Task 2: erasing singularities, analytic sets, Weierstrass polynomials. Deadline: February, 24

February 8, 2016

Problem 1. Prove the following multidimensional analogue of Hartogs' erasing singularity theorem. Let $R = (R_1, \dots, R_n)$, $R_j > 0$, $1 \leq k < n$, $r = (r_1, \dots, r_k)$, $r_s < R_s$. Set $R^k = (R_1, \dots, R_k)$, $R^{n-k} = (R_{k+1}, \dots, R_n)$. Let $V \subset \Delta_{R^{n-k}} \subset \mathbb{C}^{n-k}$ be an open subset. Let $z = (z_1, \dots, z_n)$ be coordinates on $\mathbb{C}^n$. Set $t = (z_1, \dots, z_k)$, $w = (z_{k+1}, \dots, z_n)$,

$$A = (\Delta_{R^k} \setminus \overline{\Delta_r}) \times \Delta_{R^{n-k}}, \ B = \Delta_{R^k} \times V \subset \Delta_R \subset \mathbb{C}^n, \ \Omega = A \cup B.$$

Then every function holomorphic on Ω extends holomorphically to the whole polydisk $\Delta_R = \Delta_{R^k} \times \Delta_{R^{n-k}}$.

Problem 2. Prove that every bounded function holomorphic on $\mathbb{C}^2 \setminus K$ is constant, where

a) K is a ball;
b) K is a complex line;
c) K is an arbitrary analytic subset;
d)* $K = \mathbb{R}^2 \subset \mathbb{C}^2$ is the real plane.

Problem 3. * The general Erasing Compact Singularity Theorem for a connected domain $\Omega \subset \mathbb{C}^n$ says that for every compact subset $K \Subset \Omega$ with a connected complement every function holomorphic on $\Omega \setminus K$ extends holomorphically to all of Ω. Prove it for Ω being

a) a ball $B_R = \{|z| < R\} \subset \mathbb{C}^n$;
b)** an arbitrary domain whose projection π to appropriate coordinate $n-1$-space makes it a trivial fibration by simply connected domains in $\mathbb{C}$: C^1-diffeomorphic to the direct product $D_1 \times \pi(\Omega)$.

Problem 4. Prove that every function holomorphic on the complement $\Delta_{(1,1)} \setminus S \subset \mathbb{C}^2$ extends holomorphically to all of $\Delta_{(1,1)}$, where

a) $S = \{\frac{1}{2} < |z_1| < 1\} \times \{0\}$;
b) $S = \mathbb{R}^2 \setminus \{|z_1|^2 + |z_2|^2 < \frac{1}{2}\}$;
c)* $S = \mathbb{R}^2$.

Hint to c). Consider the fibration of the space $\mathbb{C}^2$ by parabolas $iz_2 = z_1^2 + \varepsilon$. Try to adapt the proof of two-dimensional Hartogs Theorem (with argument on fibration by parallel lines) to this parabolic fibration.

Problem 5. Consider the complex projective space $\mathbb{CP}^n$ and an affine chart $\mathbb{C}^n \subset \mathbb{CP}^n$. Prove that the closure in $\mathbb{CP}^n$ of the zero set in $\mathbb{C}^n$ of arbitrary polynomial is an analytic subset in $\mathbb{CP}^n$. Deduce that the union of the infinity hyperplane and the zero set is an analytic subset in $\mathbb{CP}^n$.

Definition. A function f is *meromorphic* at a point $p \in \mathbb{C}^n$, if there exists a neighborhood $U = U(p) \subset \mathbb{C}^n$ and two holomorphic functions $g, h : U \to \mathbb{C}$ such that $f = \frac{g}{h}$ on U.

Problem 6. Let a function on a complex manifold M be holomorphic on the complement to an analytic subset $K \subset M$. Let M be equipped with a smooth Riemannian metric, for example, a domain in $\mathbb{C}^n$ equipped with Euclidean distance. Let f have at most polynomial growth at K: there exist $c, d > 0$ such that $|f(z)| < c(dist(z, K))^{-d}$ for every $z \in M \setminus K$. Prove that f is meromorphic.

Problem 7. Check and prove whether the following Weierstrass polynomials are irreducible, and factorize those that aren't as products of irreducible ones:

a) $z_1^2 - z_2^3$;
b) $z_1^3 - z_2^5$;
c) $z_1^2 - z_2^4$;
d) $z_1^4 - z_2^6$;
e)* $z_1^p - z_2^q$, the answer depends on the choice of the pair (p, q);
f)* $z_1^p + z_2^2 + z_3^2$, $p \geq 2$.

Problem 8. Check and prove whether the following germs of functions at zero are irreducible.

a) $f(z_1, z_2) = z_1^2 + z_2^2 + z_1^3 z_2^4$;
b) $f(z_1, z_2) = z_1^3 + z_2^3 + z_1^4 + z_2^5$;
c) $z_1^p + z_2 + O(|z|^2)$;
d) $z_1^3 + z_2^2 + z_2^3$.