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TOWARD A THEORY OF MONOPOLISTIC COMPETITION

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Toward a theory of monopolistic competition⁴

Abstract

We propose a general model of monopolistic competition which encompasses existing models while being flexible enough to take into account new demand and competition features. Even though preferences need not be additive and/or homothetic, the market outcome is still driven by the sole variable elasticity of substitution. We impose elementary conditions on this function to guarantee empirically relevant properties of a free-entry equilibrium. Comparative statics with respect to market size and productivity shock are characterized through necessary and sufficient conditions. Furthermore, we show that the attention to the constant elasticity of substitution (CES) based on its normative implications was misguided: constant mark-ups, additivity and homotheticity are neither necessary nor sufficient for the market to deliver the optimum outcome. Our approach can cope with heterogeneous firms once it is recognized that the elasticity of substitution is firm-specific. Finally, we show how our set-up can be extended to cope with multiple sectors.

Keywords: monopolistic competition, general equilibrium, additive preferences, homothetic preferences.

JEL classification: D43, L11, L13.

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1 Introduction

In his survey of the attempts made in the 1970s and 1980s to integrate oligopolistic competition in a general equilibrium framework, Hart (1985) argued that these efforts failed to produce a consistent and workable model. The absence of a general equilibrium model of oligopolistic competition unintentionally paved the way for the success of the constant elasticity of substitution (CES) model of monopolistic competition, developed by Dixit and Stiglitz (1977). This model has been used in so many economic fields that a large number of scholars view it as *the* model of monopolistic competition. For example, Head and Mayer (2014) observe that the CES is “nearly ubiquitous” in the trade literature. However, owing to its extreme simplicity, this model ignores several important effects that contradict some basic findings in economic theory. For example, unlike what the CES predicts, prices and firm sizes are affected by entry and market size, while markups vary with costs and consumer income. Recent empirical studies conducted at the firm level provide direct evidence for these findings.

In addition, tweaking the CES or using other specific models in the hope of getting around those difficulties unfortunately prevents a direct comparison between results. We realize that such a research strategy is motivated by its tractability, but it runs the risk of ignoring the fragility of the results. For example, by nesting quadratic preferences into a quasi-linear utility, Melitz and Ottaviano (2008) show that prices depend on market size but suppress the per capita income effect. Markups depend on per capita income under the linear expenditure system in an open economy (Simonovska, 2015), but this effect disappears in a closed economy under additive preferences (Zhelobodko *et al.*, 2012). Under indirectly additive preferences, there is an income effect, but market size has no impact on prices (Bertoletti and Etro, 2015). Prices are independent of the number of competitors in the CES model of monopolistic competition, but not in oligopolistic competition (d’Aspremont *et al.*, 1996). Therefore, the absence of a general framework makes it hard to deal with the implications of different specifications of preferences. Furthermore, Hottman *et al.* (2016) argue that 50 to 75% of the variance in U.S. firm size can be attributed to differences in what these authors call “firms’ appeal,” that is, how consumers perceive goods. As a consequence, we may safely conclude that it is time to pay more attention to the *demand side* in models of monopolistic competition. This is where we hope to contribute.

The elasticity of substitution is the keystone of the CES model of monopolistic competition. Although common wisdom holds that this concept is relevant only in the CES case, we show that this view is not justified – a variable elasticity of substitution can be used as the primitive of a general model of monopolistic competition. What makes this approach appealing is that elasticity of substitution is a simple inverse measure of the degree of differentiation across varieties. Our set-up also encompasses all existing models of monopolistic competition, including those with CES, quadratic, CARA, additive, and homothetic preferences. Working at this level of generality is desirable because it buys enough flexibility to capture a rich array

of demand and competition features. This, in turn, allows us to find *necessary and sufficient* conditions that the elasticity of substitution must satisfy for various comparative static effects to hold. This is useful for applied economists when discriminating among different specifications of preferences. In addition, modelling monopolistic competition as a non-cooperative game with a continuum of players concurs with Mas-Colell (1984, 19) who states that “the theory of monopolistic competition derives its theoretical importance not from being a realistic complication of the theory of perfect competition, but from being a simplified, tractable limit of oligopoly theory.”

Our main findings may be summarized as follows.

(i) Using the concept of Fréchet differentiability, which applies to a case where the unknowns are functions rather than vectors, we determine a general demand system that includes a wide range of special cases used in the literature. In particular, for any symmetric consumption profile, the elasticity of substitution depends on just *two* variables, namely, the consumption level and the number of available varieties. Additive and homothetic preferences may be visualized as the horizontal and vertical axes of the plane over which the elasticity of substitution is defined.

Very much like Melitz (2003), who highlights the implications of firm heterogeneity by using the CES, we can insulate the impact of different types of preferences on the market outcome when we start with symmetric firms. We first show that, in a symmetric free-entry equilibrium, a firm’s markup is equal to the inverse of the equilibrium value of the elasticity of substitution. Specifically, *the properties of an equilibrium depend on how the elasticity of substitution function behaves when the per-variety consumption and the number of firms vary.* By imposing plausible conditions on this function and by using simple analytical arguments, we are able to disentangle the various determinants of firms’ behavior.

(ii) Our set-up is especially well suited for conducting detailed comparative static analyses in that we can determine the necessary and sufficient conditions for various thought experiments undertaken in the literature. The most common experiment is studying the impact of market size on the market outcome. What market size signifies is not always clear because it compounds two variables, that is, the number of consumers and their willingness to pay for the product under consideration. The impact of population size and income level on prices, output, and the number of firms need not be the same because these two parameters (population size and income level) affect demand in different ways. An increase in population or income raises demand, thereby fostering entry and lower prices. However an income hike also increases consumers’ willingness to pay, which tends to push prices upward. The final impact is thus a priori ambiguous.

We show that a larger market results in a lower market price and bigger firms if, and only if, the elasticity of substitution responds more to a change in the number of varieties than to a change in the per-variety consumption. This is the situation in the likely case when the entry of new firms does not make the number of varieties much more differentiated. As for the number

of varieties, this increases with the population size if varieties do not become too similar when their number increases. Thus, as in most oligopoly models, monopolistic competition exhibits the standard pro-competitive effects associated with market size and entry. An increase in individual income generates effects similar, but not identical, to a population hike if, and only if, varieties become closer substitutes when their range widens. The CES is the only utility in which price and output are independent of both income and market size. However, even with identical consumers and non-strategic firms, standard assumptions about preferences are not sufficient to rule out anti-competitive effects. For this, we need additional assumptions. Our model also allows us to study the impact of a productivity or trade liberalization shock on markups. When all firms face the same shock, we show that *preferences determine the extent of the pass-through*.

(iii) Ever since Chamberlin (1933), the question of whether the market under- or over-provides diversity is one of the most studied issues in the theory of imperfect competition. It is well known that the CES is the only additive utility for which the market achieved the optimum (Dixit and Stiglitz, 1977; Dhingra and Morrow, 2015). Conventional wisdom holds that the constant markup associated with CES is necessary for this result to hold. Non-CES homothetic preferences typically generate markups that vary with the number of firms. Yet we show that, for any homothetic utility, there exists a transformation of this utility that yields another homothetic utility, which differs generically from the CES, and for which the market and optimal outcomes are the same. Therefore, we may conclude that the optimality of the market equilibrium is not driven by a constant markup. What is more, we show that homotheticity is not even required for this result to hold, as we provide an example of a non-homothetic utility where the optimum is reached. Therefore, the attention that CES has received regarding its normative implications was not warranted; constant markups, additivity, or homotheticity—three properties verified by the CES—are neither necessary nor sufficient for the market to deliver the optimum. Therefore, care is needed when the desirability of policies is assessed using monopolistic competition models based on CES preferences, as these models deliver peculiar welfare properties.

(iv) The next step is to figure out how our demand framework interacts with Melitz-like heterogeneous firms. When preferences are non-additive, the profit-maximizing strategy of a firm depends directly on the strategies chosen by all the other firms producing similar goods, which vastly increases the complexity of the problem. Despite this, we are able to show that, regardless of the distribution of marginal costs, the elasticity of substitution across varieties produced by firms which enjoy the same productivity level now depends on the number of entrants and the costs of these firms. In other words, the elasticity of substitution is now type-specific. *A general model of monopolistic competition thus remains parsimonious*. Furthermore, our approach paves the way to the study of asymmetric preferences in that a model with heterogeneous firms supplying symmetric varieties is isomorphic to a model with homogeneous

firms selling asymmetric varieties.

(v) Last, we consider a two-sector economy. The main additional difficulty stems from the fact that sector-specific expenditures depend on the upper-tier utility. Under a fairly mild assumption regarding the marginal utility, we prove the existence of an equilibrium and show that many of our results hold for a monopolistically competitive sector. All of this highlights the versatility of our model, which can be used as a building block to embed monopolistic competition in fully fledged general equilibrium models coping with various economic issues.

Related literature Different alternatives have been proposed to avoid the main pitfalls of the CES model. Behrens and Murata (2007) propose the CARA utility that captures price competition effects, while Zhelobodko *et al.* (2012) use general additive preferences to work with a variable elasticity of substitution, and thus variable markups. Vives (1999) and Ottaviano *et al.* (2002) show how the quadratic utility model avoids some of the difficulties associated with the CES model, while delivering a full analytical solution. Bilbiie *et al.* (2012) use symmetric homothetic preferences in a real business cycle model. Mrázová and Neary (2013), pursuing a different, but related, objective, study a class of demands that implies an invariant relationship between the demand elasticity and the curvature of demand schedule. Section 2.3 explains how this class fits in our general model.

In the next section, we describe the demand and supply sides of our set-up. In particular, the primitive of the model being the elasticity of substitution function, we discuss how this function may vary with the per-variety consumption and the number of varieties. In Section 3, we prove the existence and uniqueness of a free entry equilibrium and characterize its properties when firms are symmetric. Section 4 focuses on the optimality of the market outcome. Heterogeneous firms are studied in Section 5. In Section 6, we extend our model to the case of a two-sector economy. Section 7 concludes.

2 The model and preliminary results

Consider an economy with L identical consumers, one sector and one production factor – labor, which is used as the numéraire. Each consumer has y efficiency units of labor. On the supply side, there is a continuum of firms each producing a horizontally differentiated good under increasing returns. Each firm supplies a single variety and each variety is supplied by a single firm.

2.1 Consumers

Let \mathbb{N} , an arbitrarily large number, be the mass of *potential* varieties (see (21) in Section 3 for a precise definition of “arbitrarily large”). As all potential varieties are not necessarily made

available to consumers, we denote by $N \leq \mathbb{N}$ the endogenous mass of *available* varieties. Since we work with a continuum of varieties, we cannot use the standard tools of calculus. Rather, we must work in a functional space whose elements are functions, and not vectors. A *consumption profile* $\mathbf{x} \geq 0$ is a Lebesgue-measurable mapping from the space of potential varieties $[0, \mathbb{N}]$ to \mathbb{R}_+ . We denote by x_i the consumption of variety i ; for $i \in]N, \mathbb{N}]$ we set $x_i = 0$. We assume that \mathbf{x} belongs to $L_2([0, \mathbb{N}])$. The benefit of using this space is that it is fairly straightforward to impart precise content to the assumption that the utility functional is differentiable for Hilbert spaces (see (4) below). This space is rich enough for our purpose. For example, it embraces all consumption profiles that are bounded.

To start with, we give examples of preferences used in models of monopolistic competition.

1. Additive preferences (Spence, 1976; Dixit and Stiglitz, 1977; Kühn and Vives, 1999).

Preferences are *additive* over the set of varieties if

$$\mathcal{U}(\mathbf{x}) \equiv \int_0^{\mathbb{N}} u(x_i) di, \quad (1)$$

where u is differentiable, strictly increasing, strictly concave, and such that $u(0) = 0$. The CES, which has been used extensively in many fields and the CARA (Behrens and Murata, 2007) are special cases of (1). Bertolotti and Etro (2015) consider the case of *indirectly additive* preferences:

$$\mathcal{V}(\mathbf{p}; Y) \equiv \int_0^{\mathbb{N}} v(p_i/Y) di,$$

where $\mathcal{V}(\mathbf{p}; Y)$ is the indirect utility function, v is differentiable, strictly decreasing and strictly convex, Y is consumer's income, and p_i is the price for variety i . The CES is the only utility function which is both additive and indirectly additive.

2. Homothetic preferences. A tractable example of non-CES homothetic preferences used in the macroeconomic and trade literature is the *translog* (Bergin and Feenstra, 2009; Bilbiie *et al.*, 2012; Feenstra and Weinstein, 2015). There is no closed-form expression for the translog utility function. Nevertheless, by appealing to the duality principle in consumption theory, these preferences can be described by the following expenditure function:

$$\ln E(\mathbf{p}) = \ln \mathcal{U}_0 + \frac{1}{\mathbb{N}} \int_0^{\mathbb{N}} \ln p_i di - \frac{\beta}{2\mathbb{N}} \left[\int_0^{\mathbb{N}} (\ln p_i)^2 di - \frac{1}{\mathbb{N}} \left(\int_0^{\mathbb{N}} \ln p_i di \right)^2 \right].$$

A broad class of homothetic preferences is given by what is known as *Kimball's flexible aggregator*, introduced by Kimball (1995) as a production function used in the macroeconomic literature (Charie *et al.*, 2000; Smets and Wouters, 2007). A utility functional $\mathcal{U}(\mathbf{x})$ is said to be described by Kimball's flexible aggregator if there exists a strictly increasing and strictly concave function $\theta(\cdot)$ such that $\mathcal{U}(\mathbf{x})$ satisfies

$$\int_0^{\mathbb{N}} \nu \left(\frac{x_i}{\mathcal{U}(\mathbf{x})} \right) di = 1 \quad (2)$$

for any consumption bundle \mathbf{x} . Whenever $\mathcal{U}(\mathbf{x})$ satisfies (2), it is single-valued, continuous, increasing, strictly quasi-concave, and linear homogeneous.

3. Quadratic preferences. An example of preferences that are neither additive nor homothetic is the *quadratic* utility:

$$\mathcal{U}(\mathbf{x}) \equiv \alpha \int_0^{\mathbb{N}} x_i di - \frac{\beta}{2} \int_0^{\mathbb{N}} x_i^2 di - \frac{\gamma}{2} \int_0^{\mathbb{N}} \left(\int_0^{\mathbb{N}} x_i di \right) x_j dj,$$

where α , β , and γ are positive constants (Dixit, 1979; Singh and Vives, 1984; Ottaviano *et al.*, 2002; Melitz and Ottaviano, 2008).

This incomplete list of examples should be sufficient to show that authors who use monopolistic competition appeal to a range of models that display very different properties. It is a priori unclear how these functional forms relate to each other and, more importantly, it is hard to assess the robustness of the theoretical predictions derived for specific demand systems and to match them to the empirical results obtained with other demand systems. This points to the need for a more general set-up in which we can cast all these special cases and compare their properties. With this in mind, we choose to describe individual preferences by a *utility functional* $\mathcal{U}(\mathbf{x})$ defined over $L_2([0, \mathbb{N}])$.

In what follows, we make two assumptions about \mathcal{U} , which seem to be close to the “minimal” set of requirements for our model to be nonspecific while displaying the desirable features of existing models of monopolistic competition. First, for any N , the functional \mathcal{U} is *symmetric* in the sense that any Lebesgue measure-preserving mapping from $[0, N]$ into itself does not change the value of \mathcal{U} . This means that renumbering varieties has no impact on the utility level.

Second, the utility function exhibits love for variety if, for any $N \leq \mathbb{N}$, a consumer strictly prefers to consume the whole range of varieties $[0, N]$ than any subinterval $[0, k]$ of $[0, N]$, that is,

$$\mathcal{U} \left(\frac{X}{k} I_{[0, k]} \right) < \mathcal{U} \left(\frac{X}{N} I_{[0, N]} \right), \quad (3)$$

where $X > 0$ is the consumer’s total consumption of the differentiated good and I_A is the indicator of $A \subseteq [0, N]$. Consumers exhibit love for variety if $\mathcal{U}(\mathbf{x})$ is continuous and strictly quasi-concave (see Appendix 1 in the Supplemental Material).

We also impose the condition that \mathcal{U} has well-behaved marginal utilities in the following sense. For any given N , the utility functional \mathcal{U} is said to be *Fréchet differentiable* in $\mathbf{x} \in L_2([0, \mathbb{N}])$ when there exists a unique function $D(x_i, \mathbf{x})$ from $[0, \mathbb{N}] \times L_2([0, \mathbb{N}])$ to \mathbb{R} such that, for all $\mathbf{h} \in L_2$, the equality

$$\mathcal{U}(\mathbf{x} + \mathbf{h}) = \mathcal{U}(\mathbf{x}) + \int_0^N D(x_i, \mathbf{x}) h_i \, di + o(\|\mathbf{h}\|_2) \quad (4)$$

holds, $\|\cdot\|_2$ being the L_2 -norm (Dunford and Schwartz, 1988). The function $D(x_i, \mathbf{x})$ is the *marginal utility of variety i* . That $D(x_i, \mathbf{x})$ does not depend directly on $i \in [0, N]$ follows from the symmetry of preferences.⁵ From now on, we focus on utility functionals that satisfy (4) for all $\mathbf{x} \geq 0$ and such that the marginal utility $D(x_i, \mathbf{x})$ is decreasing and twice continuously differentiable with respect to $x_i \in \mathbb{R}_+$. The function $D(x_i, \mathbf{x})$ (strictly) decreases with x_i if \mathcal{U} is (strictly) concave. The reason for restricting ourselves to decreasing marginal utilities is that this property allows us to work directly with well-behaved inverse demand functions.

Let $\mathbf{p} \geq \mathbf{0}$ be a *market price profile* described by a Lebesgue-measurable mapping from the space of available varieties $[0, N]$ to \mathbb{R}_+ . We denote by p_i the price of variety i . Bewley (1972) has shown that a market price profile $\mathbf{p} \geq \mathbf{0}$ must belong to the dual of the space of consumption profiles. This condition holds here when $\mathbf{p} \in L_2([0, N])$ because this space is its own dual. In this case, the total expenditure, which is given by the inner product $\mathbf{p} \cdot \mathbf{x}$, is finite.

Under (4), the first-order conditions in $L_2([0, N])$ are similar to the corresponding conditions in a finite-dimensional space. Therefore, maximizing the utility functional $\mathcal{U}(\mathbf{x})$ subject to (i) the budget constraint

$$\mathbf{p} \cdot \mathbf{x} = \int_0^N p_i x_i \, di = Y, \quad (5)$$

and (ii) the availability constraint

$$x_i \geq 0 \text{ for all } i \in [0, N] \quad \text{and} \quad x_i = 0 \text{ for all } i \in]N, N]$$

yields the following inverse demand function for variety i :

$$p_i = \frac{D(x_i, \mathbf{x})}{\lambda} \quad \text{for all } i \in [0, N], \quad (6)$$

where λ is the Lagrange multiplier of the consumer's optimization problem. Expressing λ as a function of Y and \mathbf{x} , we obtain

$$\lambda(Y, \mathbf{x}) = \frac{\int_0^N x_i D(x_i, \mathbf{x}) \, di}{Y}, \quad (7)$$

which is the marginal utility of income at the consumption profile \mathbf{x} under income Y .

As a large share of the literature focuses on additive or homothetic preferences, it is important to provide a characterization of the corresponding demands. In the first case, Goldman and

⁵The definition of a Fréchet-differentiable function must be changed when the marginal utility grows very fast in the neighborhood of $x_i = 0$. We show in Appendix 2 in the Supplemental Material how to deal with this case.

Uzawa (1964) show that preferences are additive iff the marginal rate of substitution between varieties i and j , $D(x_i, \mathbf{x})/D(x_j, \mathbf{x})$, depends only upon the consumptions x_i and x_j . In the second, preferences are homothetic iff the marginal rate of substitution between varieties i and j depends only upon the consumption ratios \mathbf{x}/x_i and \mathbf{x}/x_j (for the proof see Appendix 3 in the Supplemental Material).

2.2 Firms: first- and second-order conditions

Let Ω be the set of active firms. There are increasing returns at the firm level, but no scope economies which would induce a firm to produce several varieties. The continuum assumption distinguishes monopolistic competition from other market structures in that it is the formal counterpart of the basic idea that a firm's action has no impact on the others. As a result, by being negligible to the market, each firm may choose its output (or price) while accurately treating market variables as given. However, for the market to be in equilibrium, firms must accurately guess what these variables will be.

Firms share the same fixed cost F and the same constant marginal cost c , so that \mathbb{N} is bounded above by yL/F . In other words, to produce q_i units of its variety, firm i needs $F + cq_i$ efficiency units of labor. Hence, firm i 's profit is given by

$$\pi(q_i) = (p_i - c)q_i - F. \quad (8)$$

Since consumers share the same preferences, the consumption of each variety is the same across consumers. Therefore, product market clearing implies $q_i = Lx_i$. Firm i maximizes (8) with respect to its output q_i while the market outcome is given by a Nash equilibrium. The Nash equilibrium distribution of firms' actions is encapsulated in \mathbf{x} and λ . In the CES case, this comes down to treating the price-index parametrically, while under additive preferences the only payoff-relevant market statistic is λ .

Plugging $D(x_i, \mathbf{x})$ into (8), the program of firm i is given by

$$\max_{x_i} \pi_i(x_i, \mathbf{x}) \equiv \left[\frac{D(x_i, \mathbf{x})}{\lambda} - c \right] Lx_i - F.$$

Setting

$$D_i \equiv D(x_i, \mathbf{x}) \quad D'_i \equiv \frac{\partial D(x_i, \mathbf{x})}{\partial x_i} \quad D''_i \equiv \frac{\partial^2 D(x_i, \mathbf{x})}{\partial x_i^2},$$

for notational simplicity, the first-order condition for profit-maximization are given by

$$D_i + x_i D'_i = [1 - \bar{\eta}(x_i, \mathbf{x})] D_i = \lambda c, \quad (9)$$

where

$$\bar{\eta}(x_i, \mathbf{x}) \equiv -\frac{x_i}{D_i} D_i'$$

is the elasticity of the inverse demand for variety i . As usual, (9) may be rewritten as follows:

$$\frac{p_i - c}{p_i} = \frac{1}{\bar{\eta}(x_i, \mathbf{x})}. \quad (10)$$

Since λ is endogenous, we seek necessary and sufficient conditions for a unique (interior or corner) profit-maximizer to exist regardless of the value of $\lambda c > 0$. The argument involves two steps.

Step 1. Consider first the case where, for any \mathbf{x} , the following conditions hold:

$$\lim_{x_i \rightarrow 0} D_i = \infty \quad \lim_{x_i \rightarrow \infty} D_i = 0. \quad (11)$$

Since $\bar{\eta}(0, \mathbf{x}) < 1$, (11) implies that $\lim_{x_i \rightarrow 0} (1 - \bar{\eta})D_i = \infty$. Similarly, since $0 < (1 - \bar{\eta})D_i < D_i$, it follows from (11) that $\lim_{x_i \rightarrow \infty} (1 - \bar{\eta})D_i = 0$. Because $(1 - \bar{\eta})D_i$ is continuous, it follows from the intermediate value theorem that (9) has at least one positive solution.

Second, (11) does not hold when D_i/λ displays a finite choke price. However, it is readily verified that (9) has at least one positive solution when the choke price exceeds λc .

Step 2. The first-order condition (9) is sufficient if the profit function π_i is strictly quasi-concave in x_i . If the maximizer of π is positive and finite, the profit function is strictly quasi-concave in x_i for any positive value of λc iff the second derivative of π is negative at any solution to the first-order condition. Since firm i treats λ parametrically, the second-order condition is given by

$$x_i D_i'' + 2D_i' < 0. \quad (12)$$

This condition means that firm i 's marginal revenue $(x_i D_i' + D_i)L/\lambda$ is strictly decreasing in x_i . It is satisfied when D_i is concave, linear or not “too” convex in x_i . Furthermore, (12) is also a necessary and sufficient condition for the profit function to be strictly quasi-concave for all $\lambda c > 0$, for otherwise a value of λc would exist such that the marginal revenue curve intersects the horizontal line λc more than once. Observe also that (12) means that the revenue function is strictly concave. Since the marginal cost is independent of x_i , this in turn implies that π_i is strictly concave in x_i . When firms are quantity-setters, the profit function π_i is strictly concave in x_i if this function is strictly quasi-concave in x_i (for the proof see Appendix 4 in the Supplemental Material). Therefore, the profit function π_i is strictly quasi-concave in x_i for all values of λc iff

(A) *firm i 's marginal revenue decreases in x_i .*

Observe that (A) is equivalent to the well-known condition obtained by Caplin and Nale-

buff (1991) for a firm's profits to be quasi-concave in its own price, which is stated as follows: the Marshallian demand for variety i is (-1) -convex in p_i (we show in Appendix A that the Marshallian demand is here well defined). Since the Caplin-Nalebuff condition is the least stringent for a firm's profit to be quasi-concave under price-setting firms, **(A)** is therefore the least demanding condition when firms compete in quantities.

A sufficient condition commonly used in the literature is as follows (Krugman, 1979; Vives, 1999):

(AA) *the elasticity of the inverse demand $\bar{\eta}(x_i, \mathbf{x})$ increases in its first argument.*

It is readily verified that **(AA)** is equivalent to

$$-x_i \frac{D_i''}{D_i'} < 1 + \bar{\eta}(x_i, \mathbf{x}).$$

Since the expression (12) may be rewritten as follows:

$$-x_i \frac{D_i''}{D_i'} < 2.$$

while $\bar{\eta} < 1$ it must be that **(AA)** implies **(A)**.

2.3 The elasticity of substitution

Definition The elasticity of substitution can be defined to cope with general preferences. In what follows, we treat the elasticity of substitution as the primitive of our approach to monopolistic competition. This allows us to show how the comparative static results are driven by the demand side. To achieve our goal, we use an infinite-dimensional version of the elasticity of substitution function given by Nadiri (1982, p.442). Since \mathbf{x} is defined up to a zero measure set, it must be that the cross-price elasticity between any two varieties is negligible:

$$\frac{\partial D_i(x_i, \mathbf{x})}{\partial x_j} = \frac{\partial D_j(x_j, \mathbf{x})}{\partial x_i} = 0.$$

In this case, the elasticity of substitution between varieties i and j for a given \mathbf{x} is given by

$$\bar{\sigma}(x_i, x_j, \mathbf{x}) = -\frac{D_i D_j (x_i D_j + x_j D_i)}{x_i x_j (D_i' D_j^2 + D_j' D_i^2)}.$$

Setting $x_i = x_j = x$ implies $D_i = D_j$ and $D_i' = D_j'$. Therefore, we obtain

$$\bar{\sigma}(x, x, \mathbf{x}) = \frac{1}{\bar{\eta}(x, \mathbf{x})}. \tag{13}$$

Evaluating $\bar{\sigma}$ at a symmetric consumption pattern, where $\mathbf{x} = xI_{[0,N]}$, yields

$$\sigma(x, N) \equiv \bar{\sigma}(x, x, xI_{[0,N]}).$$

Hence, regardless of the structure of preferences, at any symmetric consumption pattern *the elasticity of substitution depends only upon the individual consumption and the mass of varieties*. In other words, the dimensionality of the problem is reduced to two variables. When firms are heterogeneous, the consumption pattern is no longer symmetric, but we will see in Section 4 how the elasticity of substitution is still applicable.

Given these, we may consider the function $\sigma(x, N)$ as the *primitive* of the model. There are two more reasons for making this choice. First, we will see that what matters for the properties of the symmetric equilibrium is the behavior of $\sigma(x, N)$ in x and N . More precisely, we will show that the behavior of the market outcome can be characterized by necessary and sufficient conditions stating how σ varies with x and N . Rather than using the partial derivatives of σ , it will be more convenient to work with the elasticities $\mathcal{E}_x(\sigma)$ and $\mathcal{E}_N(\sigma)$. Specifically, the signs of these two expressions ($\mathcal{E}_x(\sigma) \geq 0$ and $\mathcal{E}_N(\sigma) \leq 0$) and their relationship ($\mathcal{E}_x(\sigma) \geq \mathcal{E}_N(\sigma)$) will allow us to completely characterize the market outcome.

Second, since the elasticity of substitution is an inverse measure of the degree of product differentiation across varieties, we are able to appeal to the theory of product differentiation to choose the most plausible assumptions regarding the behavior of $\sigma(x, N)$ with respect to x and N and to check whether the resulting predictions are consistent with empirical evidence.

Remark. Our approach could be equivalently reformulated by considering the manifold $(\sigma, \mathcal{E}_x(\sigma), \mathcal{E}_N(\sigma))$, which is parameterized by the variables x and N . Being generically a two-dimensional surface in \mathbb{R}^3 , this manifold boils down to a one-dimensional locus in Mrázová and Neary (2013). The one-dimensional case encompasses a wide variety of demand systems, including those generated by additive preferences ($\mathcal{E}_N(\sigma) = 0$).

2.3.1 Examples

To develop more insights about the behavior of σ as a function of x and N , we give below the elasticity of substitution for the different types of preferences discussed above.

(i) When the utility is additive, we have:

$$\frac{1}{\sigma(x, N)} = r(x) \equiv -\frac{xu''(x)}{u'(x)}, \quad (14)$$

which means that σ depends only upon the per-variety consumption when preferences are additive.

(ii) When preferences are homothetic, $D(x, \mathbf{x})$ evaluated at a symmetric consumption

profile depends solely on the mass N of available varieties. Setting

$$\varphi(N) \equiv \eta(1, N)$$

and using (13) yields

$$\frac{1}{\sigma(x, N)} = \varphi(N). \quad (15)$$

For example, under translog preferences, we have $\varphi(N) = 1/(1 + \beta N)$.

Since the CES is additive, the elasticity of substitution is independent of N . Furthermore, since the CES is also homothetic, it must be that

$$r(x) = \varphi(N) = \frac{1}{\sigma}.$$

It is, therefore, no surprise that the constant σ is the only demand parameter that drives the market outcome under CES preferences.

Using (14) and (15), it is readily verified that $\mathcal{E}_N(\sigma) = 0$ when preferences are additive, $\mathcal{E}_x(\sigma) = 0$ when preferences are homothetic, while $\mathcal{E}_x(\sigma) = \mathcal{E}_N(\sigma)$ under indirectly additive preferences.

2.3.2 How does $\sigma(x, N)$ vary with x and N ?

While our framework allows for various patterns of σ , it should be clear that they are not equally plausible. This is why most applications of monopolistic competition focus on subclasses of utilities to cope with particular effects. For instance, Bilbiie *et al.* (2012) use the translog expenditure function to capture the pro-competitive impact of entry on markups, since $\sigma(x, N) = 1 + \beta N$ increases with the number of varieties. On the same grounds, working with additive preferences Krugman (1979) assumes “without apology” that $\sigma(x, N) = 1/r(x)$ decreases with individual consumption x .

Admittedly, making “realistic” assumptions on how the elasticity of substitution varies with x and N is not an easy task. That said, it is worth recalling with Stigler (1969) that “it is often impossible to determine whether assumption A is more or less realistic than assumption B, except by comparing the agreement between their implications and the observable course of events.” This is what we do below.

Spatial and discrete choice models of product differentiation suggest that varieties become closer substitutes when the number of competing varieties rises (Tirole, 1988; Anderson *et al.*, 1995). Therefore, $\mathcal{E}_N(\sigma) \geq 0$ seems to be a reasonable assumption.

In contrast, how σ varies with x is a priori less clear. The empirical evidence strongly suggests the pass-through of a cost change triggered by a trade liberalization or productivity shock is smaller than 100 percent (De Loecker *et al.*, 2015; Amiti *et al.*, 2015). Which assumption

on σ leads to this result? The intuition is easy to grasp when preferences are additive, that is, $m(x) = 1/\sigma(x)$. Incomplete pass-through amounts to saying that p/c increases when c decreases, which means that firms have more market power or, equivalently, varieties are more differentiated. As firms facing a lower marginal cost produce more, the per capita consumption increases. Therefore, it must be that $\sigma(x)$ decreases with x . In the case of general symmetric preferences, we will show below that there is incomplete pass-through iff $\mathcal{E}_x(\sigma) < 0$. Because $\mathcal{E}_x(\sigma) = 0$ under homothetic preferences, the pass-through must be equal to 100%.

All of this suggests the following conditions:

$$\mathcal{E}_x(\sigma) \leq 0 \leq \mathcal{E}_N(\sigma). \quad (16)$$

Yet, the empirical evidence about the implications of entry and costs effects is not entirely conclusive. For this reason, we relax (16) by assuming that

$$\mathcal{E}_x(\sigma) \leq \mathcal{E}_N(\sigma) \quad (17)$$

holds for all $x > 0$ and $N > 0$. This condition is appealing because it boils down to $\mathcal{E}_x(\sigma) \leq 0$ and $\mathcal{E}_N(\sigma) = 0$ for additive preferences, whereas $\mathcal{E}_x(\sigma) = 0$ and $\mathcal{E}_N(\sigma) \geq 0$ for homothetic preferences. What is more, (17) is less stringent than either of these conditions. Therefore, we see it as a fairly natural condition to be satisfied for preferences that need not be additive or homothetic.

3 Market equilibrium

3.1 Existence and uniqueness of a symmetric free-entry equilibrium

We first determine prices, outputs and profits when the mass of firms is fixed, and then find N by using the zero-profit condition. When N is exogenously given, the market equilibrium is given by the functions $\bar{q}(N)$, $\bar{p}(N)$ and $\bar{x}(N)$ defined on $[0, N]$, which satisfy the following four conditions: (i) no firm i can increase its profit by changing its output, (ii) each consumer maximizes her utility subject to her budget constraint, (iii) the product market clearing condition

$$\bar{q}_i = L\bar{x}_i \quad \text{for all } i \in [0, N]$$

and (iv) the labor market balance

$$c \int_0^N q_i di + NF = yL \quad (18)$$

hold, where we have assumed that each consumer is endowed with y efficiency units of labor. The study of market equilibria where the number of firms is exogenous is to be viewed as an intermediate step toward monopolistic competition, where the number of firms is pinned down by free entry and exit.

Since we focus here on symmetric free-entry equilibria, we find it reasonable to study symmetric market equilibria, which means that the functions $\bar{\mathbf{q}}(N)$, $\bar{\mathbf{p}}(N)$ and $\bar{\mathbf{x}}(N)$ become the scalars $\bar{q}(N)$, $\bar{p}(N)$ and $\bar{x}(N)$. For this, consumers must have the same income, which holds when profits are uniformly distributed across consumers. In this case, the budget constraint (5) must be replaced by the following expression:

$$\int_0^N p_i x_i di = Y \equiv y + \frac{1}{L} \int_0^N \pi_i di. \quad (19)$$

where the unit wage has been normalized to 1. Being negligible to the market, each firm accurately treats Y as a given.

As each firm faces the same demand, the function $\pi(x_i, \mathbf{x})$ is the same for all i . In addition, **(A)** implies that $\pi(x_i, \mathbf{x})$ has a unique maximizer for any \mathbf{x} . As a result, the market equilibrium must be symmetric. Using $\pi_i \equiv (p_i - c)Lx_i - F$, (19) boils down to the labor market balance (18), which yields the only candidate symmetric equilibrium for the per-variety consumption:

$$\bar{x}(N) = \frac{y}{cN} - \frac{F}{cL}. \quad (20)$$

Therefore, $\bar{x}(N)$ is positive iff the following inequality holds:

$$N \leq Ly/F. \quad (21)$$

The product market clearing condition implies that the candidate equilibrium output is

$$\bar{q}(N) = \frac{yL}{cN} - \frac{F}{c}.$$

Using (10) and (13) shows that there is a unique candidate equilibrium price given by

$$\bar{p}(N) = c \frac{\sigma(\bar{x}(N), N)}{\sigma(\bar{x}(N), N) - 1}. \quad (22)$$

Clearly, if N is so large that (21) is violated, no interior equilibrium exists. Accordingly, we have the following result: *If both **(A)** and (21) hold, then there exists a unique market equilibrium. Furthermore, this equilibrium is symmetric.*

The pricing rule (22) may be rewritten as

$$\bar{m}(N) \equiv \frac{\bar{p}(N) - c}{\bar{p}(N)} = \frac{1}{\sigma(\bar{x}(N), N)}, \quad (23)$$

which shows that, for any given N , the equilibrium markup $\bar{m}(N)$ varies inversely with the elasticity of substitution. The intuition is easy to grasp. We know from industrial organization that product differentiation relaxes competition. When the elasticity of substitution is lower, varieties are poorer substitutes, thereby endowing firms with more market power. It is, therefore, no surprise that firms have a higher markup when σ is lower. It also follows from (23) that the way σ varies with x and N shapes the properties of market outcome. In particular, this demonstrates that assuming a constant elasticity of substitution amounts to adding very strong restraints on the way the market works.

Combining (20) and (22), the equilibrium operating profits earned by a firm when there are N firms are given by

$$\bar{\pi}(N) = \frac{c}{\sigma(\bar{x}(N), N) - 1} L \bar{x}(N). \quad (24)$$

It is legitimate to ask how $\bar{\pi}(N)$ varies with the mass of firms. There is no straightforward answer to this question. However, the expression (24) suffices to show how the market outcome reacts to the entry of new firms depends on how the elasticity of substitution varies with x and N . This confirms why static comparative statics may yield ambiguous results in different set-ups.

We now pin down the equilibrium value of N by using the zero-profit condition. Therefore, a consumer's income is equal to her sole labor income: $Y = y$. A *symmetric free-entry equilibrium* (SFE) is described by the vector (q^*, p^*, x^*, N^*) , where N^* solves the zero-profit condition

$$\pi^*(N) = F, \quad (25)$$

while $q^* = \bar{q}(N^*)$, $p^* = \bar{p}(N^*)$ and $x^* = \bar{x}(N^*)$. The Walras Law implies that the budget constraint $N^* p^* x^* = y$ is satisfied. Without loss of generality, we restrict ourselves to the domain of parameters for which $N^* < Ly/F$.

Combining (23) and (25), we obtain a single equilibrium condition given by

$$\bar{m}(N) = \frac{NF}{Ly}. \quad (26)$$

When preferences are non-homothetic, (20) and (22) show that L/F and y enter the function $\bar{m}(N)$ as two distinct parameters. This implies that L and y have a different impact on the equilibrium markup, while a hike in L is equivalent to a drop in F . However, when preferences are homothetic, it is well known that the effects of L and y on the equilibrium are the same.

For (25) to have a unique solution N^* for *all* values of $F > 0$, it is necessary and sufficient that $\bar{\pi}(N)$ strictly decreases with N . Differentiating (24) with respect to N and using (20) and (25), we obtain

$$\bar{\pi}'(N) = -\frac{y}{cN^2}[\sigma(x, N) - 1] \left[1 + \mathcal{E}_N(\sigma(x, N)) - \frac{\sigma(x, N)}{\sigma(x, N) - 1} \mathcal{E}_x(\sigma(x, N)) \right] \Big|_{x=\bar{x}(N)}.$$

The second term on the right-hand side of this expression is positive iff

$$\mathcal{E}_x(\sigma(x, N)) < \frac{\sigma(x, N) - 1}{\sigma(x, N)} [1 + \mathcal{E}_N(\sigma(x, N))]. \quad (27)$$

Therefore, $\bar{\pi}(N)$ strictly decreases with N for all L , y , c , and F iff (27) holds. This implies the following proposition.

Proposition 1. *Assume (A). Then, there exists a free-entry equilibrium for all $c > 0$ and $F > 0$ iff (27) holds for all $x > 0$ and $N > 0$. Furthermore, this equilibrium is unique, stable and symmetric.*

Thus, we may safely conclude that the set of assumptions required to bring into play monopolistic competition must include (27). This condition allows one to work with preferences that display a great deal of flexibility. Indeed, σ may decrease or increase with x and/or N . Evidently, (27) is satisfied when (16) holds. More generally, (17) and (27) define a range of possibilities which is broader than the one defined by (16). These conditions define the striped area in Figure 1, in which (16) is described by the fourth quadrant.

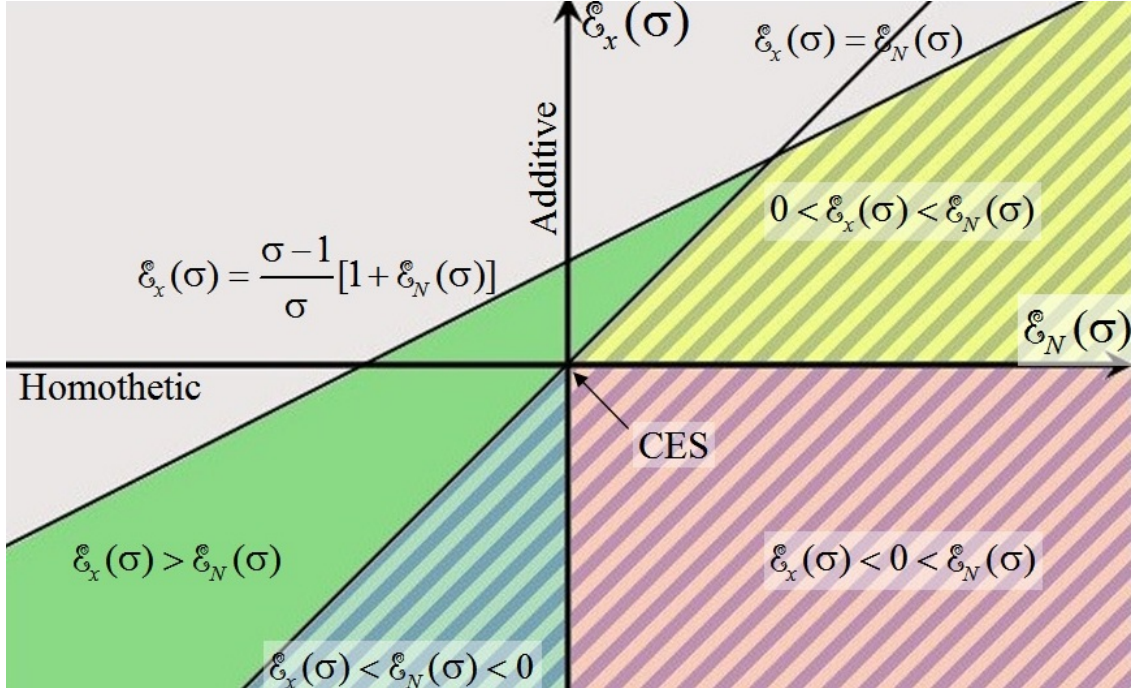


Fig. 1. The space of preferences

Note that (27) amounts to $\mathcal{E}_x(\sigma) < (\sigma - 1)/\sigma$ for additive preferences, and to $\mathcal{E}_x(\sigma) < \sigma - 1$ for indirectly additive preferences. Both conditions mean that σ cannot increase “too fast” with

x . When preferences are homothetic, (27) holds iff $\mathcal{E}_N(\sigma)$ exceeds -1 , which means that varieties cannot become too differentiated when their number increases. All these inequalities will play a critical role in the comparative statics analysis performed below, which demonstrates their relevance (see Table 1).

Local conditions It is legitimate to ask what Proposition 1 becomes when (27) does not hold for all $x > 0$ and $N > 0$. In this case, several stable SFEs may exist, so that Propositions 2-4 discussed below hold true for small shocks at any stable SFE. Of course, when there is a multiplicity of equilibria, different patterns may arise at different equilibria because the functions $\mathcal{E}_x(\sigma)$ and $\mathcal{E}_N(\sigma)$ need not behave in the same way at each stable equilibrium.

3.2 Comparative statics

In this subsection, we study the impact of a shock in the GDP on the SFE. A higher total income may stem from a larger population L , a higher per capita income y , or both. Next, we discuss the role of productivity. To achieve our goal, it is convenient to use markup as the endogenous variable. Setting $m \equiv FN/(Ly)$, we may rewrite the equilibrium condition (26) in terms of m only:

$$m\sigma \left(\frac{F}{cL} \frac{1-m}{m}, \frac{Ly}{F} m \right) = 1. \quad (28)$$

Note that (28) involves the four structural parameters of the economy: L , y , c and F . Furthermore, it is readily verified that the left-hand side of (28) increases with m iff (27) holds. Therefore, to study the impact of a specific parameter, we only have to find out how the corresponding curve is shifted. In our comparative static analysis, we will refrain from following an encyclopaedic approach in which all cases are systematically explored.

3.2.1 The impact of population size

Let us first consider the impact of L on the market price p^* . Differentiating (28) with respect to L , we find that the left-hand side of (28) is shifted upwards under an increase in L iff (17) holds. As a consequence, the equilibrium markup m^* , whence the equilibrium price p^* , decreases with L . This is in accordance with Handbury and Weinstein (2015) who observe that the price level for food products falls with city size. Second, the zero-profit condition implies that L always shifts p^* and q^* in opposite directions. Therefore, firm sizes are larger in bigger markets, as suggested by the empirical evidence provided by Manning (2010).

How does N^* change with L ? Differentiating (24) with respect to L , we obtain

$$\left. \frac{\partial \bar{\pi}}{\partial L} \right|_{N=N^*} = \left[\frac{cx\sigma}{(\sigma-1)^3} (\sigma-1-\mathcal{E}_x(\sigma)) \right] \Big|_{x=x^*, N=N^*}. \quad (29)$$

Since the first term in the right-hand side of this expression is positive, (29) is positive iff the following condition holds:

$$\mathcal{E}_x(\sigma) < \sigma - 1. \quad (30)$$

In this case, a population growth triggers the entry of new firms. Otherwise, the mass of varieties falls with the population size. Indeed, when $\mathcal{E}_x(\sigma)$ exceeds $\sigma - 1$, increasing the individual consumption makes varieties much closer substitutes, which intensifies competition. Under such circumstances, the benefits associated with diversity are low, implying that consumers value more the volumes they consume. This in turn leads a certain number of firms going out of business. Furthermore, when the mass of firms increases with L , the labor market balance condition implies that N^* rises less than proportionally because q^* also increases with L . Observe also that (27) implies (30) when preferences are additive, while (30) holds true under homothetic preferences because $\mathcal{E}_x(\sigma) = 0$.

The following proposition comprises a summary.

Proposition 2. *If $\mathcal{E}_x(\sigma)$ is smaller than $\mathcal{E}_N(\sigma)$ at the SFE, then a larger population results in a lower markup and larger firms. Furthermore, the mass of varieties increases with L iff (30) holds in equilibrium.*

What happens when $\mathcal{E}_x(\sigma) > \mathcal{E}_N(\sigma)$ at the SFE? In this event, a larger population results in a higher markup, smaller firms, a more than proportional rise in the mass of varieties, and lower per-variety consumption. In other words, a larger market would generate anti-competitive effects, which do not seem very plausible.

3.2.2 The impact of individual income

We now come to the impact of the per capita income on the SFE. One expects a positive shock on y to trigger the entry of new firms because more labor is available for production. However, consumers have a higher willingness-to-pay for the incumbent varieties and can afford to buy each of them in a larger volume. Therefore, the impact of y on the SFE is a priori ambiguous.

Differentiating (28) with respect to y , we see that the left-hand side of (28) is shifted downwards by an increase in y iff $\mathcal{E}_N(\sigma) > 0$. In this event, the equilibrium markup decreases with y . To check the impact of y on N^* , we differentiate (24) with respect to y and get after simplification:

$$\left. \frac{\partial \bar{\pi}(N)}{\partial y} \right|_{N=N^*} = \left. \frac{L}{N} \frac{\sigma - 1 - \sigma \mathcal{E}_x(\sigma)}{(\sigma - 1)^2} \right|_{x=x^*, N=N^*}.$$

Hence, $\partial \bar{\pi}(N^*)/\partial y > 0$ iff the following condition holds:

$$\mathcal{E}_x(\sigma) < \frac{\sigma - 1}{\sigma}, \quad (31)$$

a condition more stringent than (30). Thus, if $\mathcal{E}_N(\sigma) > 0$, then (31) implies (27). As a consequence, we have:

Proposition 3. *If $\mathcal{E}_N(\sigma) > 0$ at the SFE, then higher per capita income results in a lower markup and bigger firms. Furthermore, the mass of varieties increases with y iff (31) holds in equilibrium.*

Thus, when entry renders varieties less differentiated, the mass of varieties need not rise with income. This is because the decline in prices is too strong for more firms to operate at a larger scale.

3.2.3 The impact of firm productivity

Firm productivity (trade barriers) is typically measured by marginal costs (trade costs). To uncover the impact on the market outcome of a productivity shock common to all firms, we conduct a comparative static analysis of the SFE with respect to c and show that *the nature of preferences determines the extent of the pass-through*. The left-hand side of (28) is shifted downwards under a decrease in c iff

$$\mathcal{E}_x(\sigma) < 0 \tag{32}$$

holds. In this case, both the equilibrium markup m^* and the equilibrium mass of firms $N^* = (yL/F) \cdot m^*$ increases with c . In other words, when $\mathcal{E}_x(\sigma) < 0$, the pass-through is smaller than 100%. This is because varieties becomes more differentiated, which relaxes competition.

It must be kept in mind that the price change occurs through the following three channels. First, when facing a change in its own marginal cost, a firm changes its price more or less proportionally by balancing the impact of the cost change on its markup and market share. Second, since all firms face the same cost change, they all change their prices, which affects the toughness of competition and, thereby, the prices set by the incumbents. Third, as firms change their pricing behavior, the number of firms in the market changes, changing the markup of the active firms. Under homothetic preferences, the markup remains the same regardless of the productivity shock, implying that the pass-through is 100%. Indeed, we have seen that the markup function $m(\cdot)$ depends only upon N , and thus (26) does not involve c as a parameter.

Rewriting (28) as

$$m\sigma \left(\frac{q}{L}, \frac{Ly}{F}m \right) = 1$$

and totally differentiating this expression yields

$$\mathcal{E}_x(\sigma) \frac{dq^*}{q^*} + [1 + \mathcal{E}_N(\sigma)] \frac{dm^*}{m^*} = 0. \tag{33}$$

Since dm^* and $\mathcal{E}_x(\sigma)$ have opposite signs under a positive productivity shock, dq^* and $1 + \mathcal{E}_N(\sigma)$

must have the same sign for (33) to hold. In other words, a drop in c leads to an increase in q^* iff $\mathcal{E}_N(\sigma) > -1$ holds.

The following proposition comprises a summary.

Proposition 4. *If firms face a drop in their marginal production cost, (i) the market price decreases, (ii) the markup and number of firms increase iff $\mathcal{E}_x(\sigma) < 0$ holds in equilibrium, and (iii) firms are larger iff $\mathcal{E}_N(\sigma) > -1$ in equilibrium.*

This proposition has an important implication. If the data suggest a pass-through smaller than 100%, then it must be that $\mathcal{E}_x(\sigma) < 0$. In this case, (30) and (31) must hold, thereby a bigger or richer economy displays more diversity than a smaller or poorer one.

Remark. When $\mathcal{E}_x(\sigma) > 0$, the pass-through exceeds 100%, so that p^* decreases more than proportionally with c . As noticed in 2.3.2, though rather uncommon, a pass-through larger than 100% cannot be ruled out a priori.

3.2.4 Summary

Let us pause and recall our main results. We have found a necessary and sufficient condition for the existence and uniqueness of SFE (Proposition 1) and provided a complete characterization of the effect of market size or a productivity shock (Propositions 2 to 4). Given that (16) implies (27), (17) and (31), we may conclude as follows: *if (16) holds, a unique SFE exists (Proposition 1), a larger market or a higher income leads to lower markups, bigger firms and a larger number of varieties (Propositions 2 and 3), while the pass-through is incomplete (Proposition 4).* However, Propositions 1-4 still hold under conditions more general than (16).

Although we consider general symmetric preferences, the market outcome is governed *only* by the behavior of the elasticity of substitution $\sigma(x, N)$ and by $\mathcal{E}_x(\sigma)$ and $\mathcal{E}_N(\sigma)$. Table 1 summarizes the main results of this section.

Variables	Shocks		
	Increase in L	Increase in y	Decrease in c
Markups \downarrow	$\mathcal{E}_x(\sigma) < \mathcal{E}_N(\sigma)$	$\mathcal{E}_N(\sigma) > 0$	$\mathcal{E}_x(\sigma) > 0$
Firm sizes \uparrow	$\mathcal{E}_x(\sigma) < \mathcal{E}_N(\sigma)$	$\mathcal{E}_N(\sigma) > 0$	$\mathcal{E}_N(\sigma) > -1$
Number of firms \uparrow	$\mathcal{E}_x(\sigma) < \sigma - 1$	$\mathcal{E}_x(\sigma) < (\sigma - 1)/\sigma$	$\mathcal{E}_x(\sigma) < 0$

Table 1. Market equilibrium: the role of $\sigma(x, N)$

4 When does free-entry deliver the social optimum?

In this section, our aim is to delve deeper into a variety of issues discussed in the literature on optimum product diversity. It is well known that the comparison of the social optimum and market outcome often leads to ambiguous conclusions (Spence, 1976).

Dhingra and Morrow (2015) show that the two outcomes are identical under additive preferences iff preferences are CES. Given the prevalence of these preferences in quantitative models, it is legitimate to question the validity of prescriptions derived in such models. This peculiarity of the CES model should lead researchers to test against alternative specifications the policy recommendations obtained by using the CES.

What happens when preferences are homothetic? Without loss of generality, we assume that U is homogeneous of degree one in \mathbf{x} . In the case of symmetric consumption profiles $\mathbf{x} = xI_{[0,N]}$, we have

$$U(xI_{[0,N]}) \equiv \phi(N, x) = x\phi(N, 1).$$

Setting $\psi(N) \equiv \phi(N, 1)/N$ and $X \equiv xN$, we obtain

$$\phi(X, N) = X\psi(N). \tag{34}$$

Hence, at a symmetric consumption pattern $x_i = x$, homothetic preferences are *separable* in the total consumption X and the mass N of varieties. Preferences exhibit a love for variety iff $\psi(N)$ increases with N .

4.1 The social optimum

The planner aims to solve the following optimization problem:

$$\max U(\mathbf{x}) \quad \text{s.t.} \quad L = c \int_0^N q_i di + NF \quad \text{and} \quad q_i = Lx_i.$$

Using symmetry, the socially optimal outcome is given by the solution to

$$\max_{X,N} X\psi(N) \quad \text{s.t.} \quad L = cLX + NF \tag{35}$$

subject to

- (i) the labor balance condition

$$c \int_0^N q_i + fN = L,$$

(ii) and the availability constraints:

$$x_i = 0 \text{ for all } i \in]N, \mathbb{N}].$$

It is reasonable to assume that X and N are substitutes. This is so iff $1/\psi(N)$ is convex. It is readily verified that this condition is equivalent to the inequality:

$$\frac{\psi''(N)}{\psi'(N)} < 2 \frac{\psi'(N)}{\psi(N)}. \quad (36)$$

The following result provides a characterization of the optimum (for the proof see Appendix 5 in the Supplemental Material).

Proposition 5. *Assume consumers are variety-lovers while X and N are substitutes. Then, there exists a unique social optimum. Furthermore, the optimum involves a positive range of varieties iff*

$$\frac{L}{F} > \lim_{N \rightarrow 0} \frac{\psi(N)}{\psi'(N)}. \quad (37)$$

Otherwise, the optimum is given by the corner solution

$$X^O = \frac{1}{c} \quad N^O = 0. \quad (38)$$

Observe that the optimum may involve the supply of a single variety even when consumers are variety-lovers. Indeed, the labor balance constraint may be rewritten as follows:

$$X = \frac{1}{c} - \frac{F}{cL}N.$$

Therefore, the unique solution of the social planner's problem is the corner solution given by (38) iff the slope $F/(cL)$ exceeds the slope of the indifference curve at $(0, 1/c)$ in the plane (N, X) . Put differently, the marginal rate of substitution between X and N is too small for more than one variety to be produced.

4.2 Is there over- or under-provision of diversity?

4.2.1 When do the equilibrium and optimum coincide under homothetic preferences?

The ratio of the first-order conditions of (35) is given by

$$X \frac{\psi'(N)}{\psi(N)} = \frac{F}{cL}. \quad (39)$$

Using $X \equiv xN$, we may rewrite (39) as follows:

$$\mathcal{E}_\psi(N) \equiv N \frac{\psi'(N)}{\psi(N)} = \frac{F}{cLx}. \quad (40)$$

As for the market equilibrium condition (26), it may be reformulated as follows:

$$\frac{\bar{m}(N)}{1 - \bar{m}(N)} = \frac{F}{cLx}. \quad (41)$$

Comparing (40) and (41) shows that the social optimum and the market equilibrium are identical iff

$$\mathcal{E}_\psi(N) = \frac{\bar{m}(N)}{1 - \bar{m}(N)}, \quad (42)$$

while it is readily verified that there is excess (insufficient) variety iff the right-hand side term of (42) is larger (smaller) than the left-hand side term.

Clearly, (42) is unlikely to be satisfied unless some strong restrictions are imposed on preferences. The general belief is that this condition holds only for the CES. Yet, we find it natural to ask whether there are other homothetic preferences for which the SFE is optimal. In the next proposition, we show that there exists a mapping from the set of homothetic preferences into itself such that the SFE and the optimum coincide for an infinite set of homothetic preferences, which includes the CES (see Appendix B for a proof). However, as shown by Example 1, there are homothetic preferences that do not satisfy this property, even when markups are constant along the diagonal. Moreover, as shown by Example 1, there exist homothetic preferences that do not satisfy this property, even when markups are constant along the diagonal. As a consequence, working with a subset of homothetic preferences may generate versatile welfare properties, which means that care is needed when drawing policy recommendations based on models that use homothetic preferences and monopolistic competition.

Proposition 6. *For any homothetic utility $U(\mathbf{x})$, there exists an homothetic utility $V(\mathbf{x})$ that is generically non-CES such that the market equilibrium and optimum coincide for the homothetic utility given by $[U(\mathbf{x})V(\mathbf{x})]^{1/2}$.*

Example 1. The equivalence does not hold for all homothetic preferences, even when the markup is constant along the diagonal.

Consider the following class of generalized CES preferences:

$$\mathcal{U}(\mathbf{x}) = \mathbb{E} \left[\ln \left(\int_0^N x_i^\rho di \right) \right], \quad (43)$$

where $0 < \rho < 1$ is distributed according to the probability cumulative distribution $H(\rho)$ over $[0, 1]$. When this distribution is degenerate, (43) is equivalent to the standard CES. The idea behind (43) is that consumers are unsure about the degree of differentiation across varieties.

It is readily verified that the elasticity of substitution $\bar{\sigma}(x, \mathbf{x})$ is now given by

$$\bar{\sigma}(x, \mathbf{x}) = \frac{\mathbb{E} \left(\rho \frac{x^{\rho-1}}{\int_0^{\mathbb{N}} x_j^{\rho} dj} \right)}{\mathbb{E} \left[(\rho - \rho^2) \frac{x^{\rho-1}}{\int_0^{\mathbb{N}} x_j^{\rho} dj} \right]}, \quad (44)$$

which is variable, implying that (43) is non-CES. Regardless of the shape of the distribution $H(\rho)$, (43) describes a strictly convex symmetric preference over $L_2([0, \mathbb{N}])$.⁶ Therefore, as above, it is legitimate to focus on symmetric outcomes.

Evaluating $\bar{\sigma}(x, \mathbf{x})$ at a symmetric outcome $\mathbf{x} = xI_{[0, N]}$ yields

$$\sigma(x, N) = \frac{\mathbb{E}(\rho)}{\mathbb{E}(\rho) - \mathbb{E}(\rho^2)} = \frac{\mathbb{E}(\rho)}{\mathbb{E}(\rho) - [\mathbb{E}(\rho)]^2 - \text{Var}(\rho)}, \quad (45)$$

the value of which depends on the distribution H . Hence, at any symmetric outcome, everything work as if preferences were CES with a constant elasticity of substitution given by (45). Following the line of Appendix C, the SFE can be shown to be optimal iff

$$\text{Var}(\rho) = 0.$$

This amounts to assuming that the distribution of ρ is degenerate. If not, the SFE is *not* optimal.

While Example 1 shows that a constant elasticity of substitution is not a necessary condition for the optimality of the equilibrium, Proposition 6 relies on homotheticity, a property shared by the CES. It is therefore legitimate to ask whether homotheticity is a necessary condition for the SFE to be optimal. Example 2 below shows that it is not.

Example 2. The equivalence may hold for non-homothetic preferences displaying a variable markup. This is shown for the following class of non-homothetic preferences (see Appendix B):

$$\mathcal{U}(\mathbf{x}) = \mathbb{E} \left[\int_0^{\mathbb{N}} (\ln x_i^r + 1)^{\frac{r}{r-1}} di \right], \quad (46)$$

where $r \equiv \text{Var}(\rho)/\mathbb{E}(\rho)$. Observe that the constant r is positive and smaller than 1 because ρ is distributed over the interval $[0, 1]$. When the distribution is degenerate, (46) boils down to the CES with the elasticity of substitution equal to $1/(1 - \rho)$.

To sum up, a constant markup is neither a necessary nor a sufficient condition for the market equilibrium to be optimal.

Remark. Examples 1 and 2 shed further light on σ as the primitive of the model (see

⁶Symmetry holds because any Lebesgue-measure preserving mapping of $[0, \mathbb{N}]$ into itself preserves the value of $\ln \left(\int_0^{\mathbb{N}} x_i^{\rho} di \right)$ for any $\rho \in]0, 1[$.

Section 2.3). We know that standard CES preferences imply that σ is constant everywhere, hence along the diagonal. Example 1 shows that there exist non-CES symmetric preferences for which σ is constant along the diagonal. Regarding now Example 2, (A.12) in Appendix C implies

$$\sigma(x, N) = \frac{r \ln x + 1}{r \ln x + 1 - \mathbb{E}(\rho)}.$$

Therefore, even though (46) is neither homothetic nor additive, along the diagonal σ depends solely on x as in the case of additive preferences.

4.2.2 The optimal shifter

Another way to approach the diversity issue is to proceed along the line suggested by Dixit and Stiglitz who argued in their working paper that the mass of available varieties could enter the utility functional as a specific argument. Given $\phi(X, N)$, it is reasonable to map this function into another homothetic preference $\mathbb{A}(N)\phi(X, N)$, where $\mathbb{A}(N)$ is a *shifter* that depends on N only. Observe that the utility $\mathbb{A}(N)U(\mathbf{x})$ is homothetic and generates the same equilibrium outcome as $U(\mathbf{x})$, for the elasticity of substitution $\sigma(N)$ is unaffected by introducing the shifter $\mathbb{A}(N)$. An example of shifter used in the literature is given by the augmented-CES:

$$\mathcal{U}(\mathbf{x}, N) \equiv N^\nu \left(\int_0^N x_i^{\frac{\sigma-1}{\sigma}} di \right)^{\sigma/(\sigma-1)}. \quad (47)$$

In Benassy (1996), ν is a positive constant that captures the consumer benefit of a larger number of varieties. The idea is to separate the love-for-variety effect from the competition effect generated by the degree of product differentiation, which is inversely measured by the elasticity of substitution σ . Blanchard and Giavazzi (2003) take the opposite stance by assuming σ in (47) to increase with N and setting $\nu = -1/[\sigma(N) - 1]$. Under this specification, increasing the number of varieties does not raise consumer welfare but intensifies competition among firms.

To determine the shifter $\mathbb{A}(N)$ that guarantees optimal product diversity, we observe that (42) is to be rewritten as follows in the case of $\mathbb{A}(N)\phi(X, N)$:

$$\mathcal{E}_{\mathbb{A}}(N) + \mathcal{E}_{\psi}(N) = \frac{m(N)}{1 - m(N)}. \quad (48)$$

For this expression to hold, $\mathbb{A}(N)$ must be the solution to the linear differential equation in N

$$\frac{dA}{dN} = \left[\frac{m(N)}{1 - m(N)} - \frac{N}{\psi(N)} \frac{d\psi}{dN} \right] \frac{A(N)}{N},$$

which has a unique solution up to a positive constant. Therefore, *there always exists a shifter*

$A(N)$ such that (48) holds for all N iff $U(\mathbf{x})$ is replaced with $A(N)U(\mathbf{x})$. The shifter aligns the optimum to the equilibrium, which remains the same.

Furthermore, it is readily verified that there is excess (insufficient) variety iff the right-hand side term of (48) is larger (smaller) than the left-hand side term. We can even go one step further. If we use the shifter $N^\nu \mathbb{A}(N)$, there is growing under-provision of varieties when the difference $\nu-1/(\sigma(N^*)-1) < 0$ falls, but growing over-provision when $\nu-1/(\sigma(N^*)-1) > 0$ rises. Therefore, for any positive or negative number Δ there exists a shifter such that $N^* - N^o = \Delta$. In other words, by taking a power transformation of $N^\nu \phi(N, x)$, we can render the discrepancy between the equilibrium and the optimum arbitrarily large, or arbitrarily small, by changing the value of ν .

In sum, by choosing the appropriate shifter, the gap between the market equilibrium and the social optimum can be made equal to any arbitrary positive or negative constant.

5 Heterogeneous firms

It is natural to ask whether the approach developed in this paper can cope with Melitz-like heterogeneous firms. In this event, the consumption pattern ceases to be symmetric, making the problem infinitely dimensional. Yet, all firms of a given type supply the same output. As shown below, making the elasticity of substitution type-specific allows us to use σ for studying heterogeneous firms at the cost of one only additional dimension, i.e. the firm's type.

In what follows, we build on Asplund and Nocke (2006), but use a one-period framework á la Melitz and Ottaviano (2008). The mass of potential firms is given by \mathbb{N} . Prior to entry, risk-neutral firms face uncertainty about their marginal cost while entry requires a sunk cost F_e . Once this cost is paid, firms observe their marginal cost drawn randomly from the continuous probability distribution $\Gamma(c)$ defined over \mathbb{R}_+ , with a density $\gamma(c)$. After observing its type c , each entrant decides to produce or not, given that an active firm must incur a fixed production cost F . Under such circumstances, the mass of entrants, N_e , typically exceeds the mass of operating firms, N . Even though varieties are differentiated from the consumer's point of view, firms sharing the same marginal cost c behave in the same way and earn the same profit at equilibrium. As a consequence, we may refer to any variety/firm by its c -type only.

The equilibrium conditions are as follows:

(i) the profit-maximization condition for c -type firms:

$$\max_{x_c} \Pi_c(x_c, \mathbf{x}) \equiv \left[\frac{D(x_c, \mathbf{x})}{\lambda} - c \right] Lx_c - F; \quad (49)$$

(ii) the zero-profit condition for the cutoff firm:

$$(p_{\hat{c}} - \hat{c})q_{\hat{c}} = F,$$

where \hat{c} is the cutoff cost. At equilibrium, firms are sorted out by decreasing order of productivity, which implies that the mass of active firms is equal to $N \equiv N_e \Gamma(\hat{c})$;

(iii) the product market clearing condition:

$$q_c = Lx_c$$

for all $c \in [0, \hat{c}]$;

(iv) the labor market clearing condition:

$$N_e F_e + N_e \int_0^{\hat{c}} (F + cq_c) d\Gamma(c) = yL;$$

(v) firms enter the market until the expected profits net of the entry cost F_e are zero:

$$\int_0^{\hat{c}} \Pi_c(x_c, \mathbf{x}) d\Gamma(c) = F_e.$$

Since the distribution Γ is given, the profit-maximization condition implies that the equilibrium consumption profile is entirely determined by the set of active firms, which is fully described by \hat{c} and N_e . In other words, a variety supplied by an active firm can be viewed as a point in the set

$$\Omega \equiv \{(c, \nu) \in \mathbb{R}_+^2 \mid c \leq \hat{c}; \nu \leq N_e \gamma(c)\}.$$

In the case of homogeneous firms, the variable N is sufficient to describe the set of active firms, so that $\Omega = [0, N]$.

It follows from (49) and the envelope theorem that firms with a higher productivity earn higher profits. so that there is perfect sorting across firm types at any equilibrium. The first-order conditions for any c_i and c_j imply

$$\frac{D(x_{c_i}, \mathbf{x}) [1 - \bar{\eta}(x_{c_i}, \mathbf{x})]}{D(x_{c_j}, \mathbf{x}) [1 - \bar{\eta}(x_{c_j}, \mathbf{x})]} = \frac{c_i}{c_j}. \quad (50)$$

Condition **(A)** of Section 2.2 implies that, for any given \mathbf{x} , a firm's marginal revenue $D(x, \mathbf{x}) [1 - \bar{\eta}(x, \mathbf{x})]$ decreases with x regardless of its marginal cost. Therefore, it follows from (50) that $x_i > x_j$ iff $c_i < c_j$. In other words, more efficient firms produce more than less efficient firms. Furthermore, since $p_i = D(x, \mathbf{x})/\lambda$ and D decreases in x for any given \mathbf{x} , more efficient firms sell at lower prices than less efficient firms. For the markups, (50) yields

$$\frac{p_i/c_i}{p_j/c_j} = \frac{1 - \bar{\eta}(x_{c_j}, \mathbf{x})}{1 - \bar{\eta}(x_{c_i}, \mathbf{x})}.$$

Consequently, more efficient firms enjoy higher markups – as in De Loecker and Warzynski (2012) – iff $\bar{\eta}(x, \mathbf{x})$ increases with x , i.e., **(AA)** holds. Therefore, if **(A)** holds more efficient firms produce larger outputs and charge lower prices than less efficient firms. In addition, *more*

efficient firms have higher markups iff **(AA)** holds.

Very much as in 3.1 where N is treated parametrically, we assume for the moment that \hat{c} and N_e are given, and consider the game in which the corresponding active firms compete in quantities. Because we work with general preferences, the quantity game cannot be solved pointwise. Indeed, the profit-maximizing output of a c -type firm depends on what the other types of firms do. We show in Appendix 6 of the Supplemental Material that, for any \hat{c} and N_e , an equilibrium $\bar{\mathbf{x}}(\hat{c}, N_e)$ of the quantity game exists. Observe that the counterpart of $\bar{\mathbf{x}}(\hat{c}, N_e)$ in the case of symmetric firms is $\bar{x}(N)$ given by (20). Furthermore, because all the c -type firms sell at the same price, which depends on c , the consumption of a variety is c -specific, which makes the consumption of the corresponding c -type varieties c -specific.

The operating profits of a c -type firm made at an equilibrium $\mathbf{x}^*(\hat{c}, N_e)$ of the quantity game are as follows:

$$\bar{\pi}_c(\hat{c}, N_e) \equiv \max_{x_c} \left[\frac{D(x_c, \bar{\mathbf{x}}(\hat{c}, N_e))}{\lambda(\bar{\mathbf{x}}(\hat{c}, N_e))} - c \right] Lx_c,$$

which is the counterpart of $\bar{\pi}(N)$ in the case of heterogeneous firms. Note that the perfect sorting of firms implies that $\bar{\pi}_c(\hat{c}, N_e)$ decreases with c .

A free-entry equilibrium with heterogeneous firms is defined by a pair (\hat{c}^*, N_e^*) which satisfies the zero-expected-profit condition for each firm:

$$\int_0^{\hat{c}} [\bar{\pi}_c(\hat{c}, N_e) - F] d\Gamma(c) = F_e, \quad (51)$$

and the cutoff condition

$$\bar{\pi}_{\hat{c}}(\hat{c}, N_e) = F. \quad (52)$$

Thus, regardless of the nature of preferences and the distribution of marginal costs, the heterogeneity of firms amounts to replacing the variable N by the two variables \hat{c} and N_e because $N = \Gamma(\hat{c})N_e$ when $\bar{x}(N)$ is replaced by $\bar{\mathbf{x}}(\hat{c}, N_e)$. As a consequence, *the complexity of the problem increases from one to two dimensions*.

Dividing (51) by (52) yields the following new equilibrium condition:

$$\int_0^{\hat{c}} \left[\frac{\bar{\pi}_c(\hat{c}, N_e)}{\bar{\pi}_{\hat{c}}(\hat{c}, N_e)} - 1 \right] d\Gamma(c) = \frac{F_e}{F}. \quad (53)$$

5.1 Making the elasticity of substitution type-specific

When firms are symmetric, the sign of $\mathcal{E}_N(\sigma)$ plays a critical role in comparative statics. Since firms of a given type are symmetric, the same holds here. The difference is that the mass of active firms is now determined by the two endogenous variables \hat{c} and N_e . As a consequence,

understanding how the mass of active firms responds to a population hike requires studying the way the left-hand side of (53) varies with \hat{c} and N_e . Let $\sigma_c(\hat{c}, N_e)$ be the equilibrium value of the elasticity of substitution between any two varieties supplied by c -type firms:

$$\sigma_c(\hat{c}, N_e) \equiv \bar{\sigma}[\bar{x}_c(\hat{c}, N_e), \bar{\mathbf{x}}(\hat{c}, N_e)].$$

In this case, we may rewrite $\bar{\pi}_c(\hat{c}, N_e)$ as follows:

$$\bar{\pi}_c(\hat{c}, N_e) = \frac{c}{\sigma_c(\hat{c}, N_e) - 1} L\bar{x}_c(\hat{c}, N_e), \quad (54)$$

which is the counterpart of (24). Hence, by making σ type-specific, we are able to use the elasticity of substitution for studying heterogeneous firms at the cost of one additional dimension, i.e. the firm's type c .

Using the envelope theorem and the profit-maximization condition (49), we obtain:

$$\mathcal{E}_c(\bar{\pi}_c(\hat{c}, N_e)) = 1 - \sigma_c(\hat{c}, N_e). \quad (55)$$

Combining this with (54) allows us to rewrite the equilibrium conditions (52) and (53) as follows:

$$\frac{\hat{c}}{\sigma_{\hat{c}}(\hat{c}, N_e) - 1} L\bar{x}_{\hat{c}}(\hat{c}, N_e) = F, \quad (56)$$

and

$$\int_0^{\hat{c}} \left[\exp \left(\int_c^{\hat{c}} \frac{\sigma_z(\hat{c}, N_e) - 1}{z} dz \right) - 1 \right] d\Gamma(c) = \frac{F_e}{F}. \quad (57)$$

Let $\hat{c} = g(N_e)$ be the locus of solutions to (56) and $\hat{c} = h(N_e)$ the locus of solutions to (57).⁷ A free-entry equilibrium (\hat{c}^*, N_e^*) is the intersection point of the two loci $\hat{c} = g(N_e)$ and $\hat{c} = h(N_e)$ in the (N_e, \hat{c}) -plane, and thus the properties of the equilibrium (\hat{c}^*, N_e^*) depend only upon the slopes of these two curves, which in turn are determined by the behavior of $\sigma_c(\hat{c}, N_e)$. In particular, if $\sigma_c(\hat{c}, N_e)$ increases with \hat{c} , the left-hand side of (53) increases with \hat{c} . Intuitively, when \hat{c} increases, the mass of firms increases as less efficient firms stay in business, which intensifies competition and lowers markups. In this case, the selection process is tougher. This is not the end of the story, however. Indeed, the competitiveness of the market also depends on how N_e affects the degree of differentiation across varieties.

⁷We give below a sufficient condition for the left-hand side of (57) to be monotone in \hat{c} when h is well defined.

5.2 Properties of the free-entry equilibrium

The elasticity of substitution being the keystone of our approach, it is legitimate to ask whether imposing some simple conditions on σ_c (similar to those used in Section 3) can tell us something about the slope of $g(N_e)$. The left-hand side of (57) increases with N_e iff $\sigma_c(\hat{c}, N_e)$ increases in N_e . This amounts to assuming that, for any given cutoff \hat{c} , the relative impact of entry on the low-productivity firms (i.e., the small firms) is larger than the impact on the high-productivity firms. As implied by (55), $\mathcal{E}_c(\bar{\pi}_c(\hat{c}, N_e))$ decreases in N_e iff $\sigma_c(\hat{c}, N_e)$ increases in N_e . This leads us to impose an additional condition which implies that firms face a more competitive market when the number of active firms rises.

(B) *The equilibrium profit of each firm's type decreases in \hat{c} and N_e .*

The intuition behind this assumption is easy to grasp: a larger number of entrants or a higher cutoff leads to lower profits, for the mass of active firms rises. When firms are symmetric, the equilibrium operating profits depend only upon the number N of active firms (see (24)). Thus, **(B)** amounts to assuming that these profits decrease with N . Using Zhelobodko et al. (2012), it is readily verified that any additive preference satisfying **(A)** also satisfies **(B)**.

As implied by **(B)**, $g(N_e)$ is downward-sloping in the (N_e, \hat{c}) -plane. Furthermore, it is shifted upward when L rises. As for $h(N_e)$, it is independent of L but its slope is a priori undetermined. Three cases may arise. First, if the locus $h(N_e)$ is upward-sloping, there exists a unique free-entry equilibrium, and this equilibrium is stable. Furthermore, both N_e^* and \hat{c}^* increase with L (see Figure 2a). Second, under the CES preferences, $h(N_e)$ is horizontal, which implies that N_e^* rises with L while \hat{c}^* remains constant.

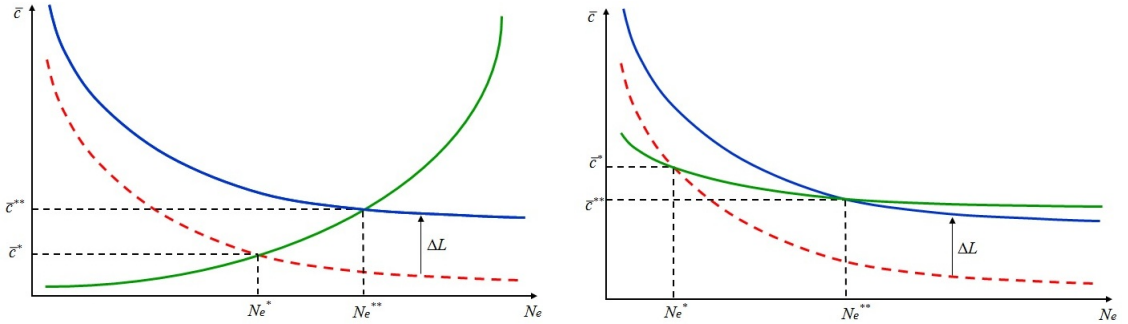


Fig. 2. Cutoff and market size

Last, when $h(N_e)$ is downward-sloping, two subcases must be distinguished. In the first, $h(N_e)$ is less steep than $g(N_e)$. As a consequence, there still exists a unique free-entry equilibrium. This equilibrium is stable and such that N_e^* increases with L , but \hat{c}^* now decreases with L (see Figure 2b). In the second subcase, $h(N_e)$ is steeper than $g(N_e)$, which implies that the equilibrium is unstable because $h(N_e)$ intersects $g(N_e)$ from below. In what follows, we focus only upon stable equilibria.

In sum, we end up with the following property:

Proposition 7. *Assume (B). Then, the equilibrium mass of entrants always increases with L .*

When $\sigma_c(\hat{c}, N_e)$ increases both with \hat{c} and N_e , the locus $h(N_e)$ is downward-sloping. Indeed, when N_e rises, so does the left-hand side of (57). Hence, since $\sigma_c(\hat{c}, N_e)$ also increases with \hat{c} , it must be that \hat{c} decreases for (57) to hold. As a consequence, we have:

Proposition 8. *Assume (B). If $\sigma_c(\hat{c}, N_e)$ increases with \hat{c} and N_e , then the equilibrium cutoff decreases with L . If $\sigma_c(\hat{c}, N_e)$ increases with \hat{c} and decreases with N_e , then \hat{c}^* increases with L .*

Given \hat{c} , we know that the number of active firms N is proportional to the number of entrants N_e . Therefore, assuming that $\sigma_c(\hat{c}, N_e)$ increases with N_e may be considered as the counterpart of (17), for (17) and $\sigma(\bar{x}(N), N)$ increasing in N can be shown to be equivalent. In this case, the pro-competitive effect generated by entry exacerbates the selection effect across firms. In response to a hike in L , the two effects combine to induce the exit of the least efficient active firms. This echoes Melitz and Ottaviano (2008) who show that a trade liberalization shock gives rise to a similar effect under quadratic preferences. In the present set-up, the impact of population size on the number of entrants remains unambiguous. In contrast, the cutoff cost behavior depends on how the elasticity of substitution $\sigma_c(\hat{c}, N_e)$ varies with N_e . In other words, even for plausible preferences generating pro-competitive effects, predictions regarding the direction of the firms' selection are inherently fragile.⁸

Note that what we said in Section 3.2 about local versus global conditions equally applies here. Indeed, when $\sigma_c(\hat{c}, N_e)$ increases with \hat{c} and N_e in the neighbourhood of the equilibrium, the above argument can be repeated to show that the equilibrium cutoff decreases with L for small changes in L . Note also that the pass-through is still complete under homothetic preferences when firms are heterogeneous.

Heterogeneous firms or asymmetric preferences. The assumption of symmetric preferences puts a strong structure on substitution between variety pairs. Without affecting the nature of our results, this assumption can be relaxed to capture a more realistic substitution pattern. We have seen that varieties sharing the same marginal cost c may be viewed as symmetric, whereas varieties produced by c_i -type and c_j -type firms are asymmetric when c_i and c_j obey different substitution patterns. As a consequence, *a model with heterogeneous firms supplying symmetric varieties is isomorphic to a model with symmetric firms selling varieties whose degree of differentiation varies with their type c .*

To illustrate, consider the case in which preferences are asymmetric in the following way: the utility functional $\mathcal{U}(\mathbf{x})$ is given by

⁸Results are ambiguous when $\sigma_c(\hat{c}, N_e)$ decreases with \hat{c} . In this case, the left-hand side of (57) may be non-monotone in \hat{c} . As a result, the mapping $h(N_e)$ may cease to be single-valued, which potentially leads to the existence of multiple equilibria. However, note that at any specific equilibrium, the behavior of \hat{c} with respect to L depends solely on whether $h(N_e)$ is locally upward-sloping or downward-sloping.

$$\mathcal{U}(\mathbf{x}) = \tilde{\mathcal{U}}(\mathbf{a} \cdot \mathbf{x}), \quad (58)$$

where $\tilde{\mathcal{U}}$ is a symmetric functional that satisfies (4), \mathbf{a} is a weight function and $\mathbf{a} \cdot \mathbf{x}$ is defined pointwise by $(\mathbf{a} \cdot \mathbf{x})_i \equiv a_i x_i$ for all $i \in [0, \mathbb{N}]$. The preferences (58) can be made symmetric by changing the units in which the quantities of varieties are measured. Indeed, for any $i, j \in [0, \mathbb{N}]$ the consumer is indifferent between consuming a_i/a_j units of variety i and one unit of variety j . Therefore, by using the change of variables $\tilde{x}_i \equiv a_i x_i$ and $\tilde{p}_i \equiv p_i/a_i$, we can reformulate the consumer's program as follows:

$$\max_{\tilde{\mathbf{x}}} \tilde{\mathcal{U}}(\tilde{\mathbf{x}}) \quad \text{s.t.} \quad \int_0^N \tilde{p}_i \tilde{x}_i di \leq Y.$$

In this case, by rescaling prices, quantities and costs by the weights a_i , the model now involves symmetric preferences but heterogeneous firms. Hence, there is a one-to-one mapping between models with symmetric preferences and heterogeneous firms, and models with asymmetric preferences of type (58) and symmetric firms. In this case, the elasticity of substitution is a -specific and there exists a cut-off variety \hat{a} such that market forces select only the varieties that have a weight exceeding \hat{a} .

6 Two-sector economy

Following Dixit and Stiglitz (1977), we consider a two-sector economy involving a differentiated good supplied under increasing returns and monopolistic competition, and a homogeneous good – or a Hicksian composite good – supplied under constant returns and perfect competition. Both goods are normal. Labor is the only production factor and is perfectly mobile between sectors. Consumers share the same preferences given by $U(\mathcal{U}(\mathbf{x}), x_0)$ where the functional $\mathcal{U}(\mathbf{x})$ satisfies the properties stated in Section 2, while x_0 is the consumption of the homogeneous good. The upper-tier utility U is strictly quasi-concave, continuously differentiable, strictly increasing in each argument, and such that the demand for the differentiated product is always positive.⁹

Choosing the unit of the homogeneous good for the marginal productivity of labor to be equal to 1, the equilibrium price of the homogeneous good is equal to 1. Since profits are zero at the SFE, the budget constraint is given by

$$\int_0^N p_i x_i di + x_0 = E + x_0 = y, \quad (59)$$

where the expenditure E on the differentiated good is endogenous because competition across firms affects the relative price of this good.

⁹Our results hold true if the choke price is finite but sufficiently high.

Using the first-order condition for utility maximization yields

$$p_i = \frac{U'_1(\mathcal{U}(\mathbf{x}), x_0)}{U'_2(\mathcal{U}(\mathbf{x}), x_0)} D(x_i, \mathbf{x}).$$

Let p be the price of the differentiated good. Along the diagonal $x_i = x$, the above condition becomes

$$p = S(\phi(x, N), x_0) D(x, xI_{[0, N]}), \quad (60)$$

where S is the marginal rate of substitution between the differentiated and homogeneous goods:

$$S(\phi, x_0) \equiv \frac{U'_1(\varphi(x, N), x_0)}{U'_2(\varphi(x, N), x_0)}$$

and

$$\varphi(x, N) \equiv \mathcal{U}(xI_{[0, N]}).$$

The quasi-concavity of the upper-tier utility U implies that the marginal rate of substitution decreases with $\varphi(x, N)$ and increases with x_0 . Therefore, for any given (p, x, N) , (60) has a unique solution $\bar{x}_0(p, x, N)$, which is the income-consumption curve. The two goods being normal, this curve is upward sloping in the plane (x, x_0) .

For any given $x_i = x$, the love for variety implies that the utility level increases with the number of varieties. However, it is reasonable to suppose that the marginal utility D of an additional variety decreases. To be precise, we assume that

(C) *for all $x > 0$, the marginal utility D weakly decreases with the number of varieties.*

Observe that **(C)** holds for additive and quadratic preferences. Since $\varphi(x, N)$ increases in N , S decreases. As D weakly decreases in N , it must be that x_0 increases for the condition (60) to be satisfied. In other words, $\bar{x}_0(p, x, N)$ increases in N .

We are now equipped to determine the relationship between x and m by using the zero-profit condition when firms are symmetric. Since by definition $m \equiv (p - c)/p$, for any given p the zero-profit and product market clearing conditions yield the per-variety consumption as a function of m only:

$$x = \frac{F}{cL} \frac{1 - m}{m}. \quad (61)$$

Plugging (61) and $p = c/(1 - m)$ into \bar{x}_0 , we may rewrite $\bar{x}_0(p, x, N)$ as a function of m and N only:

$$\hat{x}_0(m, N) \equiv \bar{x}_0 \left(\frac{c}{1 - m}, \frac{F}{cL} \frac{1 - m}{m}, N \right).$$

Furthermore, substituting (61) and $p = c/(1 - m)$ into the budget constraint (59) and

solving for N , we obtain the income y at which consumers choose the quantity $\hat{x}_0(m, N)$ of the homogeneous good:

$$N = \frac{Lm}{F} [y - \hat{x}_0(m, N)]. \quad (62)$$

Since \bar{x}_0 and \hat{x}_0 vary with N identically, \hat{x}_0 also increases in N . Therefore, (62) has a unique solution $\hat{N}(m)$ for any $m \in [0, 1]$.

Moreover, (62) implies that $\partial \hat{N} / \partial y > 0$, while $\partial \hat{N} / \partial L > 0$ because the income-consumption curve is upward slopping. In other words, if the price of the differentiated product is exogenously given, an increase in population size or individual income leads to a wider range of varieties.

Since $\hat{N}(m)$ is the number of varieties in the two-sector economy, the equilibrium condition (28) must be replaced with the following expression:

$$m\sigma \left[\frac{F}{cL} \frac{1-m}{m}, \hat{N}(m) \right] = 1. \quad (63)$$

The left-hand side $m\sigma$ of (63) equals zero for $m = 0$ and exceeds 1 when $m = 1$. Hence, by the intermediate value theorem, the set of SFEs is non-empty. Moreover, it has an infimum and a supremum, which are both SFEs because the left-hand side of (63) is continuous. In what follows, we denote the corresponding markups by m_{inf} and m_{sup} ; if the SFE is unique, $m_{\text{inf}} = m_{\text{sup}}$. Therefore, the left-hand side of (63) must increase with m in some neighborhood of m_{inf} , for otherwise there would be an equilibrium to the left of m_{inf} , a contradiction. Similarly, the left-hand side of (63) increases with m in some neighborhood of m_{sup} .

Since $\partial \hat{N} / \partial y > 0$, (63) implies that an increase in y shifts the locus $m\sigma$ upward iff $\mathcal{E}_N(\sigma) > 0$, so that the equilibrium markups m_{inf} and m_{sup} decrease in y . The same holds in response to a hike in population size.

Summarizing our results, we come to a proposition.

Proposition 9. *Assume (C). Then, the set of SFEs is non-empty. Furthermore, (i) an increase in individual income leads to a lower markup and bigger firms at the infimum and supremum SFEs iff $\mathcal{E}_N(\sigma) > 0$ and (ii) an increase in population size yields a lower markup and bigger firms at the infimum and supremum SFEs if $\mathcal{E}_x(\sigma) < 0$ and $\mathcal{E}_N(\sigma) > 0$.*

This extends to a two-sector economy what Propositions 2 and 3 state in the case of a one-sector economy. Proposition 9 also shows that the elasticity of substitution keeps its relevance for studying monopolistic competition in a multisector economy. In contrast, studying how N^* changes with L or y is a harder problem because the equilibrium number of varieties depends on the elasticity of substitution between the differentiated and homogeneous goods.

7 Concluding remarks

We have shown that monopolistic competition can be considered as the marriage between oligopoly theory and the negligibility hypothesis, thus confirming Mas-Colell's (1984) intuition. Using the concept of elasticity of substitution, we have provided a complete characterization of the market outcome and its comparative statics in terms of prices, firm size, and mass of firms/varieties. Somewhat ironically, the concept of elasticity of substitution, which has vastly contributed to the success of the CES model of monopolistic competition, is relevant in the case of general preferences, both for symmetric and heterogeneous firms. The fundamental difference is that the elasticity of substitution ceases to be constant and now varies with the key-variables of the setting under study. We take leverage on this to make predictions about the impact of market size and productivity shocks on the market outcome.

Furthermore, we have singled out our most preferred set of assumptions and given disarmingly simple necessary and sufficient conditions for the standard comparative statics effects to hold true. We have also shown that relaxing these assumptions does not jeopardize the tractability of the model. Future empirical studies should shed light on the plausibility of the assumptions discussed in this paper by checking their respective implications. It would be unreasonable, however, to expect a single set of conditions to be universally valid.

Last, although monopolistic competition is unable to replicate the rich array of findings obtained in industrial organization, it is our contention that models such as those presented in this paper avoid several of the limitations imposed by the partial equilibrium analyses of oligopoly theory. Although we acknowledge that monopolistic competition is the limit of oligopolistic equilibria, we want to stress that monopolistic competition may be used in different settings as a substitute for oligopoly models when these appear to be unworkable.

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Appendix

A. When the set Ω of active firms is given, the Marshallian demand $\mathcal{D}(p_i, \mathbf{p}, Y)$ is well defined.

Recall that (i) a quasi-concave Fréchet differentiable functional is weakly upper-semicontinuous, and (ii) an upper-semicontinuous functional defined over a bounded weakly closed subset of a Hilbert space has a maximizer. Since the budget set is bounded and weakly closed, while $L_2(\Omega)$ is a Hilbert space, existence is proven. Uniqueness follows from the strict quasi-concavity of \mathcal{U} and the convexity of the budget set. Hence, there exists a unique utility-maximizing consumption profile $\mathbf{x}^*(\mathbf{p}, Y)$ (Dunford and Schwartz, 1988). Plugging $\mathbf{x}^*(\mathbf{p}, Y)$ into (6) – (7) and solving (6) for x_i yields

$$x_i = \mathcal{D}(p_i, \mathbf{p}, Y),$$

which is weakly decreasing in its own price.¹⁰

B. *Proof of Proposition 6.*

(i). Given $[U(\mathbf{x})V(\mathbf{x})]^{1/2}$, the marginal utility of a variety $i \in [0, N]$ is given by

$$\frac{1}{2}D_U(x_i, \mathbf{x}) \left[\frac{V(\mathbf{x})}{U(\mathbf{x})} \right]^{1/2} + \frac{1}{2}D_V(x_i, \mathbf{x}) \left[\frac{U(\mathbf{x})}{V(\mathbf{x})} \right]^{1/2},$$

where D_U (respectively, D_V) is the marginal utility associated with U (respectively, V). Computing the elasticity $\bar{\eta}(x_i, \mathbf{x})$ of the inverse demand and using

$$\bar{\sigma}(x, \mathbf{x}) = \frac{1}{\bar{\eta}(x_i, \mathbf{x})},$$

where $\bar{\sigma}(x, \mathbf{x})$ is the elasticity of substitution between varieties i and j at $x_i = x_j = x$, we get

$$\bar{\sigma}(x, \mathbf{x}) = \frac{\frac{D_U(x, \mathbf{x})}{U(\mathbf{x})} + \frac{D_V(x, \mathbf{x})}{V(\mathbf{x})}}{\frac{D_U(x, \mathbf{x})}{U(\mathbf{x})} \bar{\eta}_U(x, \mathbf{x}) + \frac{D_V(x, \mathbf{x})}{V(\mathbf{x})} \bar{\eta}_V(x, \mathbf{x})},$$

where $\bar{\eta}_U$ ($\bar{\eta}_V$) is elasticity of D_U (D_V) at x . Evaluating $\bar{\sigma}(x, \mathbf{x})$ at a symmetric consumption pattern $x = xI_{[0, N]}$ with N available varieties yields

$$\frac{1}{\sigma(N)} = \frac{1}{2} \left[\frac{1}{\sigma_U(N)} + \frac{1}{\sigma_V(N)} \right], \quad (\text{A.1})$$

where σ_U (σ_V) is the elasticity of substitution associated with U (V). Furthermore, it is readily

¹⁰Since D is continuously decreasing in x_i , there exists at most one solution of (6) with respect to x_i . If there is a finite choke price ($D(0, \mathbf{x}^*)/\lambda < \infty$), there may be no solution. To encompass this case, the Marshallian demand should be formally defined by $\mathcal{D}(p_i, \mathbf{p}, y) \equiv \inf\{x_i \geq 0 \mid D(x_i, \mathbf{x}^*)/\lambda(y, \mathbf{x}^*) \leq p_i\}$.

verified that the function $\psi(N)$ associated with (A.1) is given by

$$\psi(N) = [\psi_U(N)\psi_V(N)]^{1/2}. \quad (\text{A.2})$$

Plugging (A.1) and (A.2) in (42), the optimum and equilibrium are identical iff

$$\frac{2\sigma_U(N) + 2\sigma_V(N)}{2\sigma_U(N)\sigma_V(N) - \sigma_U(N) - \sigma_V(N)} = \mathcal{E}_{\psi_U}(N) + \mathcal{E}_{\psi_V}(N), \quad (\text{A.3})$$

Since we assume $U(\mathbf{x})$ to be given, $\sigma_U(N)$ and $\mathcal{E}_{\psi_U}(N)$ are both given functions of N .

We now determine $V(\mathbf{x})$ by using the class of preferences described by the Kimball's flexible aggregator (2): there exists an increasing and convex function $\nu(\cdot)$ such that for any consumption pattern \mathbf{x} we have

$$\int_0^N \nu\left(\frac{x_i}{V}\right) di = 1, \quad (\text{A.4})$$

where $V = V(\mathbf{x})$. Evaluating (A.4) at a symmetric pattern $\mathbf{x} = xI_{[0,N]}$ implies that

$$\psi_V(N) = \frac{1}{N\nu^{-1}(1/N)}. \quad (\text{A.5})$$

Setting

$$z \equiv \nu^{-1}(1/N), \quad (\text{A.6})$$

(A.5) becomes

$$\psi_V = \frac{\nu(z)}{z}.$$

Hence,

$$\mathcal{E}_{\psi_V}(N) = \frac{1}{\mathcal{E}_\nu(z)} - 1. \quad (\text{A.7})$$

It is readily verified that

$$\frac{1}{\sigma_V(N)} = r_\nu(z) \equiv -\frac{z\nu''(z)}{\nu'(z)}. \quad (\text{A.8})$$

Using (A.5), (A.7) and (A.8) shows that (A.3) becomes a non-linear second-order differential equation in $\nu(z)$ where z is given by (A.6):

$$\nu''(z) = -\frac{\nu'(z)}{z} \frac{\left[2\sigma_U\left(\frac{1}{\nu(z)}\right) - 1\right] \left[\mathcal{E}_{\psi_U}\left(\frac{1}{\nu(z)}\right) + \frac{\nu(z) - z\nu'(z)}{z\nu'(z)}\right] - 2}{\left[\mathcal{E}_{\psi_U}\left(\frac{1}{\nu(z)}\right) + \frac{\nu(z) - z\nu'(z)}{z\nu'(z)} + 2\right] \sigma_U\left(\frac{1}{\nu(z)}\right)}. \quad (\text{A.9})$$

The Picard-Lindelöf theorem implies that (A.9) has a solution when \mathcal{E}_{ψ_U} and σ_U are well-behaved functions.

(ii) We show that $U^{1/2}V^{1/2}$ is generically described by non-CES preferences. The argument goes by contradiction. Assume that $U^{1/2}V^{1/2}$ is a CES utility whose elasticity of substitution is σ . This assumption together with (A.1) and (A.4) imply that

$$\frac{2}{\sigma} = \frac{1}{\sigma_U[1/\nu(z)]} - \frac{\nu'(z)}{\nu''(z)z}. \quad (\text{A.10})$$

Combining (A.9) and (A.10) shows that $U^{1/2} \cdot V^{1/2}$ is CES only if

$$-\frac{\nu''(z)z}{\nu'(z)} = \frac{2}{\sigma} - \frac{1}{\sigma_U[1/\nu(z)]} = \frac{\left[2\sigma_U\left(\frac{1}{\nu(z)}\right) - 1\right] \left[\mathcal{E}_{\psi_U}\left(\frac{1}{\nu(z)}\right) + \frac{\nu(z) - z\nu'(z)}{z\nu'(z)}\right] - 2}{\left[\mathcal{E}_{\psi_U}\left(\frac{1}{\nu(z)}\right) + \frac{\nu(z) - z\nu'(z)}{z\nu'(z)} + 2\right] \sigma_U\left(\frac{1}{\nu(z)}\right)},$$

or, equivalently,

$$\nu'(z) = \frac{\nu(z)}{z} \left[\frac{\sigma + 1}{\sigma - 1} - \mathcal{E}_{\psi_U}\left(\frac{1}{\nu(z)}\right) \right]. \quad (\text{A.11})$$

Observe that (A.9) is a second-order differential equation whose space of solutions is generically a two-dimensional manifold, for the solution is pinned down by fixing the values of two arbitrary integration constants. In contrast, (A.11) is a first-order differential equation, which has a unique solution up to one integration constant. Therefore, to guarantee that $U^{1/2} \cdot V^{1/2}$ is a non-CES preference, it is sufficient to choose $\nu(z)$ that satisfies (A.9) but not (A.11). Q.E.D.

C. Example 2.

Consider (46) and set:

$$u(x; \rho) \equiv (\ln x_i^r + 1)^{\frac{\rho}{r}},$$

$$\epsilon_u(x; \rho) \equiv \frac{u_x(x; \rho)x}{u(x; \rho)}, \quad r_u(x; \rho) \equiv -\frac{u_{xx}(x; \rho)x}{u_x(x; \rho)}.$$

Following the line of Appendix B and using (46), the elasticity of substitution $\bar{\sigma}(x, \mathbf{x})$ between varieties i and j when $x_i = x_j = x$ is given by

$$\bar{\sigma}(x, \mathbf{x}) = \frac{\mathbb{E} \left[\frac{u'(x; \rho)}{\int_0^N u(x; \rho) dj} \right]}{\mathbb{E} \left[\frac{u'(x; \rho)}{\int_0^N u(x; \rho) dj} r_u(x; \rho) \right]}.$$

Evaluating $\bar{\sigma}(x, \mathbf{x})$ at a symmetric consumption pattern $x = xI_{[0, N]}$ with N available varieties yields

$$\sigma(x, N) = \frac{\mathbb{E}[\epsilon_u(x; \rho)]}{\mathbb{E}[\epsilon_u(x; \rho)r_u(x; \rho)]} = \frac{r \ln x + 1}{r \ln x + 1 - \mathbb{E}(\rho)}. \quad (\text{A.12})$$

Combining (A.12) with the SFE conditions (18), (23) and (25), we find that the equilibrium individual consumption level must solve

$$\frac{\mathbb{E}[\epsilon_u(x; \rho)(1 - r_u(x; \rho))]}{\mathbb{E}[\epsilon_u(x; \rho)]} = \frac{cLx}{cLx + F}. \quad (\text{A.13})$$

As for the optimality condition, it is readily verified to be

$$\mathbb{E}[\epsilon_u(x; \rho)] = \frac{cLx}{cLx + F}. \quad (\text{A.14})$$

Computing $\epsilon_u(x; \rho)$ and $r_u(x; \rho)$ and setting $r \equiv \text{Var}(\rho)/\mathbb{E}(\rho)$, the left-hand sides of (A.13) and (A.14) imply

$$\mathbb{E}[\epsilon_u(x; \rho)] = \frac{\mathbb{E}(\rho)}{r \ln x + 1} = \frac{\mathbb{E}[\epsilon_u(x, \rho)(1 - r_u(x, \rho))]}{\mathbb{E}[\epsilon_u(x, \rho)]}.$$

Therefore, the equilibrium condition (A.13) coincides with the optimality condition (A.14). Q.E.D.

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