Strategic Network Formation and Network Allocation Rules

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Part 1 - Characteristics of Networks

Content of the course

- Introduction, background and fundamentals of network analysis - representing and measuring networks, centrality measures
- Strategic network formation - pairwise stability and efficiency, the connections model and its dynamic version, the co-author model, positive and negative externalities in networks, small worlds in an islands-connections model, general tension between stability and efficiency
- Network games and allocation rules - Myerson value in network settings, egalitarian allocation rule, component-wise egalitarian allocation rule, flexible network allocation rules, allocation rule for dynamic random networks

Social and economic networks

- Interdisciplinary field:
  economics, sociology, psychology, mathematics, statistics, computer sciences, physics, biology ...
- Different approaches to the analysis of social networks:
  theoretical models, empirical works, experiments
- Central role for modeling different phenomena:
  transmission of information (e.g., job opportunities), learning, influence, opinion formation, contagion, trade of goods and services, business interactions, financial networks, scientific collaboration, political interactions, criminal activities, ...

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Representing Networks

Preliminaries on networks (1/6)

- A network is represented by a graph (N, g), where
  - $N = \{1, 2, ..., n\}$ set of nodes (agents, players, vertices)
  - $g = [g_{ij}]$ real-valued $n \times n$ matrix (adjacency matrix)
- g_{ij} - relationship between i and j (possibly weighted and/or directed), also referred to as a link ij or an edge
- G = collection of all possible networks on n nodes
- We assume that graphs are simple, i.e., $g_{ii} = 0$ for all $i \in N$ (no loops) and $g_{ij} \in [0, 1]$ (no multiple edges).
- A network is directed if $g_{ij} \neq g_{ji}$ for some $i, j \in N$, and undirected otherwise.
- In what follows we consider an unweighted network g with

$$g_{ij} = \begin{cases} 1 & \text{if there is a link between } i \text{ and } j \\ 0 & \text{otherwise,} \end{cases}$$

and we assume that $g_{ij} = g_{ji}$ for all $i, j \in N$.

Preliminaries on networks (2/6)

- Another (equivalent) way of representing a network: (N, g), where
  - g is the set of links (subsets of N of size 2)
  - $ij \in g$ iff $g_{ij} = 1$
- $g = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Leftrightarrow g = \{12, 23\}$
- Notation:

  $g + ij$: network obtained by adding ij to g

  $g - ij$: network obtained by deleting ij from g

  $g' \subset g \quad \Leftrightarrow \quad \{ij \mid ij \in g'\} \subset \{ij \mid ij \in g\}$
- Given $S \subset N$ and g, $g|_S$ denotes g restricted to S:

$$[g|_S]_{ij} = \begin{cases} 1 & \text{if } i \in S, j \in S, g_{ij} = 1 \\ 0 & \text{otherwise,} \end{cases}$$

- Given $S \subset N$, g^S is the complete network on the nodes S (viewed as network on N).

Preliminaries on networks (3/6)

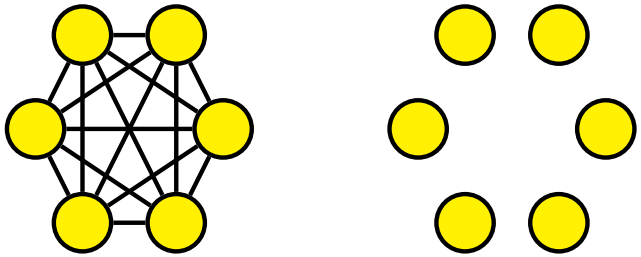
- $N_i(g)$ = neighborhood (set of neighbors) of i in g

$$N_i(g) = \{j \in N : g_{ij} = 1\}$$

- $\eta_i(g)$ = degree of i in g = number of i's neighbors in g, i.e.,

$$\eta_i(g) = |N_i(g)|$$

- A network g is regular if for some $\eta \in \{0, 1, ..., n-1\}$, $\eta_i(g) = \eta$ for each $i \in N$.
- g^N = complete network (regular network with $\eta = n - 1$)
- $g^\emptyset$ = empty network (regular network with $\eta = 0$)



Preliminaries on networks (4/6)

- How can one node be reached from another one in g?
  - Walk = sequence of links $i_1 i_2, \cdots, i_{K-1} i_K$ such that $g_{i_k i_{k+1}} = 1$ for each $k \in \{1, \cdots, K-1\}$
    (a node or a link may appear more than once)
  - Trail = walk in which all links are distinct
  - Path = trail in which all nodes are distinct
  - Cycle = trail with at least 3 nodes in which the initial node and the end node are the same.
- Geodesic between two nodes is a shortest path between them.
- $l_{ij}(g)$ = geodesic distance between i and j in g
  If there is a path between i and j in g, then

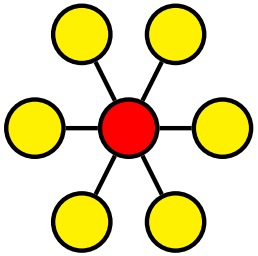
  $l_{ij}(g)$ = the number of links in a shortest path between i and j

$$l_{ij}(g) = \min_{\text{paths } P \text{ from } i \text{ to } j} \sum_{kl \in P} g_{kl}.$$

  If there is no path between i and j in g, we set $l_{ij}(g) = \infty$.

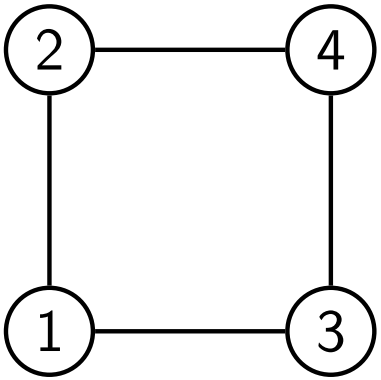
Preliminaries on networks (5/6)

- A network is connected if there exists a path between any pair of nodes $i, j \in N$ (i.e., if it consists of a single component).
- The components of a network are the distinct maximal connected subgraphs.
- Two nodes belong to the same component if and only if there exists a path between them.
- A link ij is a bridge in g if $g - ij$ has more components than g.
- A tree is a connected network that has no cycles.
- A forest is a network such that each component is a tree.
- Any network that has no cycles is a forest.
- A star is a connected network in which there exists some node i (center) such that every link in the network involves i.



Preliminaries on networks (6/6)

g^k = kth power of g
$g^0 := \mathbb{I}$ with $\mathbb{I} = n \times n$ identity matrix, where
g^k_{ij} = number of walks of length k that exist between i and j in g.



$$g = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \quad g^2 = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 2 & 0 & 0 & 2 \end{bmatrix} \quad g^3 = \begin{bmatrix} 0 & 4 & 4 & 0 \\ 4 & 0 & 0 & 4 \\ 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \end{bmatrix}$$

E.g. walks of length 3 between 1 and 2: $(12, 24, 42)$, $(13, 34, 42)$, $(12, 21, 12)$, $(13, 31, 12)$.

Measuring Networks

Some characteristics of networks (1/5)

- While small networks can be easily illustrated, large networks are more difficult to describe.
- It is important to be able to compare networks and classify them according to their properties.
- Some characteristics of a network:
  - Degree distribution
  - Diameter and average path length
  - Cliquishness and clustering
  - Centrality

Some characteristics of networks (2/5)

- The degree distribution of a network is a description of the relative frequencies of nodes that have different degrees.
- $P(\eta)$ = fraction of nodes that have degree η under a degree distribution P, where
  P can be a frequency distribution (if describing data) or a probability distribution (for random networks).
- E.g., A network is regular of degree k if $P(k) = 1$ and $P(\eta) = 0$ for all $\eta \neq k$.
- The diameter of a network is the largest distance between any two nodes in the network.
- How diameter can vary across networks with (almost) the same number of nodes and links?

Some characteristics of networks (3/5)

- Average path length between nodes - the average is taken over geodesics; it is bounded above by the diameter, sometimes can be much shorter than the diameter.
- For networks that are not connected, one often reports the diameter and the average path length in the largest component and specifies if it is a giant component (unique largest component, if there is one).
- A clique is a maximal completely connected subnetwork (≥ 3 nodes) of a given network.
- Can a node be part of several cliques? Yes!
- One measure of cliquishness is to count the number and size of the cliques in a network.
- The clique structure is very sensitive to slight changes in a network.

Some characteristics of networks (4/5)

- The most common way of measuring some aspect of cliquishness is based on transitive triples or clustering.
- The individual clustering for a node i

$$Cl_i(g) = \frac{|\{jk \in g \mid k \neq j, j \in N_i(g), k \in N_i(g)\}|}{|\{jk \mid k \neq j, j \in N_i(g), k \in N_i(g)\}|}$$

We set $Cl_i(g) = 0$ if i has no more than one link.

$$Cl_i(g) = \frac{|\{jk \in g \mid k \neq j, j \in N_i(g), k \in N_i(g)\}|}{\eta_i(g)(\eta_i(g) - 1)/2}$$

- The overall clustering

$$Cl(g) = \frac{\sum_i |\{jk \in g \mid k \neq j, j \in N_i(g), k \in N_i(g)\}|}{\sum_i |\{jk \mid k \neq j, j \in N_i(g), k \in N_i(g)\}|}$$

Some characteristics of networks (5/5)

- The average clustering coefficient

$$Cl^{Avg}(g) = \frac{\sum_i Cl_i(g)}{n}$$

- $Cl(g)$ and $Cl^{Avg}(g)$ can be very different.
- Components, cliques, clusters - what a difference?
- See the file with the examples.

Centrality Measures

Motivation

- Given nodes that represent agents (players) and links that represent relationships between the agents (communication, influence, dominance ...), the following questions may appear:
  - How central is a node (player) in the network?
  - What is his position and prestige?
  - How influential is his opinion?
  - To which degree is the agent successful and powerful in collective decision making?
  - ...
- Centrality measures can be useful for the analysis of the information flows, bargaining power, infection transmission, influence, etc.
- The aim of this part is to present the main (basic) centrality and prestige measures.

Standard measures of centrality

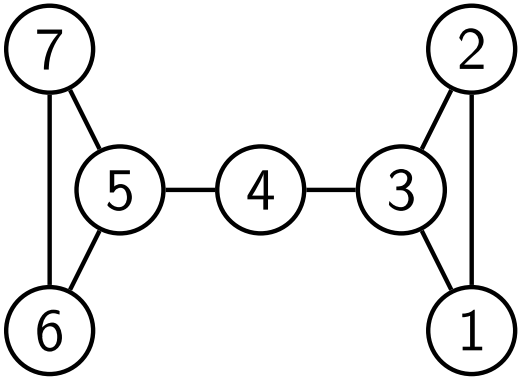
- The concept of centrality captures a kind of prominence of a node in a network.
- Since the late 1940's a variety of different centrality measures that focus on specific characteristics inherent in prominence of an agent have been developed.
- Measures of centrality can be categorized into the following main groups (Jackson (2008)):
  (1) Degree centrality - how connected a node is
  (2) Closeness centrality - how easily a node can reach other nodes
  (3) Betweenness centrality - how important a node is in terms of connecting other nodes
  (4) Prestige- and eigenvector-related centrality - how important, central, or influential a node's neighbors are.
- For extended surveys, see e.g. Jackson (2008), Goyal (2007), Wasserman & Faust (1994), Freeman (1979), Everett & Borgatti (2005).

Degree centrality of a node

- The degree centrality (Shaw (1954), Nieminen (1974)): How connected is a node in terms of direct connections?
- The degree centrality $C_i^d(g)$ of node i in network g is given by

$$C_i^d(g) = \frac{\eta_i(g)}{n-1} = \frac{|N_i(g)|}{n-1} \in [0,1]$$

- Index of the node's communication activity: the more ability to communicate directly with others, the higher the centrality.



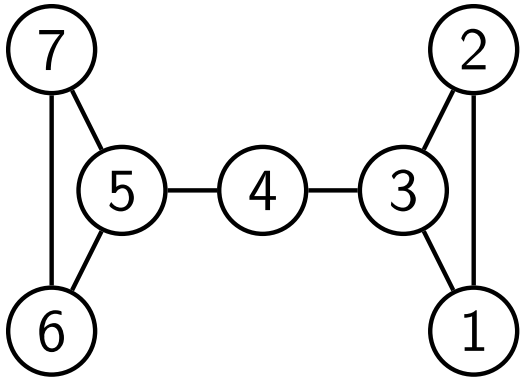
$C_i^d(g) = 0.5$ for $i \in \{3,5\}$, $C_i^d(g) = 0.33$ for $i \notin \{3,5\}$.

Degree centrality of a network

- Let i^* be a node which attains the highest degree centrality $C^d_{i^*}(g)$ in g. The degree centrality $C^d(g)$ of network g is

$$C^d(g) = \frac{\sum_{i=1}^{n}\left[C^d_{i^*}(g) - C^d_i(g)\right]}{\max_{g' \in G}\left[\sum_{i=1}^{n}\left[C^d_{i^*}(g') - C^d_i(g')\right]\right]} = \frac{\sum_{i=1}^{n}\left[C^d_{i^*}(g) - C^d_i(g)\right]}{n-2}$$

- $C^d(g) = 1$ if g is a star, and $C^d(g) = 0$ if g is a regular network.



$C^d_i(g) = \frac{1}{2}$ for $i \in \{3,5\}$, $C^d_i(g) = \frac{1}{3}$ for $i \notin \{3,5\}$

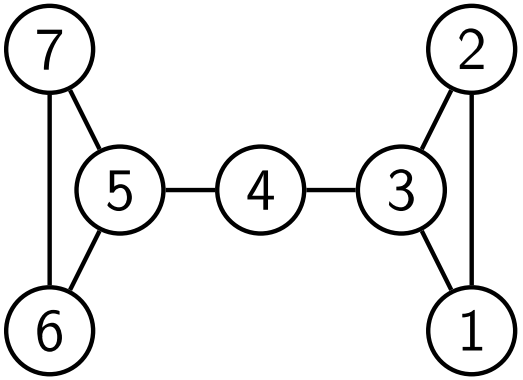
$C^d(g) = \frac{1}{5} \cdot 5 \cdot \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6}$

Closeness centrality of a node

- The closeness centrality (Beauchamp (1965), Sabidussi (1966)) is based on proximity:
  How easily can a node reach other nodes in a network?
- The closeness centrality $C_i^c(g)$ of node i in network g is

$$C_i^c(g) = \frac{n-1}{\sum_{j \neq i} l_{ij}(g)}$$

- Measure of the node's independence or efficiency: the possibility to communicate with many others depends on a minimum number of intermediaries.



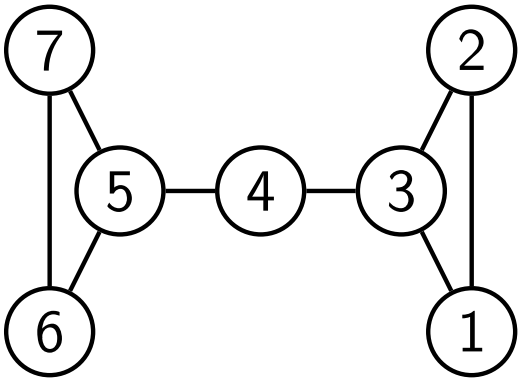
$C_4^c(g) = 0.60$, $C_3^c(g) = C_5^c(g) = 0.55$, $C_i^d(g) = 0.4$ otherwise.

Closeness centrality of a network

- Let i^* be a node which attains the highest closeness centrality $C^c_{i^*}(g)$ in g. The closeness centrality $C^c(g)$ of network g is

$$C^c(g) = \frac{\sum_{i=1}^{n}[C^c_{i^*}(g) - C^c_i(g)]}{\max_{g' \in G}\left[\sum_{i=1}^{n}[C^c_{i^*}(g') - C^c_i(g')]\right]} = \frac{\sum_{i=1}^{n}[C^c_{i^*}(g) - C^c_i(g)]}{(n-2)(n-1)/(2n-3)}$$

- $C^c(g) = 1$ if g is a star, and $C^c(g) = 0$ if g is a cycle.



$C^c_4(g) = 0.60,\ C^c_3(g) = C^c_5(g) = 0.55$

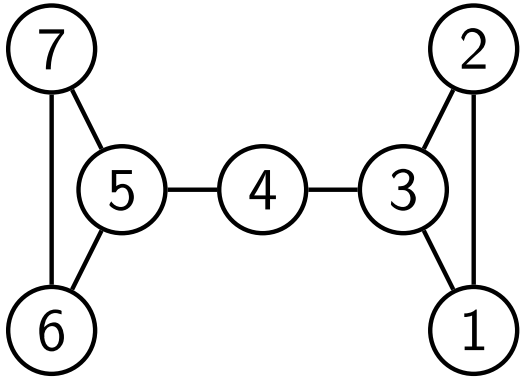
$C^c_i(g) = 0.4$ for $i \in \{1, 2, 6, 7\}$

$C^c(g) = 0.33.$

Decay centrality of a node

- We introduce a decay parameter δ, with $0 < \delta < 1$, and consider the proximity between a given node and each other node weighted by the decay.
- The decay centrality of node i in network g is

$$C_i^{dc}(g, \delta) = \sum_{j \neq i} \delta^{l_{ij}(g)}$$



For $\delta = 0.5$, $C_i^{dc}(g, 0.5) = 2$ for $i \in \{3, 4, 5\}$, $C_i^{dc}(g, 0.5) = 1.5$ for $i \in \{1, 2, 6, 7\}$.

Betweenness centrality of a node (1/2)

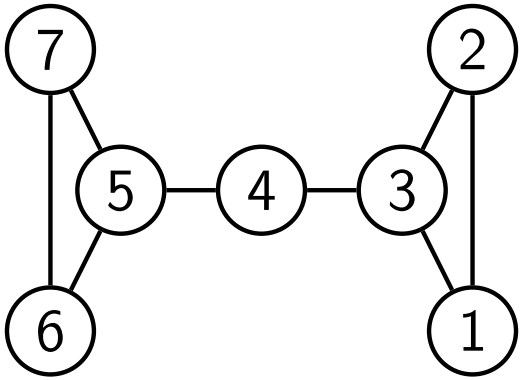
- The betweenness centrality (Bavelas (1948), Freeman (1977, 1979)):
  How important is a node in terms of connecting other nodes?
- The betweenness centrality $C_i^b(g)$ of node i in network g is

$$C_i^b(g) = \frac{2}{(n-1)(n-2)} \sum_{k \neq j : i \notin \{k,j\}} \frac{P_i(kj)}{P(kj)}$$

  $P_i(kj)$ = number of geodesics between k and j containing $i \notin \{k,j\}$
  $P(kj)$ = total number of geodesics between k and j
- Index of the potential of a node for control of communication: the possibility to intermediate in the communications of others is of importance.
- If g is a star, then $C_i^b(g) = 1$ for i being the center and $C_i^b(g) = 0$ otherwise.

Betweenness centrality of a node (2/2)

$$C_i^b(g) = \frac{2}{(n-1)(n-2)} \sum_{k \neq j : i \notin \{k,j\}} \frac{P_i(kj)}{P(kj)}$$



$C_4^b(g) = 0.60$

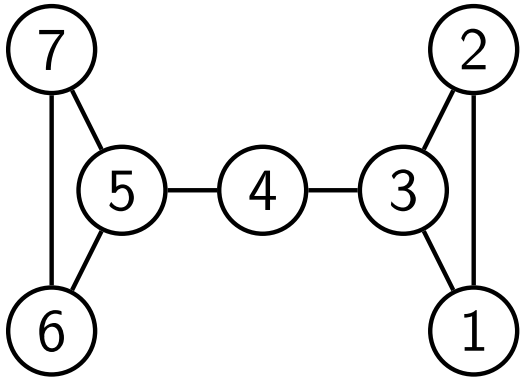
$C_3^b(g) = C_5^b(g) = 0.53$

$C_i^b(g) = 0$ for $i \in \{1, 2, 6, 7\}$

Betweenness centrality of a network

- Let i^* be a node which attains the highest betweenness centrality $C^b_{i^*}(g)$ in g.
  The betweenness centrality $C^b(g)$ of network g is

$$C^b(g) = \frac{\sum_{i=1}^{n} \left[C^b_{i^*}(g) - C^b_i(g) \right]}{n-1}$$



$C^b_4(g) = 0.60,\ C^b_3(g) = C^b_5(g) = 0.53$

$C^b_i(g) = 0$ for $i \in \{1, 2, 6, 7\}$

$C^b(g) = 0.42.$

Katz prestige

- Measures of centrality that are based on the idea that a node's importance is determined by the importance of its neighbors.
- The Katz prestige $C_i^{PK}(g)$ of node i in g is defined as

$$C_i^{PK}(g) = \sum_{j \neq i} g_{ij} \frac{C_j^{PK}(g)}{\eta_j(g)}$$

  If j has more relationships, then i gets less prestige from being connected to j. This definition is self-referential.
- Calculating $C^{PK}(g)$ - finding the unit eigenvector of $\widetilde{g}$:

$$C^{PK}(g) = \widetilde{g} C^{PK}(g)$$

$$(\mathbb{I} - \widetilde{g}) C^{PK}(g) = \mathbf{0}$$

  $\widetilde{g}$ - the normalized adjacency matrix g with $\widetilde{g}_{ij} = \frac{g_{ij}}{\eta_j(g)}$,
  we set $\widetilde{g}_{ij} = 0$ for $\eta_j(g) = 0$.
  $C^{PK}(g)$ - the $n \times 1$ vector of $C_i^{PK}(g)$, $i \in N$,
  $\mathbb{I}$ - the $n \times n$ identity matrix, $\mathbf{0}$ - the $n \times 1$ vector of 0's.

Eigenvector centrality

- If we do not normalize g, we get the eigenvector centrality $C^e(g)$ associated with g (Bonacich (1972)).
- The centrality of a node is proportional to the sum of the centrality of its neighbors.

$$\lambda C_i^e(g) = \sum_j g_{ij} C_j^e(g)$$

$$\lambda C^e(g) = g C^e(g)$$

  and thus $C^e(g)$ is an eigenvector of g and λ is the corresponding largest eigenvalue of matrix g.
- The Katz prestige can be seen as a kind of eigenvector centrality with the network adjacency matrix being weighted.

Second prestige measure of Katz

- $C^{PK2}(g, a)$ = the second prestige measure of Katz (1953)
- Introducing an attenuation parameter a to adjust the measure for the lower 'effectiveness' of longer walks in a network.
- The prestige of a node is a weighted sum of the walks that emanate from it, and a walk of length k is of worth a^k, where $0 < a < 1$. The vector of prestige of nodes is

$$C^{PK2}(g, a) = ag\mathbf{1} + a^2g^2\mathbf{1} + \cdots + a^kg^k\mathbf{1} + \cdots$$

  where $\mathbf{1}$ is the $n \times 1$ vector of 1's.
- Each entry of the vector $g^k\mathbf{1}$ is the total number of walks of length k that emanate from each node, and $g\mathbf{1}$ is simply the vector of degrees of nodes.
- For a sufficiently small, $C^{PK2}(g, a)$ is finite and

$$C^{PK2}(g, a) - agC^{PK2}(g, a) = ag\mathbf{1}$$

$$C^{PK2}(g, a) = (\mathbb{I} - ag)^{-1}\, ag\mathbf{1}.$$

Bonacich centrality

- A two-parameter family of prestige measures which can be seen as a direct extension of $C^{PK2}(g, a)$.
- An agent can have some status which does not depend on its connections to others.
- Bonacich centrality (Bonacich (1987)) is given by

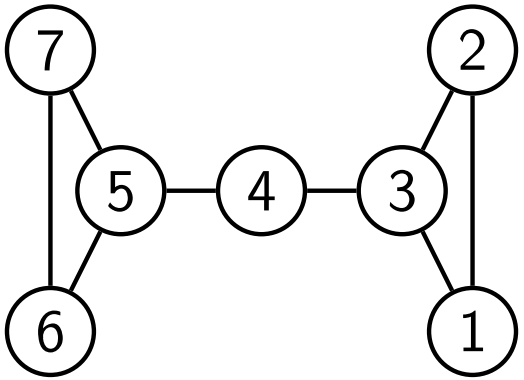
$$C^B(g, a, b) = ag\mathbf{1} + abg^2\mathbf{1} + \cdots + ab^k g^{k+1}\mathbf{1} + \cdots$$

$$C^B(g, a, b) = (\mathbb{I} - bg)^{-1} ag\mathbf{1}$$

where a and b are parameters, and b is sufficiently small.

  - b captures how the value of being connected to another node decays with distance.
  - a captures the base value on each node.
  - For $b = 0$, $C^B(g, a, b)$ takes into account only walks of length 1 and reduces to $ad_i(g)$.
  - For $b > 0$, $C^B(g, a, b)$ takes into account more distant interactions.
  - Obviously $C^{PK2}(g, a)$ and $C^B(g, a, b)$ coincide when $a = b$.

Example (ctd)



Centrality measures ↓ Nodes →	1,2,6,7	3,5	4
Degree, Katz prestige	0.33	0.50	0.33
Closeness	0.40	0.55	0.60
Decay centrality, $\delta = 0.5$	1.5	2.0	2.0
Decay centrality, $\delta = 0.75$	3.1	3.7	3.8
Decay centrality, $\delta = 0.25$	0.59	0.84	0.75
Betweenness	0	0.53	0.60
Eigenvector centrality	0.47	0.63	0.54
Second Katz prestige, $a = 1/3$	3.1	4.3	3.5
Bonacich centrality, $a = 1$, $b = 1/3$	9.4	13	11
Bonacich centrality, $a = 1$, $b = 1/4$	4.9	6.8	5.4

See the file with the examples

In this example, the most central are

- either agent 4
- or agents 3 and 5.

Different centrality measures capture different aspects of centrality, and therefore can have highest values for different individuals.