An Introduction to the Choquet integral

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Outline

Set functions and games

- X: finite universe. *Set function on X:* $\xi : 2^X \to \mathbb{R}$.
- A set function can be
  1. *Additive* if $\xi(A \cup B) = \xi(A) + \xi(B)$ for every disjoint $A, B \in 2^X$;
  2. *Monotone* if $\xi(A) \leqslant \xi(B)$ whenever $A \subseteq B$;
  3. *Grounded* if $\xi(\varnothing) = 0$;
  4. *Normalized* if $\xi(X) = 1$.
- A *game* $v : 2^X \to \mathbb{R}$ is a grounded set function.
- $\mathcal{G}(X)$: set of games on X
- *conjugate* $\overline{\xi}$ of ξ:

$$\overline{\xi}(A) = \xi(X) - \xi(A^c) \qquad (A \in 2^X).$$

- Note that
  1. If $\xi(\varnothing) = 0$, then $\overline{\xi}(X) = \xi(X)$ and $\overline{\overline{\xi}} = \xi$;
  2. If ξ is additive, then $\overline{\xi} = \xi$ (ξ is *self-conjugate*).

Measures and capacities

- A *measure* is a nonnegative and additive set function
- A normalized measure is called a *probability measure*.
- A *capacity* $\mu : 2^X \to \mathbb{R}$ is a grounded monotone set function, i.e., $\mu(\varnothing) = 0$ and $\mu(A) \leqslant \mu(B)$ whenever $A \subseteq B$.
- $\mathcal{MG}(X)$: set of capacities on X
- $\mathcal{MG}_0(X)$: set of normalized capacities on X

Two different interpretations

- **capacities/games as a means to represent the importance/power/worth of a group:**
  - X: set of persons, usually called players, agents, voters, experts, decision makers, etc.
  - $A \subseteq X$: *coalition*, group of persons, who cooperate to achieve some common goal
  - $\mu(A)$: to what extent the goal is achieved by A
- **capacities as a means to represent uncertainty:**
  - X: set of possible outcomes of some experiment. It is supposed that X is exhaustive, and that each experiment produces a single outcome.
  - $A \subseteq X$: event
  - $\mu(A)$: uncertainty that the event A contains the outcome of an experiment, with $\mu(A) = 0$ indicating total uncertainty, and $\mu(A) = 1$ indicating that there is no uncertainty.

Examples

Example

Let X be a set of firms. Certain firms may form a coalition in order to control the market for a given product. Then $\mu(A)$ may be taken as the annual benefit of the coalition A. $\Diamond$

Example

Let X be a set of voters in charge of electing a candidate for some important position (president, director, etc.) or voting a bill by a yes/no decision. Before the election, groups of voters may agree to vote for the same candidate (or for yes or no). In many cases (presidential elections, parliament, etc.), these coalitions correspond to the political parties or to alliances among them. If the result of the election is in accordance with the wish of coalition A, the coalition is said to be winning, and we set $\mu(A) = 1$, otherwise it is loosing and $\mu(A) = 0$. $\Diamond$

Example

Let X be a set of workers in a factory, producing some goods. The aim is to produce these goods as much as possible in a given time (say, in one day). Then $\mu(A)$ is the number of goods produced by the group A in one day. Since the production needs in general the collaboration of several workers with different skills, it is likely that $\mu(A) = 0$ if A is a singleton or a too small group. $\Diamond$

Examples

Example

David throws a dice, and wonders what number will show. Here $X = \{1, 2, 3, 4, 5, 6\}$, and $\mu(\{1, 3, 5\})$ quantifies the uncertainty of obtaining an odd number. $\diamondsuit$

Example

A murder has been committed. After some investigation, it is found that the guilty is either Alice, Bob or Charles. Then $X = \{\text{Alice, Bob, Charles}\}$, and $\mu(\{\text{Bob, Charles}\})$ quantifies the degree to which it is "certain" (the precise meaning of this word being conditional on the type of capacity used) that the guilty is Bob or Charles. $\diamondsuit$

Example

Glenn is an amateur of antique chinese porcelain. He enters a shop and sees a magnificent vase, wondering how old (and how expensive) this vase could be. Then X is the set of numbers from, say -3000 to 2014, i.e., the possible date expressed in years A.C. when the vase was created. For example, $\mu([1368, 1644])$ indicates tho what degree it is certain that it is a vase of the Ming period. $\diamondsuit$

Example

Leonard is planning to go to the countryside tomorrow for a picnic. He wonders if the wheather will be favorable or not. Here X is the set of possible states of the weather, like "sunny", "rainy", "cloudy", and so on. For example, $\mu(\{\text{sunny, cloudy}\})$ indicates to what degree of certainty it will not rain, and so if the picnic is conceivable or not. $\diamondsuit$

Properties

1. v is *superadditive* if for any $A, B \in 2^X$, $A \cap B = \varnothing$,
$$v(A \cup B) \geqslant v(A) + v(B).$$
(*subadditive* if the reverse inequality holds)
2. v is *supermodular* if for any $A, B \in 2^X$,
$$v(A \cup B) + v(A \cap B) \geqslant v(A) + v(B).$$
(*submodular* if the reverse inequality holds)
3. v *k-monotone* ($k \geqslant 2$) if for $A_1, \dots, A_k \in 2^X$,
$$v\Big(\bigcup_{i=1}^{k} A_i\Big) \geqslant \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcap_{i \in I} A_i\Big).$$
v is *totally monotone* if it is k-monotone for any $k \geqslant 2$
4. v *k-alternating* ($k \geqslant 2$) if for $A_1, \dots, A_k \in 2^X$,
$$v\Big(\bigcap_{i=1}^{k} A_i\Big) \leqslant \sum_{\substack{I \subseteq \{1,\dots,k\} \\ I \neq \varnothing}} (-1)^{|I|+1} v\Big(\bigcup_{i \in I} A_i\Big).$$
v is *totally alternating* if it is k-alternating for any $k \geqslant 2$.

Properties

Theorem

Let v be a game on X. The following holds.

1. *v superadditive $\Rightarrow \overline{v} \geqslant v$.*
2. *v is k-monotone (resp., k-alternating) for some $k \geqslant 2$ if and only if $\overline{v}$ is k-alternating (resp., k-monotone). In particular, v is supermodular (resp., submodular) if and only if $\overline{v}$ is submodular (resp., supermodular).*
3. *$v \geqslant 0$ and supermodular implies that v is monotone.*

0-1 capacities

- A *0-1-capacity* is a capacity valued on $\{0, 1\}$.
- Apart the null capacity 0, all 0-1-capacities are normalized.
- In game theory, 0-1-capacities are called *simple games*.
- A set A is a *winning coalition* for μ if $\mu(A) = 1$.
- A 0-1-capacity μ is uniquely determined by the antichain of its minimal winning coalitions.
- The number of antichains in 2^X with $|X| = n$ is the Dedekind number $M(n)$

n	$M(n)$
0	2
1	3
2	6
3	20
4	168
5	7581
6	7828354
7	2414682040998
8	56130437228687557907788

Unanimity games

- Let $A \subseteq X$, $A \neq \varnothing$. The *unanimity game centered on A* is the game u_A defined by

$$u_A(B) = \begin{cases} 1, & \text{if } B \supseteq A \\ 0, & \text{otherwise.} \end{cases}$$

- Unanimity games are 0-1-valued capacities.

Possibility and necessity measures

- A *possibility measure* or *maxitive measure* on a X is a normalized capacity Π on X satisfying

$$\Pi(A \cup B) = \max(\Pi(A), \Pi(B)) \text{ for all } A, B \in 2^X$$

- A *necessity measure* or *minitive measure* is a normalized capacity N satisfying

$$N(A \cap B) = \min(N(A), N(B)) \text{ for all } A, B \in 2^X$$

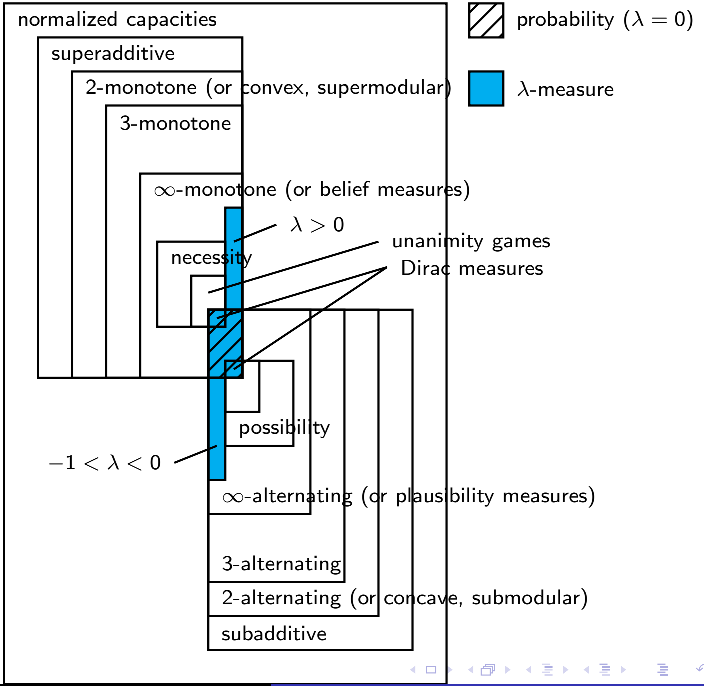
- The conjugate of a possibility measure (resp., a necessity measure) is a necessity measure (resp., a possibility measure).

Belief and plausibility measures

- A *belief measure* is a totally monotone normalized capacity.
- A *plausibility measure* is a totally alternating normalized capacity.
- the conjugate of a belief measure is a plausibility measure, and vice versa
- Possibility and necessity measures are particular cases of belief and plausibility measures
- A particular case: the *λ-measure* ($\lambda > -1$) is a normalized capacity satisfying for every $A, B \in 2^X$, $A \cap B = \varnothing$

$$\mu(A \cup B) = \mu(A) + \mu(B) + \lambda\mu(A)\mu(B)$$

- A λ-measure is a belief measure if and only if $\lambda \geqslant 0$, and is a plausibility measure otherwise



normalized capacities
superadditive
2-monotone (or convex, supermodular)
3-monotone
∞-monotone (or belief measures)
$\lambda > 0$
unanimity games
Dirac measures
necessity
possibility
$-1 < \lambda < 0$
∞-alternating (or plausibility measures)
3-alternating
2-alternating (or concave, submodular)
subadditive
probability ($\lambda = 0$)
λ-measure

Definition

Let ξ be a set function on X. The *Möbius transform* or *Möbius inverse* of ξ is a set function m^ξ on X defined by

$$m^\xi(A) = \sum_{B\subseteq A} (-1)^{|A\setminus B|}\xi(B) \tag{1}$$

for every $A \subseteq X$.

Given m^ξ, it is possible to recover ξ by the formula

$$\xi(A) = \sum_{B\subseteq A} m^\xi(B) \qquad (A \subseteq X). \tag{2}$$

Properties

1. v is additive if and only if $m^v(A) = 0$ for all $A \subseteq X$, $|A| > 1$. Moreover, we have $m^v(\{i\}) = v(\{i\})$ for all $i \in X$.
2. v is monotone if and only if
$$\sum_{i \in L \subseteq K} m^v(L) \geqslant 0 \qquad (K \subseteq X, \quad i \in K).$$
3. Let $k \geqslant 2$ be fixed. v is k-monotone if and only if
$$\sum_{L \in [A,B]} m^v(L) \geqslant 0 \qquad (A, B \subseteq X, \quad A \subseteq B, \quad 2 \leqslant |A| \leqslant k).$$
4. If v is k-monotone for some $k \geqslant 2$, then $m^v(A) \geqslant 0$ for all $A \subseteq X$ such that $2 \leqslant |A| \leqslant k$.
5. v is a nonnegative totally monotone game if and only if $m^v \geqslant 0$.

k-additive games

Definition

A game v on X is said to be *k-additive* for some integer $k \in \{1, \dots, |X|\}$ if $m^v(A) = 0$ for all $A \subseteq X$, $|A| > k$, and there exists some $A \subseteq X$ with $|A| = k$ such that $m^v(A) \neq 0$.

- A game v is *at most k-additive* for some $1 \leqslant k \leqslant |X|$ if it is k'-additive for some $k' \in \{1, \dots, k\}$
- The set of k-additive games on X (resp., capacities, etc.,) is denoted by $\mathcal{G}^k(X)$ (resp., $\mathcal{MG}^k(X)$, etc.)
- We denote by $\mathcal{G}^{\leqslant k}(X), \mathcal{MG}^{\leqslant k}(X)$ the set of at most k-additive games and capacities.

$$\begin{aligned}\mathcal{G}(X) &= \mathcal{G}^1(X) \cup \mathcal{G}^2(X) \cup \cdots \cup \mathcal{G}^{|X|}(X)\\ &= \mathcal{G}^1(X) \cup \mathcal{G}^{\leqslant 2}(X) \cup \cdots \cup \mathcal{G}^{\leqslant |X|}(X)\end{aligned}$$

The vector space of games

For any nonempty $A \subseteq X$ the *identity game δ_A centered at A* is the 0-1-game defined by

$$\delta_A(B) = \begin{cases} 1, & \text{if } A = B \\ 0, & \text{otherwise.} \end{cases}$$

Theorem

The set of identity games $\{\delta_A\}_{A \in 2^X \setminus \{\varnothing\}}$ and the set of unanimity games $\{u_A\}_{A \in 2^X \setminus \{\varnothing\}}$ are bases of $\mathcal{G}(X)$ of dimension $2^{|X|} - 1$.

- In the basis of identity games, the coordinates of a game v are simply $\{v(A)\}_{A \in 2^X \setminus \{\varnothing\}}$.
- We have for any game $v \in \mathcal{G}(X)$

$$v(B) = \sum_{A \in 2^X \setminus \{\varnothing\}} \lambda_A u_A(B) = \sum_{A \subseteq B, A \neq \varnothing} \lambda_A \qquad (B \subseteq X).$$

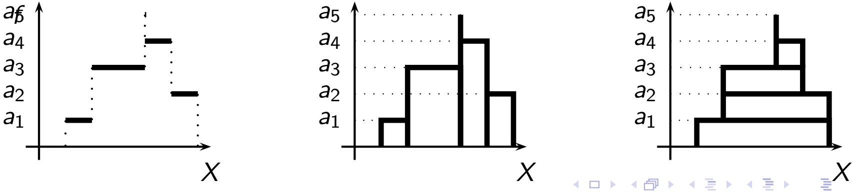
It follows that the coefficients of a game v in the basis of unanimity games are its Möbius transform: $\lambda_A = m^v(A)$ for all $A \subseteq X, A \neq \varnothing$

Outline

Simple functions

- X nonempty set
- A function $f : X \to \mathbb{R}$ is *simple* if its range ran$f$ is a finite set.
- We assume ran$f = \{a_1, \ldots, a_n\}$, supposing $0 \leqslant a_1 < a_2 < \cdots < a_n$. Then

$$
\begin{aligned}
f &= \sum_{i=1}^{n} a_i 1_{\{x \in X \,:\, f(x) = a_i\}} \\
&= \sum_{i=1}^{n} (a_i - a_{i-1}) 1_{\{x \in X \,:\, f(x) \geqslant a_i\}}
\end{aligned}
$$



The decumulative distribution function

- X nonempty set, $\mathcal{F}$ algebra on X (closed under finite union and complementation)
- f is *$\mathcal{F}$-measurable* if $\{x : f(x) > t\}$ and $\{x : f(x) \geqslant t\}$ belong to $\mathcal{F}$ for all $t \in \mathbb{R}$.
- $B(\mathcal{F})$: set of bounded $\mathcal{F}$-measurable functions; $B^+(\mathcal{F})$: set of bounded $\mathcal{F}$-measurable nonnegative functions
- Let $f \in B(\mathcal{F})$, μ a capacity on $\mathcal{F}$. The *decumulative (distribution) function* of f w.r.t. μ is

$$G_{\mu,f}(t) = \mu(\{x \in X : f(x) \geqslant t\}) \quad (t \in \mathbb{R})$$

- Properties of $G_{\mu,f}$:
  - it is a nonnegative nonincreasing function, with $G_{\mu,f}(0) = \mu(X)$;
  - $G_{\mu,f}(t) = \mu(X)$ on the interval $[0, \mathrm{ess\ inf}_\mu f]$;
  - it has a compact support, namely $[0, \mathrm{ess\ sup}_\mu f]$.

The Choquet integral

Definition

Let $f \in B^+(\mathcal{F})$ and μ be a capacity on $(X, \mathcal{F})$. The *Choquet integral* of f w.r.t. μ is defined by

$$\int f \, \mathrm{d}\mu = \int_0^\infty G_{\mu,f}(t) \, \mathrm{d}t,$$

where the right hand-side integral is the Riemann integral.

Remark: $>$ can replace $\geqslant$ in the definition of $G_{\mu,f}$.
A fundamental fact is the following.

Lemma

Let $A \in \mathcal{F}$ (i.e., 1_A is measurable). Then for every capacity μ

$$\int 1_A \, \mathrm{d}\mu = \mu(A).$$

The Sugeno integral

Definition

Let $f \in B^+(\mathcal{F})$ be a function and μ be a capacity on $(X, \mathcal{F})$. The *Sugeno integral* of f w.r.t. μ is defined by

$$\fint f \,\mathrm{d}\mu = \bigvee_{t \geqslant 0} (G_{\mu,f}(t) \wedge t) = \bigwedge_{t \geqslant 0} (G_{\mu,f}(t) \vee t). \tag{3}$$

Remark 1: $>$ can replace $\geqslant$ in the definition of $G_{\mu,f}$.

Remark 2: $\fint 1_A \,\mathrm{d}\mu = \mu(A)$ for any $A \in \mathcal{F}$ holds if μ is normalized.

The case of real-valued functions

- For any $f \in B(\mathcal{F})$ we write
$$f = f^+ - f^-, \text{ with } f^+ = 0 \vee f, \quad f^- = (-f)^+.$$
- The *symmetric Choquet integral* is defined by
$$\check{\int} f \,\mathrm{d}\mu = \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\mu.$$
- Symmetry property:
$$\check{\int} (-f) \,\mathrm{d}\mu = -\check{\int} f \,\mathrm{d}\mu$$
- Homogeneity (ratio scale invariance):
$$\check{\int} \alpha f \,\mathrm{d}\mu = \alpha \check{\int} f \,\mathrm{d}\mu \qquad (\alpha \in \mathbb{R})$$

The case of real-valued functions

- The asymmetric Choquet integral is defined by
$$\int f \,\mathrm{d}\mu = \int f^+ \,\mathrm{d}\mu - \int f^- \,\mathrm{d}\overline{\mu}$$
- Positive homogeneity and translation invariance (interval scale invariance)
$$\int (\alpha f + \beta 1_X) \,\mathrm{d}\mu = \alpha \int f \,\mathrm{d}\mu + \beta\mu(X) \qquad (\alpha > 0, \beta \in \mathbb{R})$$
- Expression w.r.t. the decumulative function:
$$\int f \,\mathrm{d}\mu = \int_0^{\infty} \mu(f \geqslant t) \,\mathrm{d}t + \int_{-\infty}^{0} \big(\mu(f \geqslant t) - \mu(X)\big) \,\mathrm{d}t.$$

The Choquet integral of simple functions

- f: simple, measurable nonnegative function with $\mathrm{ran} f = \{a_1, \dots, a_n\}$, and $0 \leqslant a_1 < a_2 < \cdots < a_n$
- $A_i = \{x \in X : f(x) \geqslant a_i\}$, for $i = 1, \dots, n$
- From the decumulative function, one finds
$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} (a_i - a_{i-1})\mu(A_i),$$
letting $a_0 = 0$, and
$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} a_i \big(\mu(A_i) - \mu(A_{i+1})\big),$$
with the convention $A_{n+1} = \varnothing$.

The Sugeno integral of simple functions

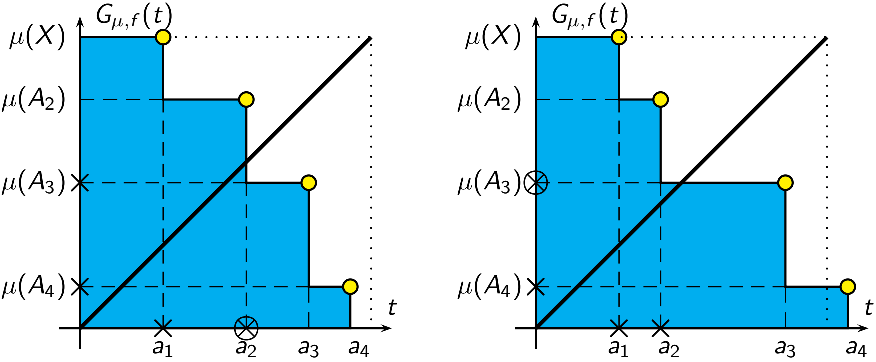
With same notations, one finds

$$\int f \,\mathrm{d}\mu = \bigvee_{i=1}^{n} \big(a_i \wedge \mu(A_i)\big)$$

$$\int f \,\mathrm{d}\mu = \bigwedge_{i=0}^{n} \big(a_i \vee \mu(A_{i+1})\big)$$

with the convention $A_{n+1} = \varnothing$ and $a_0 = 0$.

The Sugeno integral of simple functions



The Choquet integral on finite sets

- $X = \{x_1, \dots, x_n\}$
- $f : X \to \mathbb{R}_+$, $f_i = f(x_i)$
- Choose a permutation σ on X s.t. $f_{\sigma(1)} \leqslant f_{\sigma(2)} \leqslant \cdots \leqslant f_{\sigma(n)}$.
- $A^{\uparrow}_{\sigma}(i) = \{x_{\sigma(i)}, x_{\sigma(i+1)}, \dots, x_{\sigma(n)}\} \qquad (i = 1, \dots, n)$
- From the case of simple functions, we obtain directly

$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} (f_{\sigma(i)} - f_{\sigma(i-1)})\mu(A^{\uparrow}_{\sigma}(i))$$

$$\int f \,\mathrm{d}\mu = \sum_{i=1}^{n} f_{\sigma(i)}\big(\mu(A^{\uparrow}_{\sigma}(i)) - \mu(A^{\uparrow}_{\sigma}(i+1))\big)$$

with the conventions $f_{\sigma(0)} = 0$ and $A^{\uparrow}_{\sigma}(n+1) = \varnothing$.

With the same notations we obtain

$$\int f \,\mathrm{d}\mu = \bigvee_{i=1}^{n} \big(f_{\sigma(i)} \wedge \mu(A^{\uparrow}_{\sigma}(i))\big)$$

$$\int f \,\mathrm{d}\mu = \bigwedge_{i=0}^{n} \big(f_{\sigma(i)} \vee \mu(A^{\uparrow}_{\sigma}(i+1))\big)$$

Example: workers in a factory (ctd)

- $X = \{x_1, \ldots, x_n\}$ set of workers
- Each worker starts at 8:00, works continuously but they leave at different times. We denote by $f(x_i)$ the number of worked hours for x_i, and label the workers so that $f(x_1) \leqslant f(x_2) \leqslant \cdots \leqslant f(x_n)$.
- The productivity per hour of a group $A \subseteq X$ is given by $\mu(A)$.
- The total number of goods produced in a day is given by:
  - The entire group X has worked $f(x_1)$ hours;
  - Then x_1 leaves and the group $X \setminus \{x_1\} = \{x_2, \ldots, x_n\}$ works in addition $f(x_2) - f(x_1)$ hours;
  - Then x_2 leaves and the group $X \setminus \{x_1, x_2\} = \{x_3, \ldots, x_n\}$ works in addition $f(x_3) - f(x_2)$, etc.,
  - Finally only x_n remains and he works $f(x_n) - f(x_{n-1})$ hours.

  Then the total production is
  $f(x_1)\mu(X) + (f(x_2) - f(x_1))\mu(\{x_2, \ldots, x_n\}) + (f(x_3) - f(x_2))\mu(\{x_3, \ldots, x_n\}) + \cdots + (f(x_n) - f(x_{n-1}))\mu(\{x_n\}) = \int f \,\mathrm{d}\mu$.

Outline

1. Capacities and set functions
2. The Choquet and Sugeno integrals
3. **Properties**
4. Characterizations
5. Particular cases
6. The concave integral

Basic properties of the Choquet integral

- Positive homogeneity:

$$\int \alpha f \,\mathrm{d}v = \alpha \int f \,\mathrm{d}v \qquad (\alpha \geqslant 0)$$

- Monotonicity w.r.t. the integrand: for any capacity μ,

$$f \leqslant f' \Rightarrow \int f \,\mathrm{d}\mu \leqslant \int f' \,\mathrm{d}\mu \qquad (f, f' \in B(\mathcal{F}))$$

- Monotonicity w.r.t. the game for nonnegative integrands: if $f \geqslant 0$,

$$v \leqslant v' \Rightarrow \int f \,\mathrm{d}v \leqslant \int f \,\mathrm{d}v' \qquad (v, v' \in \mathcal{BV}(\mathcal{F}))$$

- Linearity w.r.t. the game:

$$\int f \,\mathrm{d}(v+\alpha v') = \int f \,\mathrm{d}v + \alpha \int f \,\mathrm{d}v' \qquad (v, v' \in \mathcal{BV}(\mathcal{F}), \alpha \in \mathbb{R})$$

Basic properties of the Choquet integral

- Boundaries: inf f and sup f are attained:

$$\inf f = \int f \,\mathrm{d}\mu_{\min}, \quad \sup f = \int f \,\mathrm{d}\mu_{\max},$$

  with $\mu_{\min}(A) = 0$ for all $A \subset X$, $A \in \mathcal{F}$, $\mu_{\min}(X) = 1$, and $\mu_{\max}(A) = 1$ for all nonempty $A \in \mathcal{F}$ (see Section **??**);
- Boundaries: for any normalized capacity μ,

$$\mathrm{ess\ inf}_\mu f \leqslant \int f \,\mathrm{d}\mu \leqslant \mathrm{ess\ sup}_\mu f$$

- Continuity

Basic properties of the Sugeno integral

Let f be a function in $B^+(\mathcal{F})$, and μ a capacity on $(X, \mathcal{F})$.

- Positive $\wedge$-homogeneity:

$$-\!\!\!\!\!\!\int (\alpha 1_X \wedge f)\,\mathrm{d}\mu = \alpha \wedge -\!\!\!\!\!\!\int f\,\mathrm{d}\mu \qquad (\alpha \geqslant 0)$$

- Positive $\vee$-homogeneity if $\mathrm{ess\ sup}_\mu f \leqslant \mu(X)$:

$$-\!\!\!\!\!\!\int (\alpha 1_X \vee f)\,\mathrm{d}\mu = \alpha \vee -\!\!\!\!\!\!\int f\,\mathrm{d}\mu \qquad (\alpha \in [0, \mathrm{ess\ sup}_\mu f]).$$

- Hat function: for every $\alpha \geqslant 0$ and for every $A \in \mathcal{F}$,

$$-\!\!\!\!\!\!\int \alpha 1_A\,\mathrm{d}\mu = \alpha \wedge \mu(A)$$

- Monotonicity w.r.t. the integrand:

$$f \leqslant f' \Rightarrow -\!\!\!\!\!\!\int f\,\mathrm{d}\mu \leqslant -\!\!\!\!\!\!\int f'\,\mathrm{d}\mu \qquad (f, f' \in B^+(\mathcal{F}))$$

Basic properties of the Sugeno integral

- Monotonicity w.r.t. the capacity:

$$\mu \leqslant \mu' \Rightarrow \fint f \,\mathrm{d}\mu \leq \fint f \,\mathrm{d}\mu' \qquad (\mu, \mu' \text{ on } (X, \mathcal{F}))$$

- Max-min linearity w.r.t. the capacity:

$$\fint f \,\mathrm{d}(\mu \vee (\alpha \wedge \mu')) = \fint f \,\mathrm{d}\mu \vee \left(\alpha \wedge \fint f \,\mathrm{d}\mu'\right) \qquad (\mu, \mu' \text{ capacities on } ($$

- Boundaries: $\inf f$ and $\sup f$ are attained:

$$\inf f = \fint f \,\mathrm{d}\mu_{\min}, \quad \sup f = \fint f \,\mathrm{d}\mu_{\max},$$

  with $\mu_{\min}(A) = 0$ for all $A \subset X$, $A \in \mathcal{F}$, and $\mu_{\max}(A) = 1$ for all nonempty $A \in \mathcal{F}$;

- Boundaries:

$$\text{ess inf}_\mu f \leqslant \fint f \,\mathrm{d}\mu \leqslant (\text{ess sup}_\mu f) \wedge \mu(X)$$

Comonotonic functions

- Two functions $f, g : X \to \mathbb{R}$ are *comonotonic* if there is no $x, x' \in X$ such that $f(x) < f(x')$ and $g(x) > g(x')$
- Equivalently when X is finite, f, g are comonotonic if there exists a permutation σ on X such that $f_{\sigma(1)} \leqslant \cdots \leqslant f_{\sigma(n)}$ and $g_{\sigma(1)} \leqslant \cdots \leqslant g_{\sigma(n)}$

Theorem

Let f, g be comonotonic functions on X (finite). Then for any game v, the Choquet integral is comonotonically additive, and the Sugeno integral is comonotonically maxitive and minitive for any capacity μ:

$$\int (f+g)\,\mathrm{d}v = \int f\,\mathrm{d}v + \int g\,\mathrm{d}v$$

$$⨍ (f \vee g)\,\mathrm{d}\mu = ⨍ f\,\mathrm{d}\mu \ \vee\ ⨍ g\,\mathrm{d}\mu$$

$$⨍ (f \wedge g)\,\mathrm{d}\mu = ⨍ f\,\mathrm{d}\mu \ \wedge\ ⨍ g\,\mathrm{d}\mu.$$

Supermodular capacities

For any game v, the following conditions are equivalent:

1. v is supermodular;
2. The Choquet integral is superadditive, that is,
$$\int (f+g)\,\mathrm{d}v \geqslant \int f\,\mathrm{d}v + \int g\,\mathrm{d}v$$
for all $f, g : X \to \mathbb{R}$
3. The Choquet integral is supermodular, that is,
$$\int (f \vee g)\,\mathrm{d}v + \int (f \wedge g)\,\mathrm{d}v \geqslant \int f\,\mathrm{d}v + \int g\,\mathrm{d}v$$
for all $f, g : X \to \mathbb{R}$;
4. The Choquet integral is concave, that is,
$$\int (\lambda f + (1-\lambda)g)\,\mathrm{d}v \geqslant \int \lambda f\,\mathrm{d}v + (1-\lambda)\int g\,\mathrm{d}v$$
for all $\lambda \in [0,1]$, $f, g : X \to \mathbb{R}$.

5. The Choquet integral yields the lower expected value on the core of v:
$$\int f \, \mathrm{d}v = \min_{\phi \in \mathrm{core}(v)} \int f \, \mathrm{d}\phi, \tag{4}$$
where $\mathrm{core}(v)$ is the set of additive games ϕ on X such that $\phi(X) = v(X)$ and $\phi(S) \geqslant v(S)$ for all $S \in 2^X$.

Theorem

The following holds:

1. $\fint (f \vee g)\,\mathrm{d}\mu = \fint f\,\mathrm{d}\mu \vee \fint g\,\mathrm{d}\mu$ *for all* f, g *if and only if* μ *is maxitive;*
2. $\fint (f \wedge g)\,\mathrm{d}\mu = \fint f\,\mathrm{d}\mu \wedge \fint g\,\mathrm{d}\mu$ *for all* f, g *if and only if* μ *is minitive.*

Let X be finite, v be a game, and $f : X \to \mathbb{R}$. Then

$$\int f \, \mathrm{d}v = \sum_{A \subseteq X} m^v(A) \bigwedge_{i \in A} f_i.$$

where m^v is the Möbius transform of v.

Outline

Characterization of the Choquet integral

Theorem

(Schmeidler 1986) Let $I : B(\mathcal{F}) \to \mathbb{R}$ be a functional. Define the set function $v(A) = I(1_A)$ on $\mathcal{F}$. The following propositions are equivalent:

1. *I is monotone and comonotonically additive;*
2. *v is a capacity, and for all $f \in B(\mathcal{F})$, $I(f) = \int f \,\mathrm{d}v$.*

Characterization of the Sugeno integral

Theorem

Let $|X| = n$, $\mathcal{F} = 2^X$, and let $I : (\mathbb{R}_+)^X \to \mathbb{R}_+$ be a functional. Define the set function $\mu(A) = I(1_A)$, $A \subseteq X$. The following propositions are equivalent:

1. *I is comonotonically maxitive, satisfies $I(\alpha 1_A) = \alpha \wedge I(1_A)$ for every $\alpha \geqslant 0$ and $A \subseteq X$ (hat function property), and $I(1_X) = 1$;*
2. *μ is a normalized capacity on X and $I(f) = \fint f \, \mathrm{d}\mu$.*

Outline

Outline

The concave integral (Lehrer 2009)

We consider $|X| = n$, $\mathcal{F} = 2^X$.

Definition

Let $f : X \to \mathbb{R}_+$ and μ be a capacity. The *concave integral* of f w.r.t. μ is given by:

$$\int^{\text{cav}} f \,\mathrm{d}\mu = \sup\left\{ \sum_{S \subseteq X} \alpha_S \mu(S) \; : \; \sum_{S \subseteq X} \alpha_S 1_S = f, \quad \alpha_S \geqslant 0, \forall S \subseteq X \right\}. \tag{5}$$

Nota: "sup" can be replaced by "max".

The example of workers in a factory revisited

- $X = \{1, 2, 3\}$, let μ on X be defined by $\mu(1) = \mu(2) = \mu(3) = 0.2$, $\mu(12) = 0.9$, $\mu(13) = 0.8$, $\mu(23) = 0.5$ and $\mu(123) = 1$
- Each worker is given an amount of time: $f_1 = 1$ for worker 1, $f_2 = 0.4$ and $f_3 = 0.6$ for workers 2 and 3
- How should the workers organize themselves in teams so as to maximize the total production while not exceeding their alloted time?
- The answer is given by the concave integral: team $\{1, 2\}$ is working 0.4 unit of time and team $\{1, 3\}$ is working 0.6 unit of time, which yields

$$0.9 \cdot 0.4 + 0.8 \cdot 0.6 = 0.84$$

The example of workers in a factory revisited

- By contrast, the Choquet integral computes the total productivity under the constraint that the teams form a specific chain, in this case the teams are $\{1,2,3\}$, $\{1,3\}$ and $\{1\}$ for durations 0.4, 0.2 and 0.4 respectively, yielding
$$0.4 + 0.8 \cdot 0.2 + 0.2 \cdot 0.4 = 0.64$$

Properties

The following properties hold for the concave integral:

1. For every capacity μ, the concave integral $\int^{\mathrm{cav}} \cdot \, \mathrm{d}\mu$ is a concave and positively homogeneous functional, and satisfies $\int^{\mathrm{cav}} 1_S \, \mathrm{d}\mu \geqslant \mu(S)$ for all $S \in 2^X$;
2. For every $f \in \mathbb{R}_+^X$ and capacity μ,
$$\int^{\mathrm{cav}} f \, \mathrm{d}\mu = \min \Big\{ I(f) \mid I : \mathbb{R}_+^X \to \mathbb{R} \text{ concave},$$
positively homogeneous, and such that $I(1_S) \geqslant \mu(S), \forall S \subseteq X \Big\}$
3. For every $f \in \mathbb{R}_+^X$ and capacity μ,
$$\int^{\mathrm{cav}} f \, \mathrm{d}\mu = \min_{P \text{ additive }, P \geqslant \mu} \int f \, \mathrm{d}P$$
4. For every $f \in \mathbb{R}_+^X$ and capacity μ,
$$\int f \, \mathrm{d}\mu \leqslant \int^{\mathrm{cav}} f \, \mathrm{d}\mu,$$
and equality holds for every $f \in \mathbb{R}_+^X$ if and only if μ is supermodular.