, we have the length squared of the gradient of W (the latter is viewed as a function of x and y, with fixed x_0 and y_0). However, we have already found out that the length of the gradient equals $1/v(x, y)$.

Chapter 2

Action functions and Hamiltonians

2.1 The least action principle

A motion of any mechanical system is described by a variational principle somewhat similar to Fermat's principle. This variational principle is called *the least action principle* and is often attributed to Hamilton (although the principle was known earlier to Lagrange). A position of a mechanical system can be associated with a point of a configuration space. We will now assume that the configuration space coincides with $\mathbb{R}^n$. Note that the case $n > 3$ is physically meaningful, since the mechanical system may comprise several points. A motion of the system corresponds to a motion of a point in the configuration space, i.e., to a smooth path $\gamma : [t_0, t_1] \to \mathbb{R}^n$. Suppose that the initial and the terminal moments t_0 and t_1, respectively, are fixed. The least action principle states that the trajectory γ of the mechanical system is such that the integral

$$\int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))\, dt$$

along this trajectory, called the *action functional*, is minimal. When looking for a minimum, we only compare smooth paths with fixed t_0, t_1, and with fixed $q_{(0)} = \gamma(t_0)$, $q_{(1)} = \gamma(t_1)$. Here L is some function on $\mathbb{R}^n \times \mathbb{R}^n$ called a *Lagrange function* (or a *Lagrangian*).

The least action principle needs some corrections similar to those necessary for Fermat's principle. In particular, we should impose not the minimal value of the action functional but rather a stationary value.

A significant difference of the least action principle from Fermat's principle is the fact that the minimization happens with fixed initial and terminal moments of time. However, one can get rid of the time parameter and state a variational principle, similar to Fermat's principle, that would just describe the shape of a trajectory in the configuration space. After the trajectory has been found, the parameterization is recovered from the conservation of energy.

The arguments of a Lagrange function will be denoted by $q \in \mathbb{R}^n$ and $\dot{q} \in \mathbb{R}^n$. The argument q has the meaning of a position (or the tuple of coordinates describing the position), and $\dot{q}$ has the meaning of a velocity. We need to stress however that $\dot{q}$ is to be understood as an independent set of variables rather than a velocity vector of a particular trajectory.

Theorem 2.1 (Euler–Lagrange equations)**.** *Consider a sufficiently smooth path $\gamma : [t_0, t_1] \to \mathbb{R}^n$ minimizing the action functional (we are currently not discussing the question of existence of such a path). Then*

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\left(\gamma(t), \frac{d}{dt}\gamma(t)\right) = \frac{\partial L}{\partial q}\left(\gamma(t), \frac{d}{dt}\gamma(t)\right).$$

If q is just a number, i.e., if $n = 1$, then $\frac{\partial L}{\partial q}$ is the usual partial derivative of the function L with respect to q. If q is a tuple of coordinates $(q_1, \dots, q_n)$, then the expression $\frac{\partial L}{\partial q}$ should be understood as the tuple of partial derivatives $(\frac{\partial L}{\partial q_1}, \dots, \frac{\partial L}{\partial q_n})$, or, better yet, as the differential of the function L restricted to some fixed values of $\dot{q}$ (if q and $\dot{q}$ are fixed, then this is a linear functional on the space $\mathbb{R}^n$). The left-hand side of the equation displayed in the theorem is obtained as follows. We first differentiate the function $L(q, \dot{q})$ by $\dot{q}$, then substitute $\gamma(t)$ for q and $\frac{d}{dt}\gamma(t)$ for $\dot{q}$, and, finally, differentiate the expression thus obtained by t (note that the obtained expression is a function of t only).

Proof. Consider a one-parameter family of paths (=parameterized curves) connecting the points $q_{(0)}$, $q_{(1)}$ and parameterized by the interval $[t_0, t_1]$. We will write $t \mapsto \gamma(t, s)$ to denote the path labeled by s. Thus, t is a parameter along the path (having the meaning of time) and s is a parameter labeling the path itself (thus, variation of this parameter corresponds to a variation of the path).

Suppose that the function $\gamma(t, s)$ is a sufficiently many times differentiable function of two variables. The differentiation by t will be written as a dot, and the differentiation by s as the symbol δ. We have

$$0 = \delta \int_{t_0}^{t_1} L(\gamma(t,s), \dot\gamma(t,s))\, dt = \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}\delta\gamma + \frac{\partial L}{\partial \dot q}\delta\dot\gamma \right) dt =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q}\delta\gamma - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q} \right)\delta\gamma \right) dt + \frac{\partial L}{\partial \dot q}\cdot \delta\gamma \,\Big|_{t_0}^{t_1} =$$

$$= \int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q} \right) \right)\delta\gamma\, dt.$$

The first equality holds for a parameter s, for which the action functional takes the minimal value on the path $t \mapsto \gamma(t,s)$ (we may assume, e.g., that this minimal value is attained at $s = 0$). This equality follows from the fact that the derivative of a differentiable function at a point of a minimum must vanish. The second equality is a differentiation under the integral sign. The product $\frac{\partial L}{\partial q}\delta q$ means the value of the linear functional $\frac{\partial L}{\partial q}$ at the vector δq. In coordinates, this value can be written as

$$\frac{\partial L}{\partial q}\delta q = \sum_{i=1}^{n} \frac{\partial L}{\partial q_i}\delta q_i.$$

The third equality is the integration by parts. In the fourth equality, we use that $\gamma(s,t_0) = q_{(0)}$ does not depend on s, and that $\gamma(s,t_1) = q_{(1)}$ does not depend on s, hence, $\delta\gamma = 0$ for $t = t_0$ and for $t = t_1$.

Observe finally that if $t \mapsto \gamma(t)$ is a sufficiently smooth path minimizing the action functional, and $t \mapsto \alpha(t)$ is any sufficiently smooth mapping of $[t_0, t_1]$ to $\mathbb{R}^n$ such that $\alpha(t_0) = \alpha(t_1) = 0$, then the family of paths

$$\gamma(t,s) = \gamma(t) + s\alpha(t),$$

is such that $\delta\gamma = \alpha$ for $s = 0$. Thus we have proved that the integral

$$\int_{t_0}^{t_1} \left(\frac{\partial L}{\partial q} - \left(\frac{d}{dt}\frac{\partial L}{\partial \dot q} \right) \right)\alpha\, dt$$

vanishes for any sufficiently smooth mapping $\alpha : [t_0, t_1] \to \mathbb{R}^n$ taking zero values at the endpoints of $[t_0, t_1]$. It follows that the integrand vanishes identically, i.e.,

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot q} = 0,$$

if, instead of q and $\dot q$ (even before differentiating by t!) we substitute $\gamma(t)$ and $\dot\gamma(t)$. □

In the proof of the theorem, we have used the following lemma:

Lemma 2.2. *Let $F : [t_0, t_1] \to \mathbb{R}^{n*}$ be a continuous mapping. If, for any sufficiently smooth mapping $\alpha : [t_0, t_1] \to \mathbb{R}^n$ such that $\alpha(t_0) = \alpha(t_1) = 0$, we have*

$$\int_{t_0}^{t_1} F(t)\alpha(t)dt = 0,$$

then F is identically equal to zero. Here $F(t)\alpha(t)$ is the value of the canonical pairing between the co-vector $F(t)$ and the vector $\alpha(t)$.

A proof of the lemma is not hard, and is left to the reader.

Let $\gamma : [t_0, t_1] \to \mathbb{R}^n$ be a smooth path minimizing the action functional among all paths satisfying $\gamma(t_0) = q_{(0)}$, $\gamma(t_1) = q_{(1)}$. We will call such a path *optimal*. The least action principle in the (uncorrected) form it was stated above states that a mechanical system always moves along an optimal path. In reality, this is not always so. A *genuine trajectory* (sometimes abbreviated as trajectory) is a path satisfying the Euler–Lagrange equations. Because of corrections to the least action principle, mechanical systems can move along genuine trajectories that are not optimal paths. The corrected least action principle can be stated as follows: a motion of a mechanical system satisfies the Euler–Lagrange equations, i.e., it is a genuine trajectory.

Example. Suppose that the Lagrangian of a mechanical system does not depend on coordinates, i.e., depends on the velocities only (this is similar to studying optical properties of a homogeneous unisotropic medium) In this case, the Lagrangian has the form $T(\dot{q})$. We assume that the mapping $\frac{\partial T}{\partial \dot{q}} : \mathbb{R}^n \to \mathbb{R}^n$ (i.e., the first differential of the function T; formally speaking, this is a mapping of the space $\mathbb{R}^n$ to the dual space $\mathbb{R}^{n*}$) is invertible. The Euler–Lagrange equations take the form

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} = 0.$$

This means that $\frac{\partial T}{\partial \dot{q}}$ is constant along trajectories, i.e., for every trajectory γ, we have $\frac{\partial T}{\partial \dot{q}}(\dot{\gamma}(t)) = const$. This and the invertibility of the mapping $\frac{\partial T}{\partial \dot{q}}$ implies that $\dot{\gamma}(t)$ does not depend on t, i.e., the genuine motion is uniform and rectilinear.

Example. Let now $L(q, \dot{q})$ be a quadratic form of $\dot{q} \in \mathbb{R}^n$, whose coefficients may depend on $q \in \mathbb{R}^n$ (sufficiently smoothly). By the Euler theorem on homogeneous functions,

$$\dot{q} \cdot \frac{\partial L}{\partial \dot{q}} = \sum_{i=1}^{n} \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2L.$$

For q and $\dot{q}$, we substitute $q(t)$ and $\dot{q}(t)$ — the position and the velocity at time t of a point moving along a genuine trajectory. Now differentiate both sides by t:

$$\ddot{q}(t) \cdot \frac{\partial L}{\partial \dot{q}} + \dot{q}(t) \cdot \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \cdot \dot{q}(t) + \frac{\partial L}{\partial \dot{q}} \cdot \ddot{q}(t) + \frac{dL}{dt}.$$

We did the following trick here: it was $2L$ in the RHS, we differentiated one L according to the chain rule, and kept the derivative of the other L as is. Using the Euler–Lagrange equations and killing similar terms in the RHS and the LHS, we obtain that $\frac{d}{dt}L = 0$. This mean that the quantity $L(q(t), \dot{q}(t))$ is constant along trajectories, i.e., does not depend on time. By definition, this means that the function $L(q, \dot{q})$ is a *first integral* of the Euler–Lagrange equations, or, to put it differently, the first integral of the motion with the Lagrangian L.

We may view $L(q, \dot{q})$ as the energy of a free particle with velocity $\dot{q}$ in a curved space. The fact that the energy depends on q has to do with the space being curved. An important fact, however, is that the kinetic energy depends quadratically on the velocity.

A free motion is characterized by the absence of the potential energy. In a flat (Euclidean) space, the kinetic energy of a particle has the form

$$L(q, \dot{q}) = \frac{m|\dot{q}|^2}{2},$$

where $m > 0$ is the mass of a particle. We may assume without loss of generality that the mass is equal to one. The number $|\dot{q}|$ is the speed — the absolute value of the velocity measured with respect to the Euclidean metric.

We now assume that, for every $q \in \mathbb{R}^n$, the quadratic form $\dot{q} \mapsto L(q, \dot{q})$ has the following property: $L(q, \dot{q}) > 0$ for all nonzero vectors $\dot{q}$. Then $L(q, \dot{q})$ is the modulus squared of the vector $\dot{q}$ with respect to some Euclidean metric. However, this metric depends on the point q. Such metrics (Euclidean metrics depending on a point) are called *Riemannian metrics.* Thus, our Lagrangian function is a Riemannian metric. In the sense of this metric, the square of the length of the vector $\dot{q}$ at the point q is equal to $L(q, \dot{q})$. It is not hard to compute the length of the curve $\gamma : [t_0, t_1] \to \mathbb{R}^n$ with respect to the metric L:

$$\mathrm{Len}(\gamma) = \int_{t_0}^{t_1} \sqrt{L(\gamma(t), \dot{\gamma}(t))}\, dt.$$

Thus the length of a curve also has the form of an action functional, however, this action corresponds not to the Lagrangian L but rather to the Lagrangian $\sqrt{L}$. Optimal paths for the Lagrangian $\sqrt{L}$ are called *shortest paths.* These are the paths of the least length among all paths connecting the given pair of points (the endpoints of a shortest path).

Observe that the length of a curve does not depend on its parameterization. In particular, if $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is a shortest path, then so is any path obtained from γ by re-parameterization, i.e., every path of the form $\tilde{\gamma} : [s_0, s_1] \to \mathbb{R}^n$, where $\tilde{\gamma}(s) = \gamma(h(s))$ for some homeomorphism $h : [s_0, s_1] \to [t_0, t_1]$. However, a shortest path γ carries a natural parameter, in which the speed is constant, i.e, $L(\gamma(t), \dot{\gamma}(t))$ does not depend on t. Genuine trajectories of the Lagrangian L are called *geodesics* of the Riemannian metric L.

Theorem 2.3. *Let L be a Riemannian metric. The geodesics of the metric L are also trajectories of the Lagrangian $\sqrt{L}$. Conversely, if a trajectory of the Lagrangian $\sqrt{L}$ (e.g., a shortest path) has the property that $L = const$ along these trajectories, then this trajectory is also a geodesic of the Riemannian metric L.*

Proof. We have:

$$\frac{\partial\sqrt{L}}{\partial q} = \frac{1}{2\sqrt{L}}\frac{\partial L}{\partial q}, \quad \frac{\partial\sqrt{L}}{\partial \dot{q}} = \frac{1}{2\sqrt{L}}\frac{\partial L}{\partial \dot{q}},$$

Consider a geodesic of the Riemannian metric L, i.e. a trajectory of L. As we know, L restricted to this trajectory does not depend on time. We need to verify that the considered trajectory satisfied the Euler–Lagrange equations for the Lagrangian $\sqrt{L}$, i.e., we have $\frac{d}{dt}\frac{\partial\sqrt{L}}{\partial\dot{q}} = \frac{\partial\sqrt{L}}{\partial q}$. The LHS equals $\frac{1}{2\sqrt{L}}\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}$, since $L = const$ with respect to t. The RHS equals $\frac{1}{2\sqrt{L}}\frac{\partial L}{\partial q}$. Thus the Euler–Lagrange equations for the Lagrangian $\sqrt{L}$ follow from those for the Lagrangian L.

The proof of the converse statement is similar. □

Thus, every shortest path admits a re-parameterization, in which it becomes a geodesic. However, not every geodesic is a shortest path. For example, geodesics in the sphere are arcs of great circles (i.e., circles obtained as sections of the sphere by planes through the center). However, an arc of a great circle is a shortest path only if its length is less than the half the length of the circle.

2.1.1 Problems

2.1. Write down a system of differential equations in polar coordinates satisfied by any straight line. More precisely, let $x(t)$, $y(t)$ be the affine coordinates of a point in a line, i.e., $x(t) = a + bt$, $y(t) = c + dt$ for some constants a, b, c, $d \in \mathbb{R}$, such that $b^2 + d^2 \neq 0$. Let $r(t)$, $\phi(t)$ denote the corresponding polar coordinates. Find differential equations on the functions $r(t)$, $\phi(t)$.

2.2. Find the trajectories of the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \frac{(\dot{x})^2 + (\dot{y})^2}{2} + y.$$

2.3. Consider the Lagrangian defined in the upper half-plane by the formula

$$L(x, y, \dot{x}, \dot{y}) = \frac{(\dot{x})^2 + (\dot{y})^2}{y^2}$$

Prove that the trajectories of this Lagrangian are half-circles centered at the horizontal line $y = 0$.

2.4. Prove that the trajectories of the Lagrangian

$$L(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 + \frac{1}{\sqrt{x^2 + y^2}}$$

are subsets of degree two algebraic curves.

2.5. A particle slides on the surface of the sphere with no friction and no external forces. Find its Lagrangian in the spherical coordinates.

2.6. Write down (differential) equations for geodesics on the paraboloid of revolution $z = x^2 + y^2$.

2.7. Prove that, for any positive R and a, the geodesics of the Riemannian metric

$$L(x, y, \dot{x}, \dot{y}) = R^2 \frac{(a^2 + x^2)\dot{y}^2 + 2xy\dot{x}\dot{y} + (a^2 + y^2)\dot{x}^2}{(a^2 + x^2 + y^2)^2}$$

are intervals of straight lines.

2.8. *A minimal surface of revolution.* Consider all surfaces of revolution, whose axis coincides with the z-axis, and whose boundary coincides with the union of two circles centered at the z-axis. It follows automatically from this requirement that the bounding circles lie on the planes perpendicular to the z-axis. Suppose that, among all such surfaces, one can find a surface of the smallest area. Find this surface.

2.2 Legendre transform and Hamiltonians

First consider a sufficiently many times differentiable function $f : \mathbb{R} \to \mathbb{R}$ of one real variable. The graph of f can be recovered from the set of all tangent lines to the graph. The idea of the Legendgre transform is passing from f to the family of all tangent lines to the graph of f. Suppose that f is *strictly convex*, i.e., the derivative f' is strictly increasing. Then, for every

$p \in \mathbb{R}$, there is a unique tangent line of the graph of f, whose slope is equal to p. We can write the equation of this tangent line as $y = px - g(p)$. Here $g(p)$ is just a constant coefficient, for every fixed p. However, this coefficient depends on p. Thus the function $p \mapsto g(p)$ defines a family of lines in the plane such that no line of this family is vertical (in our particular case, we obtain the family of all tangent lines of the graph of f). The g is called *the Legengre transform* of the function f, and is often denoted by $\hat{f}$. The term "Legendre transform" may also refer to the passage from the function f to the function $\hat{f}$.

The Legendre transform is a particular case of a more general construction that associates, say, a curve in the plane, with the family of all tangent lines of this curve. This construction is very useful in algebraic geometry (as well as in some branches of mathematical physics), and is called the *projective duality.*

People often use a different definition of the Legendre transform, which we are about to give. Observe that the graph of f lies always above any tangent line of it. This fact can be rewritten as the inequality $f(x) \geqslant px - \hat{f}(p)$, in which the equality is attained at a unique point $x(p)$ — the point of tangency. An equivalent inequality is the following:

$$\hat{f}(p) \geqslant px - f(x).$$

In this inequality, the equality is also attained at a unique point $x(p)$. Therefore, the function $\hat{f}(p)$ can be also defined by the following formula:

$$\hat{f}(p) = \min(px - f(x)).$$

Let us compute the derivative of the Legendre transform:

$$\frac{d}{dp}\hat{f}(p) = \frac{d}{dp}\left(px(p) - f(x(p))\right) = x(p) + p\frac{dx(p)}{dp} - f'(x(p))\frac{dx(p)}{dp}$$

Note however that $f'(x(p)) = p$ (the derivative of a function coincides with the slope of the corresponding tangent line). Therefore, the computation performed above shows that $\frac{d}{dp}\hat{f}(p) = x(p)$, i.e., the derivative of the Legendre transform of f at a point p coincides with the x-coordinate of the tangency point of the graph of the function f and its tangent line, whose slope equals p.

The Legendre transform is also defined for functions of many variables. Consider an open convex subset $U \subseteq \mathbb{R}^n$ and a sufficiently smooth function $f : U \to \mathbb{R}$. We say that the function f is *strictly convex* if for any two points $a, b \in \mathbb{R}^n$ the entire line segment with endpoints $(a, f(a))$ and $(b, f(b))$ lies above the graph of f, except for the endpoints. The first differential $d_{\mathbf{x}} f$ of the function f at a point $\mathbf{x} \in \mathbb{R}^n$ is a linear functional on $\mathbb{R}^n$, i.e., an element of the space $\mathbb{R}^{n*}$. Let $\hat{U} \subseteq \mathbb{R}^{n*}$ denote the set of all linear functionals of the form $d_{\mathbf{x}} f$, where $\mathbf{x} \in U$. For every element $p \in \hat{U}$, there exists a unique tangent hyperplane of the graph of f, given by $y = p \cdot \mathbf{x} - g(p)$. In this formula, the product $p \cdot \mathbf{x}$ means the value of the linear functional p at the vector $\mathbf{x}$. In coordinates, if $\mathbf{x} = (x_1, \ldots, x_n)$ and $p = (p_1, \ldots, p_n)$, then $p \cdot \mathbf{x} = p_1 x_1 + \cdots + p_n x_n$. The function $g : \hat{U} \to \mathbb{R}$ is called the *Legendre transform* of f and is often denoted by $\hat{f}$. An equivalent definition of the Legendre transform is that

$$\hat{f}(p) = \min \left(p \cdot \mathbf{x} - f(\mathbf{x}) \right).$$

The minimum is attained at a unique point $\mathbf{x}(p)$. Thus we have

$$\hat{f}(p) = p \cdot \mathbf{x}(p) - f(\mathbf{x}(p)).$$

Moreover, similarly to the case of one-variable functions, we obtain that the first differential of the function $\hat{f}$ at a point $p \in \mathbb{R}^{n*}$ coincides with the vector $\mathbf{x}(p)$ (under the natural identification of the dual space of $\mathbb{R}^{n*}$ with the space $\mathbb{R}^n$).

Now consider the Lagrangian $L(q, \dot{q})$ of some mechanical system. Suppose that, for any fixed q, this function is a strongly convex function of $\dot{q}$. We let $H(q, p)$ denote the Legendre transform of this function (as a function of $\dot{q}$ with q fixed). The function $H(q, p)$ is called the *Hamiltonian function*, or a *Hamiltonian*. By definition of the Legendre transform, the Hamiltonian is given by the formula

$$H(p, q) = p \cdot \dot{q} - L(q, \dot{q}),$$

in which $\dot{q}$ is expressed through p by the formula $p = \frac{\partial L}{\partial \dot{q}}$ (due to strict convexity, this equation is always solvable for $\dot{q}$). The Hamiltonian is defined on some open subset of $\mathbb{R}^{n*} \times \mathbb{R}^n$. We will often assume that this open subset is the whole of $\mathbb{R}^{n*} \times \mathbb{R}^n$.

The space $\mathbb{R}^{n*} \times \mathbb{R}^n$ is called the *phase space*. Consider a genuine trajectory $q(t)$ in the *configuration space* $\mathbb{R}^n$. It defines a *phase trajectory* $(p(t), q(t))$ in the phase space, where $p(t)$ is defined as $\frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t))$. The vector $p(t)$ is called the *momentum* of the system at time t.

Theorem 2.4 (Hamilton's equations)**.** *Let $q(t)$, $p(t)$ be the position and the momentum of a system at time t (we of course assume that the system is moving along a genuine trajectory). Then*

$$\dot{q}(t) = \frac{\partial H}{\partial p}(p(t), q(t)), \quad \dot{p}(t) = -\frac{\partial H}{\partial q}(p(t), q(t)).$$

Proof. The first equality follows from the formula for the first differential of a Legendre transform. Note that this equality does not make use of the Euler–Lagrange equations. It can be ultimately reduced to the definition of the momentum.

Consider the equality

$$H(p,q) = p \cdot \dot{q}(p,q) - L(q, \dot{q}(p,q)),$$

in which $\dot{q}(p,q)$ is found from the formula $p = \frac{\partial L}{\partial \dot{q}}(q, \dot{q}(p,q))$, or, more explicitly, from the formula $\dot{q}(p,q) = \frac{\partial H}{\partial p}(q,p)$. Differentiate this equality by q:

$$\frac{\partial H}{\partial q} = p \cdot \frac{\partial \dot{q}(p,q)}{\partial q} - \frac{\partial L}{\partial q}(q, \dot{q}(p,q)) - \frac{\partial L}{\partial \dot{q}}(q, \dot{q}(p,q)) \cdot \frac{\partial \dot{q}(p,q)}{\partial q}.$$

The first and the last terms in the RHS cancel each other, therefore,

$$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q}(q, \dot{q}(p,q)).$$

Now use the Euler–Lagrange equations:

$$-\frac{\partial L}{\partial q}(q(t), \dot{q}(p(t))) = -\frac{\partial L}{\partial q}(q(t), \dot{q}(t)) = -\frac{d}{dt}p(t).$$

We need to recall here that $p(t) = \frac{\partial L}{\partial \dot{q}}(q(t), \dot{q}(t))$. □

The system of Hamilton's equations is sometimes called the system of *canonical equations.*

Theorem 2.5 (Conservation of Energy)**.** *Let $q(t)$ and $p(t)$ be the position and the momentum of the system at time t. Then $H(p(t), q(t))$ does not depend on t (in other words, the mechanical energy is conserved).*

Proof. Indeed, differentiate $H(p(t), q(t))$ by t:

$$\frac{d}{dt}H(p(t), q(t)) = \frac{\partial H}{\partial q}\dot{q} + \frac{\partial H}{\partial p}\dot{p} =$$

$$= \frac{\partial H}{\partial q}\left(\frac{\partial H}{\partial p}\right) + \frac{\partial H}{\partial p}\left(-\frac{\partial H}{\partial q}\right) = 0.$$

In the second equality, we used Hamilton's canonical equations. □

One can consider Hamiltonians depending explicitly on time. For such Hamiltonians, the conservation of energy fails. It is replaced with the law $\frac{d}{dt}H(p(t), q(t), t) = \frac{\partial H}{\partial t}(p(t), q(t), t)$. In the right-hand side, the differentiation is performed only by the third argument t.

In most mechanical problems, the Lagrangian has the form

$$L(q, \dot{q}) = T_q(\dot{q}) - U(q),$$

where $T_q(\dot{q})$ is a positive definite quadratic form of $\dot{q}$, whose coefficients depend smoothly on q (i.e., a Riemannian metric on the configuration space), and the function $U(q)$ does not depend on $\dot{q}$, i.e., is simply a function on the configuration space. The physical meaning of the functions $T_q(\dot{q})$ and $U(q)$ is that the first function represents the kinetic energy of the system, and the second function represents the potential energy. The corresponding Hamiltonian has the form

$$H(q, p) = T_q(\dot{q}(p)) + U(q).$$

2.2.1 Problems

2.9. Find the Legendre transforms of the following functions:

(a) $f(x) = e^x$

(b) $f(x) = \frac{x^\alpha}{\alpha}$

Answer:
(*a*) $\hat{f}(p) = p(\log p - 1)$.
(*b*) $\hat{f}(p) = \frac{p^\beta}{\beta}$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$.

2.10. Find the Legendre transform of the function

$$f(x, y) = x^4 + y^4.$$

2.11. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable strictly convex function. Prove that the differentials of the function f at different points of the space $\mathbb{R}^n$ cannot be the same.

2.12. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function, whose second differential $d^2_{\mathbf{x}} f$ is positive definite for every point $\mathbf{x} \in \mathbb{R}^n$. Verify that, in this case, $d^2_{\mathbf{y}} \hat{f} > 0$ for all $\mathbf{y} \in \mathbb{R}^{n*}$.

2.13. Let $U \subset \mathbb{R}^n$ be an open convex set, and let $f : U \to \mathbb{R}$ be a strictly convex function. Recall that $\hat{U}$ denotes the domain of the function $\hat{f}$. Prove that for every $x \in \mathbb{R}^n$ and every $p \in \hat{U}$, we have

$$f(x) + \hat{f}(p) \geqslant p \cdot x.$$

This inequality is called the *Young inequality*. Setting $f(x) = x^\alpha/\alpha$, where $\alpha > 1$, we obtain the classical Young inequality (we use the already performed computation of the Legendre transform for f):

$$\frac{x^\alpha}{\alpha} + \frac{p^\beta}{\beta} \geqslant p \cdot x, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

2.14. Suppose that the Legendre transform $\hat{f}$ of a strictly convex sufficiently many times differentiable function f is defined on the whole of $\mathbb{R}^n$. Prove that the function $\hat{f}$ is also strictly convex, and the Legendre transform of the function $\hat{f}$ coincides with f.

2.15. Let $U \subset \mathbb{R}^n$ be an open subset such that $\lambda U \subseteq U$ for every $\lambda > 0$. The function $f : U \to \mathbb{R}$ is said to be *positively homogeneous of degree* k if

$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^k f(x_1, \ldots, x_n)$$

for every point $(x_1, \ldots, x_n) \in \mathbb{R}^n$ and every number $\lambda > 0$. Prove *Euler's theorem on homogeneous functions*: if f is a differentiable positively homogeneous function of degree k, then

$$\frac{\partial f}{\partial x} \cdot x = \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} = kf.$$

Euler's theorem on homogeneous functions implies the following.

2.16. Suppose that the Lagrangian of a mechanical system has the form

$$L(q, \dot{q}) = T_q(\dot{q}) - U(q),$$

where T_q is a quadratic form of $\dot{q}$, whose coefficients may depend (smoothly) on q, and U is a smooth function of q. (Almost all Lagrangians appearing in classical mechanics have this particular form). Then the corresponding Hamiltonian equals $T + U$, where the quadratic form T is expressed in terms of the momenta p rather than velocities $\dot{q}$.

2.17. Set $q = (q_1, \ldots, q_n)$ and $p = (p_1, \ldots, p_n)$. A coordinate q_1 is said to be *cyclic* for a Hamiltonian $H(q, p)$ if H does not depend explicitly on q_1, i.e. it is a function of $q_2, \ldots, q_n$, $p_1, \ldots, p_n$. Prove that if q_1 is a cyclic coordinate, then the corresponding momentum p_1 is conserved (i.e., does not change in time along every particular trajectory).

2.18. Consider a smooth function $u(x, y)$ satisfying the soap film equation

$$(1 + u_y^2)u_{xx} - 2u_x u_y u_{xy} + (1 + u_x^2)u_{yy} = 0.$$

Suppose that u admits a Legendre transform $\hat{u}$. Find a PDE on the function $\hat{u}$.

2.3 Action functions and Hamilton–Jacobi equations

Consider the *action function*

$$S(q_{(0)}, t_0, q_{(1)}, t_1) = \inf_{\gamma} \int_{t_0}^{t_1} L(\gamma(t), \dot{\gamma}(t))\, dt.$$

The infimum in the RHS is taken over all paths $\gamma : [t_0, t_1] \to \mathbb{R}^n$, such that $\gamma(t_0) = q_{(0)}$, $\gamma(t_1) = q_{(1)}$. If the path γ is optimal, then the action function coincides with the action functional evaluated at this particular path γ. In the sequel, when considering the action function, we will always assume than an optimal path exists. Under the similarity between classical mechanics and geometric optics, the action function corresponds to the function $W(\mathbf{x}, \mathbf{y})$ introduced above. Let us stress the difference between the action function and the action functional: in contrast to the action functional, the action function is a function of $q_{(0)}$, $q_{(1)}$ and t only; there is no dependence on a path γ.

Consider the action $S(q_{(0)}, t_0, q, t_1)$ as a function of q, i.e., fix all other arguments of this function. Let us find the differential of this function. We will use a remarkable argument borrowed from [**?**]. Let $t \mapsto \gamma(t, q)$ be an optimal path passing through $q_{(0)}$ at time t_0 and through q at time t_1. By definition, the action function equals

$$S(q_{(0)}, t_0, q, t_1) = \int_{t_0}^{t_1} L(\gamma(t, q), \dot{\gamma}(t, q))dt,$$

where, as usual, the dot means the t-derivative. Set $q = q_{(1)} + sa$ for some vector a, and differentiate the identity displayed above by s for $s = 0$. The s-derivative, as before, will be written as δ. When deriving the Euler–Lagrange equations, we have already performed this differentiation using the integration by parts. We now only recall the end result:

$$\delta S = \int_{t_0}^{t_1} \delta\gamma \left(\frac{\partial L}{\partial q} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} \right) dt + \left. \frac{\partial L}{\partial \dot{q}} \cdot \delta\gamma \right|_{t_0}^{t_1}.$$

Note that the first term in the RHS vanishes due to the Euler–Lagrange equations. The second term is equal to $p(q_{(1)}) \cdot a$. Recall that we view S as a function of q. The directional derivative of this function along the vector a at the point $q_{(1)}$ equals $p(q_{(1)}) \cdot a$. Therefore, the differential of the function S with respect to q equals $p(q)$:

$$\frac{\partial S}{\partial q} = p(q).$$

We now consider the function $S(q_{(0)}, t_0, q, t)$ as a function of q and t. Since $q_{(0)}$ and t_0 are fixed, we will simply write $S(q, t)$ instead of $S(q_{(0)}, t_0, q, t)$. In contrast to the arguments presented above, we now fix the terminal time t. In order to compute the partial derivative $\frac{\partial S}{\partial t}$, we employ the following argument, also borrowed from [**?**]. Consider a genuine trajectory $\tau \mapsto \gamma(\tau)$ passing at time $\tau = t$ through the point q. Differentiate the expression $S(\gamma(\tau), \tau)$ by τ at the point $\tau = t$. On the one hand, by definition of the action integral, the derivative of this expression is equal to the value $L(q, \dot{\gamma}(\tau))$ of the Lagrangian. On the other hand, by the chain rule,

$$L(q, \dot\gamma(\tau)) = \frac{d}{d\tau} S(\gamma(\tau), \tau) = \frac{\partial S}{\partial q}(q,t)\dot\gamma(t) + \frac{\partial S}{\partial t}(q,t)\bigg|_{t=\tau}.$$

Besides, we already know that the derivative $\frac{\partial S}{\partial q}(q,t)$ equals the momentum $p(t)$ of the system at time t. Thus, we have

$$\frac{\partial S}{\partial t}(q,t)\bigg|_{t=\tau} = L(q, \dot\gamma(t)) - p(t)\cdot\dot\gamma(t)|_{t=\tau} = -H(q, p(\tau)).$$

The formulas

$$\frac{\partial S}{\partial q} = p, \quad \frac{\partial S}{\partial t} = -H$$

can be regarded as an alternative definition of the momentum and the Hamiltonian (this is how Hamilton came up to the notion of a Hamiltonian). This definition is perhaps more natural from the geometric and mechanical viewpoints. However, for technical reasons, it is more convenient to work with the definition of a Hamiltonian through the Legendre transform.

The minus sign in the definition of the Hamiltonian $\frac{\partial S}{\partial t} = -H$ initiated a notational dispute between the physicists and the mathematicians in the 19th century. Physicists insisted. The reason for the negative sign is that the Hamiltonian measures the mechanical energy, and, for example, as you raise a stone, its energy increases rather than decreases. There was another notational dispute, in which, on the other hand, mathematicians were successful. Mathematicians used the letter q (from Latin qualitas) for the position, and the letter p (from Latin potentia) for the momentum. The physicists were doing the opposite: denoted the coordinates by p, and the momenta by q. Nowadays, the notation of the mathematicians are commonly accepted. Let us stress again that, in physical problems, $H(q,p)$ is the mechanical energy expressed through the coordinates and the momenta.

The formulas $\frac{\partial S}{\partial q} = p$, $\frac{\partial S}{\partial t} = -H$ imply the following PDE on the action function, which is called the *Hamilton–Jacobi equation*:

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}\right) = 0.$$

The Hamilton–Jacobi equation is an analog of the eikonal equation

$$\left|\frac{\partial W}{\partial \mathbf{x}}(\mathbf{x}_{(0)}, \mathbf{x})\right| = \frac{1}{v(\mathbf{x})}$$

for an optically isotropic medium.

Suppose that the action function $S(q_{(0)}, t_0, q_{(1)}, t_1)$ is known. How to find the trajectories? This problem was solved by Hamilton, who used the optical analogy. Note that the derivatives of the action function by $q_{(0)}$ and t_0 can be computed by the formulas

$$\frac{\partial S}{\partial q_{(0)}} = -p_{(0)}, \quad \frac{\partial S}{\partial t_0} = H(q_{(0)}, p_{(0)}),$$

in which $p_{(0)}$ is the momentum of the particle (system) at time t_0 provided that the motion of the particle originates in $q_{(0)}$ at time t_0 and terminates in $q_{(1)}$ at time t_1. Observe that the values of these derivatives depend only on $q_{(0)}$, t_0 and on the trajectory originating at time t_0 in $q_{(0)}$. However, they are independent of the point $q_{(1)}$ of this trajectory and of the time t_1, at which the system passes through $q_{(1)}$. Thus the shape and the parameterization of trajectories can be found from the equations

$$\frac{\partial S}{\partial q_{(0)}} = const, \quad \frac{\partial S}{\partial t_0} = const.$$

This is the content of Hamilton's method of finding the trajectories. As we will see later, in order to find the trajectories, it is not necessary to know the action function, it suffices only to know sufficiently many sufficiently independent solutions of the Hamilton–Jacobi equation.

2.3.1 Problems

2.19. Set $L(q, \dot{q}) = T_q(\dot{q}) - U(q)$, where $T_q(\dot{q})$ is a Riemannian metric, and $U(q)$ is a smooth function. Find the corresponding Hamiltonian.

2.20. Suppose that the Lagrangian $L = T(\dot{q})$ is a non-degenerate quadratic form of $\dot{q}$ (whose coefficients do not depend on q). Find the action function.

2.21. Prove that all trajectories of the Hamiltonian system in $\mathbb{R}^2$ with the Hamiltonian $H(p, q) = p^4 + e^q$ are periodic, i.e., for every trajectory, there exists a number $T > 0$ such that $q(t + T) = q(t)$ and $p(t + T) = p(t)$ for all $t \in \mathbb{R}$.

2.22. Is it true that every trajectory of every Hamiltonian system in $\mathbb{R}^2$ is periodic?

Chapter 3

Basics of symplectic geometry

3.1 Smooth manifolds and vector fields: a reminder

Prior to discussing symplectic geometry, we need to recall smooth manifolds and differential forms. We assume that the readers have seen these notions earlier. Here, we only give a brief reminder.

A *smooth manifold* of dimension n is a set M, equipped with a space of functions $C^\infty(M)$ (called smooth functions on M) with the following properties:

1. for every point $a \in M$, there exists a subset $U \subseteq M$ containing the point a and a tuple of functions $x_1, \ldots, x_n \in C^\infty(M)$, such that the mapping
$$\varphi : U \to \mathbb{R}^n, \quad \varphi(q) = (x_1(q), \ldots, x_n(q)),$$
called a *local coordinate chart*, is injective, the subset $\varphi(U) \subseteq \mathbb{R}^n$ being open. The set U is called a *coordinate neighborhood* of the point a, and the functions $x_1, \ldots, x_n$ are called local coordinates near a. We impose that the restriction of every function $f \in C^\infty(M)$ to U have the form $F \circ \varphi$, where $F : \varphi(U) \to \mathbb{R}$ is a smooth (sufficiently many times differentiable) function.

2. If a manifold M is covered by coordinate neighborhoods, and the restriction of some function $f : M \to \mathbb{R}$ to each coordinate neighborhood is a smooth function of the local coordinates, then $f \in C^\infty(M)$.

The simplest and the most important example of a smooth manifold is the space $\mathbb{R}^n$, for which $C^\infty(\mathbb{R}^n)$ is defined as the set of functions that are everywhere defined and infinitely many times differentiable.

Let $M \subset \mathbb{R}^n$ be any subset. A function $f : M \to \mathbb{R}$ is called *smooth*, if, for every point $q \in M$, there exists a neighborhood U of q in $\mathbb{R}^n$ such that the restriction of f to $M \cap U$ coincides with the restriction of some smooth function to U.

3.1. Suppose that $M \subset \mathbb{R}^n$ is given by a single equation $F = 0$, where F is a smooth function on $\mathbb{R}^n$ such that $dF \neq 0$ on M. Verify that smooth functions on M, as defined above, equip M with a smooth manifold structure.

Every smooth manifold has a natural structure of a topological space, i.e., we can define open and closed subsets. Firstly, the notion of an open subset in a coordinate neighborhood is easy to define: this is the preimage of any open subset of $\mathbb{R}^n$ under the local coordinate chart. We now say that a subset of a smooth manifold is open if its intersection with any coordinate neighborhood is open.

3.2. Verify the axioms of a topological space.

Since every smooth manifold is a topological space, we can talk about connected, compact, etc., manifolds.

Let M and N be smooth manifolds. A mapping $\Phi : M \to N$ is called a *smooth mapping* if, for every smooth function f on N, the function $f \circ \Phi$ is a smooth function on M.

3.3. Let $a \in M$ and $b = \Phi(a) \in N$. Consider local coordinates $x_1, \ldots, x_n$ in a neighborhood of a and local coordinates $y_1, \ldots, y_m$ in a neighborhood of b. Then there exist smooth functions φ_i, $i = 1, \ldots, m$ defined on some neighborhood of the point $(x_1(a), \ldots, x_n(a)) \in \mathbb{R}^n$ such that

$$y_i \circ \Phi = \varphi_i(x_1, \ldots, x_n)$$

on some coordinate neighborhood of a.

3.4. Verify that the composition of smooth mappings is smooth.

We now give more examples of smooth manifolds. Consider the n-dimensional torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be $\mathbb{Z}^n$-periodic if $f(q + a) = f(q)$ for all $q \in \mathbb{R}^n$, $a \in \mathbb{Z}^n$. Clearly, every $\mathbb{Z}^n$-periodic function descends to a function on the torus T^n. By definition, the smooth functions on the torus are the functions that come from (lift to) smooth $\mathbb{Z}^n$-periodic functions on $\mathbb{R}^n$.

We can now generalize our example, in which the manifold was given by a single equation in $\mathbb{R}^n$. We now consider the set M of points $q \in \mathbb{R}^n$ given by a system of equations

$$f_1(q) = \cdots = f_k(q) = 0.$$

Suppose that the differentials $df_1, \ldots, df_k$ are linearly independent at every point $x \in M$. Then the set M is a smooth manifold (with the notion of a smooth function on M already defined above).

We will identify a vector $\mathbf{v}$ in $\mathbb{R}^n$ with coordinates $(v_1, \ldots, v_n)$, based at a point $q \in \mathbb{R}^n$ with the operator of differentiation along $\mathbf{v}$:

$$L_v : f \mapsto L_v f(q) = \lim_{t \to 0} \frac{f(q + vt) - f(q)}{t}.$$

This operator is linear over real numbers and satisfies the Leibnitz rule. The differentiation along a vector $\mathbf{v}$ is the differentiation along any path passing through q and such that the velocity vector of the path at q equals $\mathbf{v}$. More precisely, in order to compute $L_v f$, we restrict the function f to any smooth path $\gamma : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ such that $\gamma(0) = q \quad \dot{\gamma}(0) = v$, and then differentiate the restriction by t at $t = 0$:

$$L_v f(q) = \frac{d}{dt} f(\gamma(t))|_{t=0}.$$

On any smooth manifold, vectors can also be defined as derivations. Namely, a vector X tangent to a manifold M at a point q, is defined as a functional $X : C^\infty(M) \to \mathbb{R}$ with the following properties:

1. $X(\alpha f + \beta g) = \alpha X f + \beta X g$ for all $f, g \in C^\infty(M) \quad \alpha, \beta \in \mathbb{R}$ (linearity over reals),

2. $X(fg) = f(a) X g + g(a) X f$ (the Leibnitz rule).

We now establish some simple properties of vectors:

1. $X(1) = 0$, i.e., the derivative of the constant function identically equal to one along any vector is zero. Indeed, we have:

$$X(1) = X(1 \cdot 1) = 1 \cdot X(1) + 1 \cdot X(1) = 2X(1).$$

2. $X(c) = 0$ for every constant function c. This follows from the linearity of X.

3. Let $\varphi,\ \psi \in C^\infty(M)$ be functions such that $\varphi(q) = \psi(q) = 0$. Then $X(\varphi\psi) = 0$. Indeed,

$$X(\varphi\psi) = \varphi(q) X\psi + \psi(q) X\varphi = 0.$$

4. Let $\varphi \in C^\infty(M)$ vanish identically in some neighborhood of a point q. Then $X\varphi = 0$. Indeed, consider a smooth function ψ vanishing at the point q and equal identically to one outside of a neighborhood of q, in which $\varphi = 0$. Then $\varphi = \varphi\psi$. Therefore, $X\varphi = 0$.

5. If a function $\varphi \in C^\infty(M)$ is constant in some neighborhood of the point q, then $X\varphi = 0$. Indeed, the function $\varphi - \varphi(0)$ satisfies the assumptions of the previous paragraph.

The following assertion is called *Hadamard's Lemma*:

Theorem 3.1. *Let U be an open convex subset of $\mathbb{R}^n$, containing the origin, and $f \in C^\infty(U)$. Then*

$$f(x_1, \ldots, x_n) = f(0, \ldots, 0) + \sum_{i=1}^{n} x_i g_i(x_1, \ldots, x_n),$$

where g_i are some smooth functions on U.

Proof. We have:

$$f(x) - f(0) = \int_0^1 \frac{f(tx)}{dt} dt = \int_0^1 \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i}(tx)\, dt = \sum_{i=1}^{n} x_i g_i(x),$$

where $g_i(x) = \int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt$. □

Theorem 3.2. *Consider any smooth manifold M and a vector X tangent to the manifold M at a point $a \in M$. Let $(x_1, \ldots, x_n)$ be a local coordinate system in a neighborhood of the point a such that $x_i(a) = 0$. Set $Xx_i = \alpha_i$. Then*

$$Xf = \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}.$$

Proof. We may assume that $M = \mathbb{R}^n$, $a = 0$ and $f(0) = 0$. Then, by Hadamard's Lemma, $f = \sum x_i g_i$ for some smooth functions g_i. It follows from this representation that $\frac{\partial f}{\partial x_i}(0) = g_i(0)$. We have:

$$Xf = \sum_{i=1}^{n} X(x_i) g_i(0) = \sum_{i=1}^{n} \alpha_i \frac{\partial f}{\partial x_i}(0).$$

□

It follows from Theorem 3.2 that the set T_aM of all vectors at a fixed point a of a manifold M forms a vector space of dimension n. Indeed, as can be seen from Theorem 3.2, a tangent vector at a point a is completely determined by its coordinates $\alpha_1, \ldots, \alpha_n$, moreover, the mapping $T_aM \to \mathbb{R}^n$, associating with every vector its coordinates in a given local coordinate system is linear. The space T_aM is called the *tangent space* of M at the point a.

A *vector field* X on a manifold M is a mapping assigning to every point $q \in M$ some vector $X_q \in T_qM$. A vector field is said to be *smooth*, if, in every local coordinate system $(x_1, \ldots, x_n)$, we have $Xf = \sum_{i=1}^n \alpha_i(x) \frac{\partial f}{\partial x_i}$ for every smooth function f, where $\alpha_i(x) = \alpha_i(x_1, \ldots, x_n)$ — are smooth functions of the coordinates not depending on the choice of f. In this case, we write

$$X = \sum_{i=1}^n \alpha_i \frac{\partial}{\partial x_i}.$$

In fact, instead of assuming that the vector field X has this form in every local coordinate system, it suffices to assume that the vector field has this form in one particular local coordinate system. The *coordinates* (*or components*) *of the vector field* (i.e., the functions α_i) can be different in a different local coordinate system but, if these coordinates were smooth in one system, then they remain smooth in all other systems.

In the sequel, by vector fields, we always mean smooth vector fields.

3.5. How are do components of a vector field transform under a local coordinate change? *Answer:* If $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ are two different coordinate systems, and

$$X = \sum \alpha_i(x) \frac{\partial}{\partial x_i} = \sum \beta_j(y) \frac{\partial}{\partial y_j},$$

then

$$\beta_i(y) = \sum_k \alpha_k(x) \frac{\partial y_i}{\partial x_k}.$$

This follows from the chain rule.

Another (equivalent) definition of a vector field is as follows: a smooth vector field is a mapping (also called a *first order differential operator*) $A : C^\infty(M) \to C^\infty(M)$ satisfying the following conditions:

1. linearity: $A(\alpha f + \beta g) = \alpha A(f) + \beta A(g)$ for any constant coefficients α and β,

2. the Leibnitz rule: $A(fg) = (Af)g + f(Ag)$.

A vector field X in the sense of the first definition gives rise to a first order differential operator A given by the formula $A(f) = g$, where $g(q) = X_qf$.

3.6. Prove the equivalence of these two definitions.

Vector fields have the following geometric meaning. Let $g^t : M \to M$ be a smooth family of smooth mappings, i.e., a mapping $(-\varepsilon, \varepsilon) \times M \to M$ defined by the formula $(t, a) \mapsto g^t a$ is smooth. Suppose also that g^0 coincides with the identity transformation. The family of one-to-one mappings g^t can be interpreted physically as a motion of a medium: if one of the particles constituting the medium was at time $t = 0$ in a point $g^0(q) = q$, then at time t it is located at the point $g^t(q)$. Every smooth family of smooth mappings g^t defines a vector field X that acts on functions by the formula

$$Xf(q) = \frac{d}{dt} f(g^t(q)) \,|_{t=0} \,.$$

A physical interpretation is that the vector X_q equals the velocity vector of a particle in the medium passing at time $t = 0$ through the point q.

The converse is also true: every smooth vector field can be obtained as described above. This follows from the existence and uniqueness theorem for ODEs. We recall the statement of the theorem in a form that is suitable for our purposes:

Theorem 3.3 (Existence and Uniqueness Theorem)**.** *Consider a smooth vector field A on a manifold M. For every point $q \in M$, there exists $\varepsilon > 0$ and a unique path $t \mapsto g^t(q)$ defined for all $t \in (-\varepsilon, \varepsilon)$ such that*

$$\frac{d}{dt} f(g^t(q)) = Af(g^t(q)).$$

These paths are included into a smooth family of mappings $g^t : U \to M$, where U is some neighborhood of the point q.

3.7. Suppose that t, s, $t + s \in (-\varepsilon, \varepsilon)$. Then $g^t \circ g^s = g^{t+s}$ wherever both sides are defined.

Hint: this follows from the uniqueness theorem.

Solution. Indeed, by definition of the path $g^t(g^s(q))$ (originating at the point $g^s(q)$), we have

$$\frac{d}{dt} f(g^t(g^s(q))) = Af(g^t \circ g^s(q)).$$

On the other hand, by definition of the path $g^t(q)$ (originating at the point q), we have

$$\frac{d}{dt} f(g^{t+s}(q))) = Af(g^{t+s}(q)).$$

We obtained two paths

$$t \mapsto g^t(g^s(q)), \quad t \mapsto g^{t+s}(q),$$

passing at time $t = 0$ through the same point $g^s(q)$ and tangent to the same vector field A. It follows from the uniqueness theorem that these two paths coincide. $\square$

Theorem 3.4. *Suppose that M is compact, and A is a smooth vector field on M. Then there exists a smooth mapping $\mathbb{R} \times M \to M$, $q \mapsto g^t(q)$ such that*

$$\frac{d}{dt} f(g^t(q)) = Af(g^t(q))$$

for every smooth function f on M. The mappings $g^t : M \to M$ satisfy the identity $g^t \circ g^s = g^{t+s}$.

A local version of this statement is contained in the existence and uniqueness theorem. To prove the theorem, it suffices, using the compactness, to cover the manifold with finitely many charts, in which the local statement holds. A family of smooth mappings $g^t : M \to M$, $t \in \mathbb{R}$ with the property $g^{t+s} = g^t \circ g^s$ is called a *one-parameter diffeomorphism group* of the manifold M. In general, a *diffeomorphism* of a manifold M is defined as a smooth mapping $g : M \to M$, whose inverse is also smooth. A one-parameter diffeomorphism group consists, indeed, of diffeomorphisms, since $(g^t)^{-1} = g^{-t}$. A one-parameter diffeomorphism group satisfying the conditions of Theorem 3.4 is called the *flow of the vector field A*.

As we discussed, we can think of vector fields on M as first order linear differential operators $C^\infty(M) \to C^\infty(M)$. Compositions of such operators are no longer representable by vector fields. A linear differential operator of of order $\leqslant n$ can be defined as a linear combination of operators of the form $A_1 \circ \cdots \circ A_n$, where A_i are vector fields. According to this definition, a linear differential operator can act on smooth functions. In local coordinates, this action reduces to computing a linear combination of certain (higher order) partial derivatives. Coefficients in this linear combination may be any smooth functions.

We will deal with only the first and the second order operators so far. Let D be an operator of order at most 2. We now define the *quadratic symbol* $\sigma_2(D)_q$ at a point $q \in M$. This is an element of the space $Sym^2(T_qM)$ spanned by formal products of pairs of tangent vectors at the point q. These formal products are subject to the commutativity law, i.e., we have $v \cdot w = w \cdot v$ for all v, $w \in T_qM$. Every element of the space $Sym^2(T_qM)$ can be represented as a linear combination of such products. We can identify the space $Sym^2(T_qM)$ with the space of all quadratic forms on the cotangent space T_q^*M. The element $\sigma_2(D)_q \in Sym^2(T_qM)$ is defined by the formula

$$\sigma_2(A_1 \circ B_1 + \cdots + A_m \circ B_m)_q = A_{1,q} \cdot B_{1,q} + \cdots + A_{m,q} \cdot B_{m,q}.$$

Here, e.g., $A_{1,q}$ is a vector from T_qM belonging to the vector field A_1.

Theorem 3.5. *The quadratic symbol is well-defined, i.e., if* $D = A_1 \circ B_1 + \cdots + A_m \circ B_m = 0$*, then* $\sigma_2(D) = 0$ *at all points.*

This is not hard to verify by a straightforward computation in coordinates (exercise: perform this computation!). The heart of the matter is that the quadratic symbol corresponds to the second order partial derivatives (ignoring the first and zeroth order derivatives) — recall that no derivatives of order higher than two can appear in the differential operators under consideration.

Let A and B are two vector fields. Define their *commutator* by the formula

$$[A, B] = A \circ B - B \circ A.$$

The LHS looks like a differential operator of the second order. However, $\sigma_2(A \circ B - B \circ A) = A \cdot B - B \cdot A = 0$, since the product in Sym^2 is commutative (in fact, this is just a sophisticated way of saying that mixed partial derivatives commute). It follows that $A \circ B - B \circ A$ is in fact a first order operator rather than a second order operator. But the first order operators are the same as vector fields. Thus, the commutator of two vector fields is again a vector field.

Let $\Phi : M \to N$ be a smooth mapping, and $v \in T_qM$ a vector based at some point $q \in M$. Then we can naturally define the vector $w = \Phi_*(v)$ tangent to the manifold N at the point $\Phi(q) \in N$. The vector w is defined as a derivation at the point $\Phi(q)$ that acts on a smooth function $g \in C^\infty(N)$ by the formula $w(g) = v(g \circ \Phi)$. Therefore, vectors are transported in the direction of the mapping. If Φ is a diffeomorphism, then for every smooth vector field X on M, one can define a smooth vector field $\Phi_*(X)$ on N. The vector of this field based at the point $\Phi(q) \in N$ is $\Phi_*(X_q)$, where X_q is the vector of the vector field X based at the point q. It is not hard to check that $\Phi_*[X_1, X_2] = [\Phi_* X_1, \Phi_* X_2]$. If Φ is not a diffeomorphism, then, in general, the vector field $\Phi_*(X)$ is not defined, although every individual vector tangent to M can be transported to the manifold N under Φ. The problem is that, to the same point of N, several vectors may be transported, and to some other point of N, no point is transported.

3.1.1 A geometric meaning of the commutator

Let A and B be two smooth vector fields. Let g^t and h^s denote the flows of these vector fields (they are defined at least on some open subset and at least provided that t and s have sufficiently small absolute values). Consider the commutator of these flows $h^{-s} \circ g^{-t} \circ h^s \circ g^t$. The action of the diffeomorphism $h^{-s} \circ g^{-t} \circ h^s \circ g^t$ on a point a can be viewed as the composition of the following motions: the motion for time t along the first flow, the motion for time s along the second flow, then the motion for time t against the first flow, and, finally, the motion for time s against the second flow. In general, in the end of this process, we do not return to the same point.

Introducing a local coordinate system allows to reduce the consideration to the case, where the manifold is an open subset of $\mathbb{R}^n$ containing 0, and the point a coincides with 0. In this case, the following formula holds

$$[A,B]_0 = \lim_{t,s\to 0} \frac{h^{-s}\circ g^{-t}\circ h^s\circ g^t(0)}{ts}.$$

Here we use that points of the space $\mathbb{R}^n$ are identified with vectors tangent to the space $\mathbb{R}^n$ at the point 0.

3.1.2 Problems

3.8. On the plane with coordinates x and y, find the commutator of the vector fields $\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$.

3.9. Does there exist a local diffeomorphism between a neighborhood of the point $(0,0)$ and another neighborhood of the point $(0,0)$ transporting the vector fields $\frac{\partial}{\partial x}$ and $(1+x^2)\frac{\partial}{\partial y}$ to the vector fields $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$, respectively?

3.10. Let $g^t:\mathbb{R}^2\to\mathbb{R}^2$ be a mapping defined by the formula

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x+ty^2 \\ y \end{pmatrix}.$$

Find coordinates of a vector field A such that

$$Af(x) = \frac{d}{dt} f(g^t(x))\,|_{t=0}$$

for every smooth function f.

3.11. Prove that the commutator of vector fields satisfies the following identities:

- $[A,B] = -[B,A]$,
- $[A,B+C] = [A,B]+[A,C]$,
- $[A,fB] = (Af)B + f[A,B]$ for every smooth function f (the Leibnitz rule),
- $[A,[B,C]] = [[A,B],C]+[B,[A,C]]$ (the Jacobi identity)

(The Jacobi identity is the Leibnitz rule with respect to the commutator).

3.2 Differential forms: a reminder

Let M be a smooth manifold, and $\mathrm{Vect}(M)$ the space of all smooth vector fields on M. Define a (smooth differential) *1-form* on M as a mapping $\alpha:\mathrm{Vect}(M)\to C^\infty(M)$ with the following properties:

$$\alpha(fX) = f\alpha(X), \quad \alpha(X+Y) = \alpha(X)+\alpha(Y)$$

for all smooth vector fields X, Y and all functions $f\in C^\infty(M)$.

3.12. Consider a vector field $X \in \mathrm{Vect}(M)$ and a 1-form α. Prove that the value $\alpha(X)_q$ of the function $\alpha(X)$ at a point $q \in M$ depends only on α and on the vector X_q (i.e., the vector at the point q belonging to the vector field X) but does not depend on what the vector field X looks at other points.

Solution. Indeed, it suffices to prove that $\alpha(X)_q = 0$ if $X_q = 0$. We may assume without loss of generality that the vector field X_q vanishes outside a small neighborhood of q. Indeed, we can consider a smooth function $\varphi \in C^\infty(M)$, equal to one at the point q and to zero outside of a small neighborhood of q, and use the inequality $\alpha(\varphi X)_q = \varphi(q)\alpha(X)_q = \alpha(X)_q$. Thus, if necessary, we can replace the vector field X with the field φX. Since the field X is defined locally, we can confine ourselves to consideration of a single chart, or, which is equivalent, to assume that $M = \mathbb{R}^n$. Let us write the field X in coordinates $(x_1, \ldots, x_n)$:

$$X = \sum_{i=1}^n \xi_i \frac{\partial}{\partial x_i}, \quad \xi_i \in C^\infty(\mathbb{R}^n).$$

Since $X_q = 0$, all functions ξ_i vanish at the point q. We obtain:

$$\alpha(X) = \sum_{i=1}^n \xi_i(q)\alpha\left(\frac{\partial}{\partial x_i}\right) = 0.$$

The statement is proved.

We now give an equivalent definition of a 1-form. Suppose that, for every point $q \in M$, we defined a linear functional on the tangent space T_qM of the manifold M at the point q, so that this functional depends smoothly on q (a smooth dependence needs to be defined precisely — this is left to the reader). Then there is a 1-form α such that the linear functionals $l_q : T_qM \to \mathbb{R}$ mentioned above have the form $l_q(X_q) = \alpha(X)_q$ (for the RHS to make sense, we need to extend the vector X_q to a smooth vector field X on M, no matter how).

3.13. Prove that every tangent vector (at any point of M) can be included into a smooth vector field on M.

Let $(x_1, \ldots, x_n)$ be a local coordinate system on the manifold M in a neighborhood U of some fixed point q. Recall that each of the coordinates x_i is a smooth function on an open subset U. For a 1-form α, set

$$\alpha_i(q) = \alpha\left(\frac{\partial}{\partial x_i}\right)_q.$$

This equality defines some smooth function α_i on U. The functions $\alpha_1, \ldots, \alpha_n$ are called the *coefficients* of the 1-form α in a given local coordinate system. Define the 1-form dx_i on U by the formula

$$dx_i(A) = A_i.$$

Here A is any smooth vector field on U, which in the given coordinate system is written as $A_1\frac{\partial}{\partial x_1} + \cdots + A_n\frac{\partial}{\partial x_n}$.

3.14. Prove that every smooth 1-form on U can be written as

$$\alpha = \sum_{i=1}^{n} \alpha_i \, dx_i.$$

Smooth 1-forms can be integrated over smooth curves. Consider a smooth path $\gamma : [0,1] \to M$. Then, for every $t \in [0,1]$, there is the velocity vector $\dot{\gamma}(t) \in T_{\gamma(t)}M$. If the manifold M carries a 1-form α, we can apply this form to each of the vectors $\dot{\gamma}(t)$, and thus obtain a smooth function of t. This function can be integrated over the interval $[0,1]$. The integral thus obtained in called the *integral of the form* α over the path γ:

$$\int_\gamma \alpha = \int_0^1 \alpha(\dot{\gamma}(t))dt.$$

In fact, what is important is not a particular parameterization of the curve $\gamma[0,1]$ but only the orientation: if we make a smooth change of a parameter on the curve, under which the orientation is unchanged, then the integral of a differential 1-form does not change either. If the orientation is reversed, then the integral changes sign.

3.15. Prove the claims made above.

For every smooth function f on M, we define the 1-form df by the formula:

$$df(A) = Af.$$

This form is called the *differential* of the function f. The operation of taking the differential satisfies the Leibnitz rule

$$d(fg) = g\, df + f\, dg.$$

We will now define (smooth differential) *2-*. This is a mapping α of $\mathrm{Vect}(M) \times \mathrm{Vect}(M)$ (i.e., of pairs of vector fields) to $C^\infty(M)$ with the following properties:

$$\alpha(X,Y) = -\alpha(Y,X), \quad \alpha(X, Y+Z) = \alpha(X,Y) + \alpha(X,Z),$$

$$\alpha(X, fY) = f\alpha(X,Y)$$

for all smooth vector fields X, Y, Z and all smooth function f.

Similarly to the case of 1-forms, it makes sense to talk about the value of a 2-form on a pair of vectors based at the same point, i.e., belonging to the same tangent space. This value does not depend on how these vectors are extended to smooth vector fields on the entire manifold.

If two 1-forms α and β are given, then we define their *wedge product* (also called exterior product) $\alpha \wedge \beta$, which is a 2-form. The wedge product is defined by the following formula:

$$\alpha \wedge \beta(A, B) = \alpha(A)\beta(B) - \alpha(B)\beta(A).$$

3.16. Verify that the thus defined function of pairs of vector fields is indeed a 2-form.

If $(x_1, \dots, x_n)$ is a local coordinate system on the manifold M, then any 2-form α can be written in this coordinate system as follows:

$$\alpha = \sum_{i,j=1}^{n} \alpha_{i,j}\, dx_i \wedge dx_j.$$

Here $\alpha_{i,j}$ — are certain smooth functions on an open subset of M, called the *coefficients* of the form α in the given coordinate system. The coefficients of α can be defined by the formula $\alpha_{i,j} = \alpha(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$. It is not hard to see that $\alpha_{i,j} = -\alpha_{j,i}$.

3.17. Prove the claims made above.

With every smooth 1-form β, we associate a smooth 2-form $d\beta$, called the *differential* of the form β. The differential is defined by the following formula:

$$d\beta(A, B) = A\beta(B) - B\beta(A) - \beta([A, B]).$$

3.18. Prove that $d\beta$ is indeed a 2-form.

However, the algebraic definition of the differential given above does not clarify the geometric meaning of this notion. Informally speaking, the geometric meaning is the following. A pair of tangent vectors based at the same point defines a parallelogram in the tangent space. As can be seen from the definition, the value of a 2-form on the pair of vectors depends on the orientation and the area of the corresponding parallelogram but does not depend on its shape. (by the orientation of the parallelogram, we mean in which plane it lies and how it is oriented in this plane, i.e., which is the direction of the shortest rotation from the first vector to the same vector). A parallelogram in the tangent space can be thought of (which is not entirely correct but useful) as an infinitesimal oriented 2-dimensional surface piece in the manifold. Thus, a 2-form can be understood as a special kind of function on infinitesimal surface pieces. Integrate the 1-form α over the boundary of the surface piece. We obtain a function of an infinitesimal surface piece, which in fact coincides with the 2-form $d\alpha$. The term with the commutator is a correction term responsible for the discrepancy between a small surface piece in the manifold and a piece of a tangent space.

Differential 2-forms can be integrated over two-dimensional surfaces. First consider a smooth parameterized surface $\Gamma : W \mapsto M$, where W is a domain in the plain with coordinates (u, v). The integral of the a 2-form α over this surface is defined by the following formula:

$$\int_\Gamma \alpha = \int_W \alpha\left(\frac{\partial\Gamma}{\partial u}, \frac{\partial\Gamma}{\partial v}\right) du\, dv.$$

In fact, this integral does not depend on the parameterization; it depends only on the orientation of a smooth surface. We have considered the case, where the surface is the image of some planar domain under some smooth mapping. In fact, we will need more general surfaces, which can be glued out of several such pieces. For example, we can consider an arbitrary union of convex oriented polygons (lying in a space of arbitrary dimension), and define a smooth mapping on each of these polygons M. Suppose that these mappings match on the intersections. Then such a collection of date is called a *smooth 2-chain*. The integral of α over a smooth 2-chain is defined as the sum of integrals over individual polygons.

The boundary $\partial\Gamma$ of an oriented smooth 2-chain Γ is an oriented 1-chain, i.e., a finite union of smooth oriented curves.[1] On every boundary piece, the orientation is chosen so that, as a particle traverses the boundary piece in a positive direction, the adjacent piece of the chain is on the left (the notions of left and right are determined by the orientation of the chain). The integral of a 1-form over $\partial\Gamma$ can be related to the integral of the differential of this form over Γ. This is a partial case of the Stokes formula:

$$\int_{\partial\Gamma} \alpha = \int_\Gamma d\alpha.$$

We will adopt this formula without proof. A proof of the Stokes formula is given in analysis courses and in differential geometry courses.

A differential 1-form α is said to be *closed*, if $d\alpha = 0$, and *exact*, if α coincides with the differential of some smooth function. In the space $\mathbb{R}^n$, every closed 1-form is exact. Indeed, fix an arbitrary initial point $q_{(0)}$, and consider the integral

$$f(q) = \int_{q_{(0)}}^{q} \alpha$$

This integral does not depend on the choice of a path connecting the point $q_{(0)}$ with the point q. Indeed, if we consider two different paths γ_0 and γ_1,

[1] More precisely, we need to consider not a union but rather a formal sum of curves. Curves may enter this sum with integer, not necessarily positive, coefficients. Thus we can think of a union of curves, on which some integers are written.

connecting the point $q_{(0)}$ with the point q, then the union of these paths[2] is the boundary of some 2-chain and, as follows from the Stokes formula, the integral of α over the union $\gamma_0 \cup \gamma_1$ is equal to zero. Therefore, $f(q)$ does not depend on a path. It is not hard to verify that $df = \alpha$.

Similarly to 2-forms, we can define (smooth differential) 3-forms on a manifold M as mappings

$$\alpha : \mathrm{Vect}(M) \times \mathrm{Vect}(M) \times \mathrm{Vect}(M) \to C^\infty(M),$$

that are linear over $C^\infty(M)$ w.r.t to each argument and skew-symmetric. In other terms, for any vector fields X_1, X_2, X_3, Y_1, any permutation σ of the set $\{1, 2, 3\}$, and any smooth function f, we have

$$\alpha(X_{\sigma(1)}, X_{\sigma(2)}, X_{\sigma(3)}) = \mathrm{sign}(\sigma)\alpha(X_1, X_2, X_3),$$

$$\alpha(X_1 + Y_1, X_2, X_3) = \alpha(X_1, X_2, X_3) + \alpha(Y_1, X_2, X_3),$$

$$\alpha(fX_1, X_2, X_3) = f\alpha(X_1, X_2, X_3).$$

Here $\mathrm{sign}(\sigma)$ denotes the sign of the permutation σ, which equals 1 for an even permutation and -1 for an odd permutation. For every 2-form β, we define the differential $d\beta$ by the formula

$$d\beta(X, Y, Z) = X\beta(Y, Z) + Y\beta(Z, X) + Z\beta(X, Y)-$$

$$-\beta([X, Y], Z) - \beta([Y, Z], X) - \beta([Z, X], Y).$$

Note that, together with every term, this formula includes all terms obtained from the given one by cyclic permutations of the arguments X, Y, Z. Thus it is enough to memorize just two terms: $X\beta(Y, Z)\ \ -\beta([X, Y], Z)$; all other terms are obtained from these two by cyclic permutations. It is easy to verify that the differential of every 2-form is a 3-form. The most meaningful part of this statement is that the differential of a 2-form is multilinear over smooth functions.

It is not hard now to define a smooth differential k-form for arbitrary k as a skew-symmetric form on the space of smooth vector fields that is multilinear over the algebra of smooth functions. The (exterior) differential of a smooth k-form is defined similarly to the cases of 1-forms and 2-forms; it is a smooth $(k+1)$-form. An explicit formula for the value of the differential

[2] more precisely, the formal difference of these paths, if both paths are oriented from $q_{(0)}$ to q

$d\alpha$ of a smooth k-form α on smooth vector fields X_0, $\dots$, X_k is rather involved:

$$d\alpha(X_0,\dots,X_k)=\sum_{i=0}^{k}(-1)^i X_i\alpha(X_0,\dots,\hat{X}_i,\dots,X_k)+$$

$$+(-1)^{i+1}\alpha(X_0,\dots,[X_i,X_{i+1}],\dots,X_k).$$

This formula, as well as the entire formalism of differential forms, is due to E. Cartan. If $i=k$, the last term in Cartan's formula is equal to $(-1)^k\alpha(X_1,\dots,[X_k,X_0])$. In other words, we consider indices as residues modulo $k+1$, in particular, we set $k+1=0$.

Let α be a smooth k-form, and β be a smooth l-form on the same manifold. Then we can define the *wedge product* $\alpha\wedge\beta$. This is a differential $(k+l)$-form. The value of this form on a tuple of smooth vector fields X_1, $\dots$, X_k, X_{k+1}, $\dots$, X_{k+l} can be computed by the following formula:

$$\alpha\wedge\beta(X_1,\dots,X_{k+l})=\sum\pm\alpha(X_{i_1},\dots,X_{i_k})\beta(X_{j_1},\dots,X_{j_l}).$$

Here the indices i_1, $\dots$, i_k, j_1, $\dots$, j_l run through all values such that

$$i_1<\dots<i_k,\quad j_1<\dots<j_l,\quad \{i_1,\dots,i_k,j_1,\dots,j_l\}=\{1,\dots,k+l\}.$$

The sign (plus or minus) coincides with the sign of the permutation

$$(i_1,\dots,i_k,j_1,\dots,j_l)\in S_{k+l}$$

Let us emphasize the similarity between this formula and the expansion formula for a determinant of side $k+l$, say, in the first k rows (or, which is the same, in the last l rows). Let W denote the determinant, and let $W'_{i_1,\dots,i_k}$ denote the minor obtained at the intersection of the first k rows with columns i_1, $\dots$, i_k. Similarly, we let $W''_{j_1,\dots,j_l}$ denote the minor obtained at the intersection of the last l rows with columns j_1, $\dots$, j_l. *Lagrange's formula* expresses the determinant W as the sum of products of the form $W'_{i_1,\dots,i_k}W''_{j_1,\dots,j_l}$ taken with suitable signs. Note that the term

$$\alpha(X_{i_1},\dots,X_{i_k})\beta(X_{j_1},\dots,X_{j_l})$$

enters the expression for $\alpha\wedge\beta(X_1,\dots,X_{k+l})$ with the same sign, with which the term $W'_{i_1,\dots,i_k}W''_{j_1,\dots,j_l}$ enters the expansion of the determinant W. This rule allows not to memorize two formulas: it suffices to only memorize Lagrange's formula for the expansion of a determinant, and then the formula for the product of two differential forms can be easily recovered.

We now discuss how differential forms are transported under smooth mappings. If $\Phi : M \to N$ is a smooth mapping and α is a differential form on N, then we will define differential form $\beta = \Phi^*\alpha$ on M. Thus, the differential forms are transported against the arrows representing smooth mappings. To fix the ideas, we assume that α is a 2-form. The case of a k-form for arbitrary k is similar. For any pair of smooth vector fields X and Y on the manifold M and for any point $q \in M$, we set

$$\beta(X,Y)_q = \alpha(\Phi_*(X_q), \Phi_*(Y_q))_{\Phi(q)}.$$

In the left-hand side, we have the value of the function $\beta(X,Y)$ at the point q. In the RHS, the differential form α is applied at the point $\Phi(q)$ to the pair of vectors $\Phi_*(X_q)$ and $\Phi_*(Y_q)$ obtained by transporting tangent vectors X_q, $Y_q \in T_qM$ to the tangent space $T_{\Phi(q)}N$ under the smooth mapping Φ. The form β is called the *pullback* of the form α under the mapping Φ. The pullback operation commutes with all natural operations over differential forms, in particular, with wedge products and taking differential. In other words, we have the identities

$$\Phi^*(\alpha_1 \wedge \alpha_2) = \Phi^*\alpha_1 \wedge \Phi^*\alpha_2,$$

$$\Phi^*(d\alpha) = d\Phi^*\alpha$$

for all smooth differential forms α, α_1, α_2 on the manifold N.

3.2.1 Problems

3.19. Let β be a smooth 1-form on a manifold M, and α be a smooth 2-form on the manifold M such that

$$\alpha(A,B) = A\beta(B) - B\beta(A)$$

for any pair of commuting vector fields A and B. Prove that $\alpha = d\beta$.

3.20. Verify that the differential of any differential 2-form α defined by the formula

$$d\alpha(X,Y,Z) = X\alpha(Y,Z) + Z\alpha(X,Y) + Y\alpha(Z,X) -$$

$$-\alpha([X,Y],Z) - \alpha([Z,X],Y) - \alpha(Y,[Z,X]).$$

is a smooth 3-form.

3.21. On the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with coordinates (x,y) (these coordinates are real numbers modulo integers), consider a differential form $\alpha = dx \wedge dy$. Is this form exact?

3.3 Symplectic manifolds

A *symplectic structure* on a manifold M is a 2-form ω on M with the following properties:

1. the form ω is closed, i.e., $d\omega = 0$,
2. the form ω is non-degenerate, i.e., for every 1-form α on M, there exists a vector field X on M such that $\alpha(Y) = \omega(Y, X)$ for all vector fields Y on M.

The second condition is equivalent to saying that the restriction of the form ω to every tangent space of M is a non-degenerate bilinear form (the non-degeneracy is understood in the sense of linear algebra).

3.22. Prove that the vector field X from condition (2) is necessarily unique.

We obtain a linear mapping of the 1-forms to vector fields, which, to every 1-form α, assigns a vector field X on M such that $\alpha(Y) = \omega(Y, X)$ for all vector fields Y on M. This mapping of 1-forms to vector fields will be denoted by $\mathcal{I}$.

Let H be a smooth function on the manifold M. The vector field $X_H = \mathcal{I}(dH)$ is called a *Hamiltonian vector field* generated by the Hamiltonian H. In other words, by definition, the field X_H is the unique vector field satisfying the identity

$$dH(Y) = \omega(Y, X_H)$$

for every smooth vector field Y on M.

Example. Consider the coordinate space $M = \mathbb{R}^{2n}$ with coordinates $p_1, \dots, p_n, q_1, \dots, q_n$. The 2-form

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i$$

is a symplectic structure on the manifold M. This symplectic structure is called the *standard symplectic structure* on $\mathbb{R}^{2n}$. Let us see what the Hamiltonian vector field generated by the Hamiltonian H looks like. We have

$$dH(Y) = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i(Y) + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i(Y).$$

On the other hand,

$$\omega(X, Y) = \sum_{i=1}^{n} (dp_i(X) dq_i(Y) - dq_i(X) dp_i(Y)).$$

If $X = X_H$, then we must have $dH(Y) = -\omega(X, Y)$ for all Y. In particular, the numbers $dp_i(Y)$ and $dq_i(Y)$ can be thought of as independent variables.

Equating the coefficients with these variables, we obtain

$$dp_i(X) = -\frac{\partial H}{\partial q_i}, \quad dq_i(X) = \frac{\partial H}{\partial p_i}.$$

It follows that the system of ODEs defined by the vector field X has the form

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

Thus, we obtained the standard form of a Hamiltonian system, which is the principle (although not the only) motivation for the notions introduced above. We will discuss other examples of symplectic structures later.

Let X_H be a Hamiltonian vector field corresponding to the Hamiltonian H. The flow g_H^t generated by the vector field X is called the *Hamiltonian flow* with the Hamiltonian H. By definition of a flow generated by a vector field, we have

$$\frac{d}{dt} f(g_H^t(a)) = X_H f(g_H^t(a))$$

for every point $a \in M$.

Theorem 3.6. *A Hamiltonian flow preserves the corresponding Hamiltonian:*

$$H \circ g_H^t = H \quad \forall t \in \mathbb{R}.$$

Proof. It suffices to prove that the derivative of the function $H \circ g_H^t$ with respect to t equals zero. We have:

$$\frac{d}{dt} H \circ g_H^t(a) = X_H H(g_H^t(a)) = d_{g_H^t(a)} H(X_H) = \omega_{g_H^t(a)}(X_H, X_H) = 0,$$

as desired. □

Note that Theorem 3.6 is a generalization of the energy preservation in classical mechanics to the case of arbitrary Hamiltonian flows.

Theorem 3.7. *A Hamiltonian flow preserves the symplectic form:*

$$(g_H^t)^* \omega = \omega \quad \forall t \in \mathbb{R}.$$

This is a deep and hard theorem. It provides a collection of integrals of a Hamiltonian flow that are more subtle than the Hamiltonian itself. Before we prove Theorem 3.7, we need to discuss Lie derivatives. Let X be a vector field on a manifold M. Let g^t denote the corresponding Hamiltonian flow on M. *The Lie derivative* of a differential form α on a manifold M *along a vector field* X is defined by the following formula:

$$L_X\alpha = \frac{d}{dt}(g^t)^*\alpha|_{t=0}.$$

Note that the Lie derivative commutes with taking the differential: $L_X d\beta = dL_X\beta$. The definition of the Lie derivative was given for every smooth differential k-form and every k. In particular, it makes sense for $k = 0$. Recall that smooth 0-forms are just smooth functions. In this case, we obtain the usual differentiation of a smooth function along a smooth vector field, i.e., we have $L_X f = Xf$.

3.23. Verify that the Lie derivative satisfies the Leibnitz rule with respect to the wedge product of differential forms:

$$L_X(\alpha \wedge \beta) = (L_X\alpha) \wedge \beta + \alpha \wedge (L_X\beta)$$

(in this formula, X is a smooth vector field, α and β are smooth differential forms on a smooth manifold).

3.24. Let $X_1, \ldots, X_n$ be coordinates of a smooth vector field X in some local coordinate system $(x_1, \ldots, x_n)$. Prove that $L_X(dx_i) = dX_i$ for every $i = 1, \ldots, n$.

Solution. We can assume that the manifold under consideration coincides with the space $\mathbb{R}^n$, on which some coordinate system $(x_1, \ldots, x_n)$ is fixed. Let g^t denote the flow generated by the vector field X. By definition of the Lie derivative,

$$L_X(dx_i) = \frac{d}{dt}(g^t)^*(dx_i)|_{t=0} = \frac{d}{dt}d(x_i \circ g^t)|_{t=0} = d\left(\frac{d}{dt}x_i \circ g^t|_{t=0}\right).$$

In the last equality, we used that the operation of taking the differential of a differential form commutes with the operation of differentiation with respect to a parameter. Note that

$$\frac{d}{dt}x_i \circ g^t|_{t=0} = X(x_i) = X_i.$$

The statement is thus proved.

We now define the *substitution of a vector field* X into a differential form α. To fix the ideas, we assume that α is a smooth 2-form. Then the substitution of a vector field X into the form α gives the 1-form $\iota_X\alpha$ defined by the following formula:

$$\iota_X\alpha(Y) = \alpha(X, Y).$$

Thus the vector field X is being substituted as the first argument of the form α. Similarly, we can define the substitution of a vector field into a differential k-form for arbitrary k.

The following formula relates the Lie derivative, the substitution of a vector field, and the differential:

$$L_X\alpha = d\,\iota_X\alpha + \iota_X d\alpha.$$

This formula holds for every smooth k-form α. It is called the *homotopy formula*. The homotopy formula can be verified by a direct computation in coordinates. We will now perform these computations in the case $k = 2$. By linearity, it suffices to assume that $\alpha = f dx_i \wedge dx_j$ for some smooth function f. In this case, we have:

$$L_X\alpha = (Xf)dx_i \wedge dx_j + f dX_i \wedge dx_j + f dx_i \wedge dX_j.$$

Here X_i and X_j are coordinates of the vector field X regarded as smooth functions.

In the computation displayed above, we use the Leibnitz rule and the easily verifiable fact that $L_X(dx_i) = dX_i$, cf. Problem 3.24. On the other hand, we have:

$$\begin{aligned} d\,\iota_X\alpha &= d(f(X_i\,dx_j - X_j\,dx_i)) = \\ &= df \wedge (X_i\,dx_j - X_j\,dx_i) + f(dX_i \wedge dx_j - dX_j \wedge dx_i). \end{aligned}$$

$$\iota_X d\alpha = df \wedge dx_i \wedge dx_j = (Xf)dx_i \wedge dx_j - df \wedge (X_i\,dx_j - X_j\,dx_i).$$

Comparing the three obtained equalities, we see that the homotopy formula holds.

It may be that the proof of the homotopy formula presented above is the most direct, however, it does not shed any light on a geometric meaning of this formula. For this reason, we sketch another, more geometric proof, omitting details. Consider a 2-chain σ, and let σ_t denote the image of this chain under the flow mapping g^t generated by the vector field X. By definition of the Lie derivative,

$$\int_\sigma L_X\alpha = \frac{d}{dt}\int_{\sigma_t}\alpha\bigg|_{t=0} = \lim_{t\to 0}\frac{1}{t}\left(\int_{\sigma_t-\sigma_0}\alpha\right).$$

Consider a 3-chain Σ spanned by σ_τ as τ runs through the interval $[0,t]$.[3] The boundary of the 3-chain Σ consists of the following parts: the "top" σ^t, the "bottom" σ^0, and the "side surface" δ that is spanned by the 1-chain $\partial\sigma^\tau$, when τ runs through the interval $[0,t]$, see Figure 3.1. By the Stokes

[3]Recall that a 2-chain can be thought of as a formal linear combination of elementary 2-chains, and an *elementary* 2-chain is by definition a smooth mapping from a convex polygon to the manifold. If we assume that σ is an elementary 2-chain given by a smooth mapping $\varphi : P \to M$ from a convex polygon P to the manifold M, then Σ is an elementary 3-chain $\psi : P \times [0,t] \to M$ given by the formula $\psi(a,\tau) = g^\tau \circ \varphi(a)$.

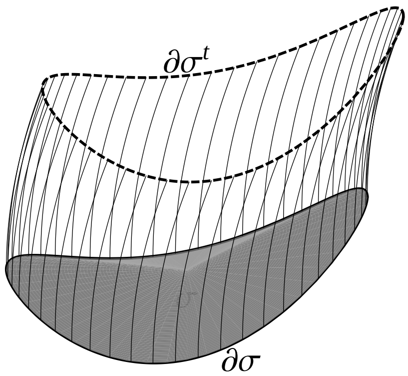

Figure 3.1: The boundary of the chain Σ. The "side surface" δ consists of trajectory segments. The "bottom" coincides with σ^0, and the top with σ^t.

formula,

$$\int_\Sigma d\alpha = \int_{\sigma^t - \sigma^0} \alpha - \int_\delta \alpha.$$

In order to verify the signs, we need to explore the orientation of the boundary of the chain Σ. The integral in the LHS can be rewritten as an integral over τ, namely,

$$\int_\Sigma d\alpha = \int_0^t d\tau \int_{\sigma^\tau} \iota_X \, d\alpha.$$

This follows immediately from the definition of the integral of a differential 3-form over a 3-chain. The integral over δ in the RHS can also be rewritten as an integral over τ; then the Stokes formula yields:

$$\int_\delta \alpha = \int_0^t d\tau \int_{\partial\sigma^\tau} \iota_X \alpha = \int_0^t d\tau \int_{\sigma^\tau} d\,\iota_X \alpha.$$

Thus we have

$$\int_{\sigma^t - \sigma^0} \alpha = \int_0^t d\tau \int_{\sigma^t} \left(\iota_X d\alpha + d\,\iota_X \alpha\right).$$

We obtain the homotopy formula if we divide this formula by t and pass to the limit as $t \to 0$, since the the 2-chain σ is arbitrary.

Proof of Theorem 3.7. In order to prove that the Hamiltonian flow with a Hamiltonian H preserves the symplectic structure ω, we need to verify that $L_{X_H}\omega = 0$. This follows from the homotopy formula:

$$L_{X_H}\omega = \iota_{X_H} d\omega + d\,\iota_{X_H}\omega = 0.$$

The first term vanishes since $d\omega = 0$ by definition of a symplectic structure. The second term vanishes since $\iota_{X_H}\omega = -dH$ by definition of of the Hamiltonian vector field X_H, and since every exact differential form is closed. $\square$

3.25. Consider the upper halfplane $y > 0$ with the symplectic structure

$$\omega = y\,dx \wedge dy.$$

Find a Hamiltonian vector field corresponding to the Hamiltonian $H(x, y) = e^x + y$.

3.26. In the space $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4), we are given the 2-form

$$\omega = 3\,dx_1 \wedge dx_2 + dx_1 \wedge dx_3 + 5\,dx_3 \wedge dx_4.$$

Verify that this 2-form is a symplectic structure. Find the skew-orthogonal complement of the plane $x_3 = x_4 = 0$ with respect to ω, i.e., the vector subspace of $\mathbb{R}^4$, consisting of all vectors v with the following property: for every vector u in the plane $x_3 = x_4 = 0$, we have $\omega(u, v) = 0$.

3.27. Consider the vector field on $\mathbb{R}^2$ with coordinates $(x, -y + \sin x)$. Is this vector field Hamiltonian with respect to the symplectic structure $\omega = dx \wedge dy$? If so, then find the Hamiltonian.

3.28. Give an example of a vector field that is not Hamiltonian, no matter which symplectic structure and which Hamiltonian function we consider.

3.29. Consider a 2-torus with coordinates $p, q \in \mathbb{R}/2\pi\mathbb{Z}$ and the function

$$H(p, q) = \sin(p)\cos(q)$$

on this 2-torus. Is it true that every solution $(p(t), q(t))$ of the Hamiltonian system

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}$$

is periodic, i.e., $p(t+T) = p(t)$, $q(t+T) = q(t)$ for some $T > 0$? If yes, then prove. If no, then give a counterexample.

3.30. On the torus $T^4 = \mathbb{R}^4/\mathbb{Z}^4$, consider the symplectic structure

$$\omega = dx_1 \wedge dx_2 + dx_3 \wedge dx_4$$

and the function $H = \sin x_1 \sin x_2 \sin x_3 \sin x_4$. Find

$$L_{v_H}\left(\sin^2(x_1)\sin^2(x_2)\sin^2(x_3)\sin^2(x_4)(dx_1 \wedge dx_2 + dx_3 \wedge dx_4)\right),$$

where v_H is the Hamiltonian vector field corresponding to the Hamiltonian H.

3.31. Let $H : M \to \mathbb{R}$ be a smooth function on a symplectic manifold M, and v_H be the Hamiltonian vector field with the Hamiltonian H. Prove that $v_H \in T \cap T^{\perp}$, where T is the tangent space at a point x to the hypersurface $\{H = \mathit{const}\}$, and $T^{\perp}$ is its skew-orthogonal complement.