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DOES INCOMPLETE INFORMATION REDUCE MANIPULABILITY?

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We consider the problem of individual manipulation under incomplete information, i.e. the whole preference profile is not known to voters. Instead, voters know the result of an opinion poll (the outcome of a poll information function π, e.g. a list of scores or a set of winners). In this case, a voter has an incentive to misrepresent his preferences (π-manipulate) if he knows that he will not become worse off and there is a chance of becoming better off. We consider six social choice rules and eight types of poll information functions differing in their informativeness. To compare manipulability, first we calculate the probability that there is a voter which has an incentive to π-manipulate and show that this measure is not illustrative in the case of incomplete information. Then we suggest considering two other measures: the probability of a successful manipulation and an aggregate stimulus of voters to manipulate which demonstrate more intuitive behaviour. We provide results of computational experiments and analytical proofs of some of the observed effects.

JEL Classification: C6, D7.

Keywords: voting theory; manipulation; manipulability index; opinion poll; incomplete information.

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1 Introduction

When a society needs to make a collective decision, one should apply a procedure aggregating individual preferences into a social choice. There are many aggregation procedures, but none are ideal, since all of them have some undesirable properties. One of these properties is the vulnerability of an aggregation rule to manipulation by voters. We say that a voter has an incentive to manipulate if he can achieve a better voting result by misrepresenting his preferences. Of course, it is better when all voters want to declare their sincere preferences, otherwise, a collective choice would be biased and, consequently, would not reflect the preferences of a society. Unfortunately, all social choice rules which have at least three possible outcomes are either manipulable or dictatorial [Gibbard, 1973], [Satterthwaite, 1975], [Gärdenfors, 1976]. This result is called Gibbard-Satterthwaite theorem.

In this regard, it is of interest to compare social choice rules in their vulnerability to manipulation. The most intuitive approach is to calculate the probability of manipulation for different rules and choose a rule which minimizes this probability. This probability of manipulation is also called Nitzan-Kelly's index, since it was first used in [Nitzan, 1985] and [Kelly, 1988]. There are a number of studies investigating social choice rules from this perspective. [Kelly, 1988] suggests considering the minimal number of manipulable preference profiles for a social choice rule satisfying some predefined properties. This research direction is continued in [Fristrup and Keiding, 1998], and a series of studies [Maus et al., 2007a, 2007b, 2007c, 2007d].

In [Kelly, 1993] the manipulability of the Borda rule is compared with the manipulability of different classes of rules satisfying some predefined properties. An extended statistical investigation of individual manipulability of social choice rules using Monte-Carlo experiments was done in [Aleskerov and Kurbanov, 1999] and continued in [Aleskerov et al., 2009, 2011, 2012]. The manipulability of approval rule and a family of k-approval rules was studied in [Peters at al., 2012] both theoretically and using simulations. The same probabilistic approach was applied to studying coalitional manipulability [Lepelley and Valognes, 2003], [Pritchard and Wilson, 2007], [Slinko, 2006].

In all these articles it is assumed that voters know each other’s sincere preferences, i.e. public information is reliable and complete. This is a rather strong assumption, but helps to simplify the comparative analysis of manipulability of social choice rules. Intuitively, incomplete information would make manipulation more difficult and rarer.

A more realistic assumption is that voters know some information from opinion polls held before voting. This information could be represented, for example, by preferences of a subset of voters, or a list of candidate scores, or the winner of the election. A mathematical model for the manipulation under poll information is presented in [Reijngoud and Endriss, 2012]. The authors show which rules and which types of public information make manipulation possible or not and study voter response to the repeated poll information.

Using this model, we investigate to what extent social choice rules are susceptible to manipulation, calculating the share of preference profiles with at least one voter having an incentive to manipulate. One of our results is that social choice rules are manipulated almost everywhere in this model if we consider manipulation with information about the winner of the election. Having this, we propose two other indices of manipulability under incomplete information. The first is the proportion of preference profiles where at least one voter has an incentive to manipulate and this manipulation is successful. The second takes into account the stimulus level of each manipulating voter, which is calculated as the probability of success. We show that these measures are more representative than the first one when studying manipulability under incomplete information.

2 The model

2.1 Definitions and notations

Let N denote a finite set of voters, $N = \{1, 2, ..., n\}$, and X be a finite set of m alternatives. The preferences of voter i are represented by a linear order P_i on X, an element of $L(X)$—the set of all linear orders. We write aP_ib if an alternative a is preferred to an alternative b for voter i. An upper contour set of an alternative a in a preference order P_i is $P_ia = \{b \in X : bP_ia\}$. Similarly, a lower contour set of a in P_i is $aP_i = \{b \in X : aP_ib\}$. A preference profile $\vec{P} = (P_1, ..., P_n) \in L(X)^N$ is an ordered set of individual preferences. A contraction of a preference profile onto the set $A \subseteq X$ is $\vec{P} / A = (P_1 / A, ..., P_n / A)$, where $P_i / A = P_i \cap (A \times A)$. A vector of positions for alternative a is $v(a, \vec{P}) = (v_1(a, \vec{P}), ..., v_m(a, \vec{P}))$, where $v_j(a, \vec{P})$ denotes the number of voters having a on the j-th position in preferences. A matrix of a weighted majority graph for a profile $\vec{P}$ is denoted by $WMG(\vec{P})$ and consists of elements

$$WMG(\vec{P})_{kl} = |\{i \in N : a_k P_i a_l\}|.$$

A matrix of a majority graph is $MG(\vec{P})$, where

$$MG(\vec{P})_{kl} = \begin{cases} 1, \text{ if } a_k P_M a_l, \\ -1, \text{ if } a_l P_M a_k. \\ 0, \text{ otherwise.} \end{cases}$$

By P_M we denote majority relation: $a_k P_M a_l$ if $WMG(\vec{P})_{kl} > WMG(\vec{P})_{lk}$.

A *social welfare function* (SWF) is the mapping $F : L(X)^N \to W(X)$, where $W(X)$ is the set of all weak orders on X. SWF aggregates individual preferences represented by a preference profile into a social ordering $F(\vec{P}) = R = P \cup I$, where P is a preference relation and I is an indifference relation. A *social choice rule* (SCR) $C_F : L(X)^N \to 2^X \setminus \varnothing$ chooses alternatives from the top of a social ordering, i.e. $C_F(\vec{P}) = \{a \in X : \neg\exists b \text{ s.t. } bPa\}$.

Many social choice rules are not resolute, i.e. their result could consist of more than one alternative. One way to deal with this is to break ties according to some rule, $T : 2^X \setminus \varnothing \to X$. We consider an alphabetic tie-breaking rule: assume some linear order on X to be predefined, $aP_T bP_T c\ldots$, and when alternatives are tied, we choose the one which dominates all others by P_T (has a higher priority). In this case we sacrifice neutrality to obtain resoluteness.

Another way is to extend voter i's preferences in such a way that he can compare all subsets of alternatives. There is a number of possible ways to do this [Karabekyan, 2009], but we restrict our attention to lexicographic preference extension methods (PEMs), *Leximin* and *Leximax* [Pattanaik, 1978]. We denote the extended preference relation of voter i by EP_i (EI_i—extended indifference relation) and say that $A\, EP_i\, B$ means “$A \in 2^X \setminus \varnothing$ is preferred to $B \in 2^X \setminus \varnothing$ by individual i”.

Leximin. Let us enumerate the elements in A and B in ascending order of preferability for individual i. That is, $A = \{a_1, ..., a_{|A|}\}$, $\forall j \in \{1, ..., |A|-1\}$, $a_j P_i a_{j+1}$ and $B = \{b_1, ..., b_{|A|}\}$, $\forall j \in \{1, ..., |B|-1\}$ $b_j P_i b_{j+1}$. Then we compare sets A and B as follows:

1. If $\exists h \in \{1, ..., \min(|A|, |B|)\}$, s.t. $\forall j \in \{1, ..., h-1\}$ $a_j = b_j$ and $a_h P_i b_h$, then $A\, EP_i\, B$.

2. If $B \subset A$, then $A\, EP_i\, B$.

Leximax. Now enumerate the elements in both subsets in descending order of preferability for individual i, $A=\{a_1,...,a_{|A|}\}$, $\forall j \in \{1,...,|A|-1\}$ $a_j P_i a_{j+1}$ and $B=\{b_1,...,b_{|A|}\}$, $\forall j \in \{1,...,|B|-1\}$ $b_j P_i b_{j+1}$. Similarly,

1. If $\exists h \in \{1,...,\min(|A|,|B|)\}$, s.t. $\forall j \in \{1,...,h-1\}$ $a_j = b_j$ and $a_h P_i b_h$, then $A\ EP_i\ B$.

2. If $A \subset B$, then $A\ EP_i\ B$.

In other words, with the Leximin method we assume that it is important for a voter to avoid worse alternatives, while under Leximax, to seek better alternatives.

The alphabetic tie-breaking rule may also be interpreted as an extended preference relation. Let $a_1 \in A$ be s.t. $\forall a_j \in A \setminus \{a_1\}$ $a_1 P_T a_j$ and $b_1 \in B$ be s.t. $\forall b_j \in B \setminus \{b_1\}$ $b_1 P_T b_j$. If $a_1 P_i b_1$, then $A\ EP_i\ B$, if $a_1 = b_1$ then $A\ EI_i\ B$.

2.2 Poll Information Functions and Manipulation

We use the model for poll information functions introduced in [Endriss and Reijngoud, 2012]. Assume that before voting, an opinion poll is carried out which reveals voters' sincere preferences, $\vec{P}$. Thus, profile $\vec{P}$ represents complete and exact information about preferences of all voters. However, for some reasons not all the information is available to voters. A poll information function (PIF) $\pi(\vec{P})$ puts into correspondence to a preference profile any piece of information about this profile. We consider the following types of PIF.

1. Profile: $\pi(\vec{P}) = \vec{P}$.

2. Anonymous profile (ballot): $\pi(\vec{P}) = \vec{p}(\vec{P}) = (n_1,...,n_{m!})$, where n_h is the number of ballots of h-th type in $\vec{P}$.

3. Position: $\pi(\vec{P}) = \vec{v}(\vec{P}) = (v(a_1,\vec{P}),...,v(a_m,\vec{P}))$ returns a vector of positions for each alternative in X.

4. Score: $\pi(\vec{P}) = \vec{S}(\vec{P}) = (S(a_1,\vec{P}),...,S(a_m,\vec{P}))$ assigns to each alternative its score according to a given SWF F. It may be defined specifically or not defined for rules not using any scoring function.

5. Rank: $\pi(\vec{P}) = F(\vec{P})$ returns a social ordering.

6. Winner: $\pi(\vec{P}) = C_F(\vec{P})$.

7. Unique winner (1Winner): $\pi(\vec{P}) = T(C_F(\vec{P}))$.

8. Weighted majority graph (WMG): $\pi(\vec{P}) = WMG(\vec{P})$.

9. Majority graph (MG): $\pi(\vec{P}) = MG(\vec{P})$.

Having information about a preference profile $\pi(\vec{P})$ and knowing his own preference order, voter i now has a set of profiles consistent with his knowledge. This set is called *information set* and defined as follows

$$W_i^{\pi(\vec{P})} = \{\vec{P}'_{-i} \in L(X)^{N\backslash\{i\}} : \pi(P_i, \vec{P}'_{-i}) = \pi(\vec{P})\}.$$

Given two PIFs π and π', if $\forall \vec{P} \in L(X)^N \quad \forall i \in N \; W_i^{\pi(\vec{P})} \subseteq W_i^{\pi'(\vec{P})}$, then π is *at least as informative as* π'. Of course, the most informative is Profile-PIF.

Then, when is a voter willing to manipulate, i.e. misrepresent his preferences in order to achieve a more preferable result? It is assumed that if there is at least one possible situation, when manipulation makes him better off and nothing changes in all other possible situations, then a voter has a dominating strategy and, thus, an incentive to manipulate under PIF π [Reijngoud and Endriss, 2012].

Definition 1. Given a SWF F and a preference profile $\vec{P}$, we say, that voter i *has an incentive to* π*-manipulate*, if there exists $\tilde{P}_i \in L(X)$ s.t.

i) $\forall \vec{P}'_{-i} \in W_i^{\pi(\vec{P})} \; C_F(\tilde{P}_i, \vec{P}'_{-i}) EP_i C_F(\vec{P})$ or $C_F(\tilde{P}_i, \vec{P}'_{-i}) EI_i C_F(\vec{P})$;

ii) $\exists \vec{P}'_{-i} \in W_i^{\pi(\vec{P})}$, s.t. $C_F(\tilde{P}_i, \vec{P}'_{-i}) EP_i C_F(\vec{P})$.

Definition 2. A SWF F (together with a SCR C_F) is called *susceptible to* π*-manipulation* if there is a profile $\vec{P} \in L(X)^N$ and a voter $i \in N$ who has an incentive to π-manipulate in $\vec{P}$.

Conditions for the susceptibility of social choice rules with alphabetic tie-breaking to π-manipulation were found in [Reijngoud and Endriss, 2012] (a generalization of the Gibbard-Satterthwaite theorem). The aim of this paper is to reveal the degree of manipulability of SCRs under different PIFs. The first measure considered is the simple probability of manipulation, i.e. how often at least one voter has an incentive to misrepresent his preferences. The sample space consists of all preference profiles, that is, they are assumed to appear equally likely (the Impartial Culture assumption).

Denote $I_1(m,n,\pi,F)$ as the probability that in a preference profile, randomly chosen from $L(X)^N$ there exists at least one voter who has an incentive to π-manipulate under SWF F.

2.3 Social Welfare Functions

In this section we introduce formal definitions for SWFs in the study. We need to specify what to consider as “scores” (Score-PIF) and how a social ordering $R = P \cup I$ is defined.

- *Scoring rules*. A scoring rule is defined by a scoring vector $s = (s_1,...,s_m)$, where s_j denotes the number of scores assigned to alternative for the j-th position in individual preferences. The total number of scores for each alternative $a_j \in X$ is calculated as a scalar product $S(a_j,\vec{P}) = \langle s, v(a_j,\vec{P})\rangle$.

Then, $R = P \cup I$ is defined as follows: $\forall a_k, a_l \in X$ [i] $a_k P a_l \leftrightarrow S(a_k,\vec{P}) > S(a_l,\vec{P})$; [ii] $a_k I a_l \leftrightarrow S(a_k,\vec{P}) = S(a_l,\vec{P})$.

- *Plurality*: $s_{Pl} = (1,0,...,0)$.
- *Veto* (Antiplurality): $s_V = (1,...,1,0)$.
- *Borda*: $s_B = (m-1,m-2,...,1,0)$.

- *Run-off procedure*. It has two stages:

[1] The plurality score is calculated for each alternative. A first-stage vector of scores is

$S^1 = (S^1(a_1),...,S^1(a_m))$,

where $S^1(a_j) = \langle s_{Pl}, v(a_j,\vec{P})\rangle$. If $\exists a_k \in X$ s.t. $S^1(a_k) > n/2$, then a social ordering is $a_j I a_h$ $\forall a_j, a_h \in X \setminus \{a_k\}$ and the procedure terminates. Otherwise, procedure moves on to the stage [2].

[2] Two alternatives with maximal number of scores are chosen;

$a_k = \arg\max_{a_j \in X}(S^1(a_j))$, $a_l = \arg\max_{a_j \in X \setminus \{a_k\}}(S^1(a_j))$.

If there are ties, they are broken according to the alphabetical tie-breaking rule T. Then a second-stage vector of scores is calculated:

$$S^2 = (S^2(a_k), S^2(a_l)),$$

Where

$$S^2(a_k) = \left\langle (1,0), v(a_k, \vec{P}/\{a_k, a_l\}) \right\rangle, \; S^2(a_l) = \left\langle (1,0), v(a_l, \vec{P}/\{a_k, a_l\}) \right\rangle.$$

The alternative with the higher score is considered better, $a_k P a_l$ if $S^2(a_k) > S^2(a_l)$ and $a_l P a_k$ if $S^2(a_l) > S^2(a_k)$. Both of them are better than all other alternatives, $\forall a_j \in X \setminus \{a_k, a_l\} \;\; a_k P a_j$, $a_l P a_j$. All other alternatives are considered as indifferent, $\forall a_j, a_h \in X \setminus \{a_k, a_l\} \;\; a_j I a_h$.

The output of Score-PIF is $S = (S^1, S^2)$.

- *Single Transferable vote* (*STV*). This is a multi-stage procedure, which we define in an iterative form.

[0] $t := 1$, $X^t := X$, $\vec{P}^t = \vec{P}$;

[1] $\forall a_j \in X^t \quad S^t(a_j) := \left\langle s_{Pl}, v(a_j, \vec{P}) \right\rangle$;

[2] If $\exists a_j \in X^t$ s.t. $S^t(a_j) > n/2$, then $\forall a_h, a_l \in X^t \setminus \{a_j\} \quad a_j P a_h$, $a_h I a_l$, the procedure terminates. $a_k = \arg\min_{a_l \in X^t}(S^t(a_l))$, $\forall a_j \in X^t \setminus \{a_k\} \;\; a_j P a_k$.

[3] $t := t+1$, $X^t := X^t \setminus \{a_k\}$, $\vec{P}^t = \vec{P} / X^t$. Go to step 1.

The output of Score-PIF is $S = (S^1, ..., S^{t*})$, where $t*$ is the number of cycles done by procedure.

- *Copeland*. A majority graph is computed. Then scores of alternatives are computed as follows

$$S(a_k, \vec{P}) = \sum_{l=1}^{m} MG(\vec{P})_{kl}.$$

Ranking $R = P \cup I$ is defined as usual: $\forall a_k, a_l \in X$ [i] $a_k P a_l \leftrightarrow S(a_k, \vec{P}) > S(a_l, \vec{P})$; [ii] $a_k I a_l \leftrightarrow S(a_k, \vec{P}) = S(a_l, \vec{P})$.

3 Computability and strong computability

A PIF gives a voter some information about a preference profile and sometimes uses a SWF to obtain it. Some of these PIFs provide all the necessary information to calculate the result of the procedure. They are, of course, those PIFs, which make use of SWF, profile-PIF, and Anonymous Profile-PIF (for anonymous SWF). A SCR is *computable from* π*-images* if there is a function $H : I \to 2^X \setminus \varnothing$, s.t. $C_F = H \circ \pi$. If the information provided by π is sufficient for voter i to compute the result of C_F for every possible preference $\tilde{P}_i \in L(X)$, then C_F is called *strongly computable from* π*-images*.

Obviously, C_F is computable from π-images iff π-PIF is at least as informative as Winner-PIF (for any two preference profiles $\forall \vec{P}, \vec{P}' \in W_i^{\pi(\vec{P})}$ the winner is the same $C_F(\vec{P}) = C_F(\vec{P}')$). Another useful observation: if π is at least as informative as π' and F is strongly computable from π'-images, then F is also strongly computable from π'-images. Now we study computability and the binary relation "being at least as informative as" for six SWFs from the previous section and illustrate them with directed graphs.

For all scoring rule $\forall \vec{P} \in L(X)^N$, $\forall i \in N$,

$$W_i^{\vec{P}} \subseteq W_i^{\vec{p}(\vec{P})} \subseteq W_i^{\vec{v}(\vec{P})} \subseteq W_i^{\vec{S}(\vec{P})} \subseteq W_i^{F(\vec{P})} \subseteq W_i^{C_F(\vec{P})} \subseteq W_i^{T(C_F(\vec{P}))}.$$

For the Runoff procedure and STV rule, Positions-PIF is dropped from this chain, since information provided by $\vec{v}(\vec{P})$ is not enough to compute the winner. Consequently, scoring rules are computable from π-images for all PIFs of this chain. Obviously, $W_i^{\vec{P}} \subseteq W_i^{WMG(\vec{P})} \subseteq W_i^{MG(\vec{P})}$ is true for any rule. However, all the considered positional social choice rules are not computable from MG-PIF (see Fig.1–3).

For the Borda rule WMG-PIF contains all the necessary information to compute the winner and is at least as informative as Score-PIF, $W_i^{WMG(\vec{P})} \subseteq W_i^{\vec{S}(\vec{P})}$. Let us prove it.

$$S_B(a_j, \vec{P}) = v_1(a_j, \vec{P}) \cdot (m-1) + \ldots + v_{m-1}(a_j, \vec{P}) \cdot 1 =$$

$$= \left(\sum_{\substack{i \in N: \\ |a_j P_i| = m-1}} 1 \right) \cdot (m-1) + \ldots + \left(\sum_{\substack{i \in N: \\ |a_j P_i| = 1}} 1 \right) \cdot 1 = \sum_{k=1}^{m-1} \sum_{\substack{i \in N: \\ |a_j P_i| = k}} k = \sum_{i \in N} |a_j P_i| =$$

$$=\sum_{i\in N}\sum_{\substack{a_l\in A\\ a_jP_ia_l}}1=\sum_{a_l\in A}\sum_{\substack{i\in N\\ a_jP_ia_l}}1=\sum_{a_l\in A}|\{i\in N: a_jP_ia_l\}|=\sum_{a_l\in A}WMG_{jl}(\vec{P})$$

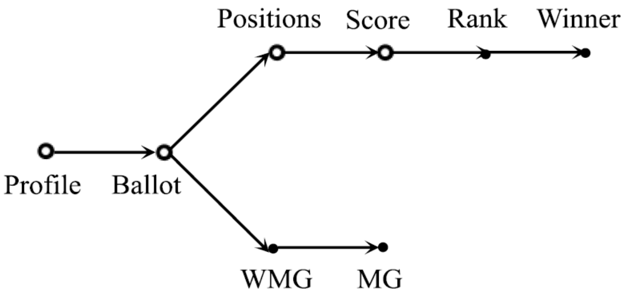


Fig. 1. Informativeness of PIFs for plurality and veto

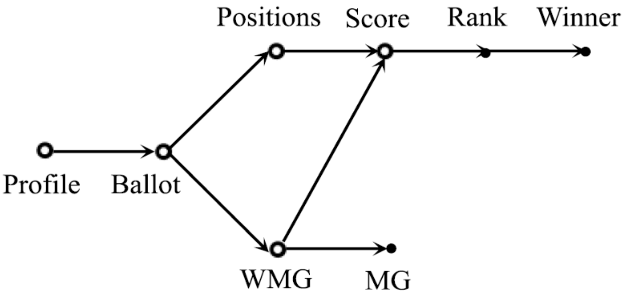


Fig. 2. Informativeness of PIFs for Borda rule

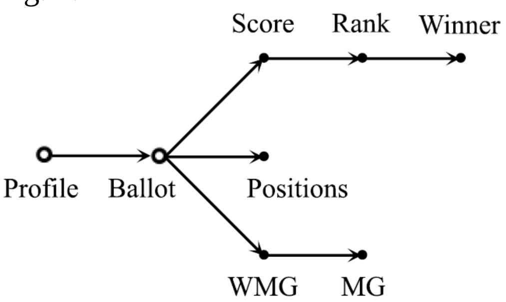


Fig. 3. Informativeness of PIFs for STV and Runoff

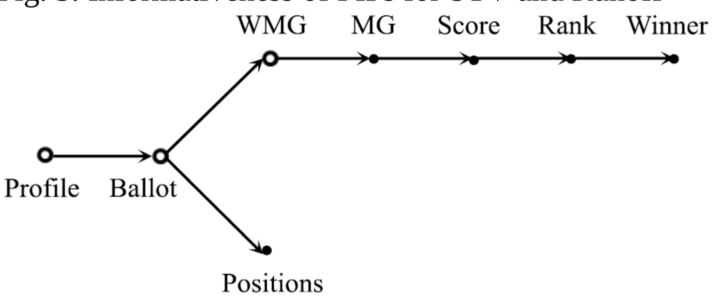


Fig. 4. Informativeness of PIFs for Copeland rule

For the Copeland rule, scores are computed from a majority graph. Thus, we have the following chain

$$W_i^{\vec{P}} \subseteq W_i^{\vec{p}(\vec{P})} \subseteq W_i^{WMG(\vec{P})} \subseteq W_i^{MG(\vec{P})} \subseteq W_i^{\vec{S}(\vec{P})} \subseteq W_i^{F(\vec{P})} \subseteq W_i^{C_F(\vec{P})} \subseteq W_i^{T(C_F(\vec{P}))}.$$

However, a majority graph does not provide sufficient information for a voter to compute the winner for any way of voting, while a weighted majority graph does.

Further we make use of strong computability. The PIFs, for which SWFs are strongly computable from π-images, are denoted by circles on Fig. 1–4. The Runoff procedure and STV are strongly computable from π-images only for Profile-PIF and Ballot-PIF. For both rules, if alternatives eliminated at the first stage in $(\tilde{P}_i, \vec{P}_{-i})$ are different from those eliminated in $\vec{P}$, then it is impossible to compute a second-stage scoring vector for $(\tilde{P}_i, \vec{P}_{-i})$, and thus, to compute the winner.

Scoring rules are strongly computable from Score-images. Let us explain this by the short argument. Suppose voter i receives information about the score of each alternative, $\vec{S}(\vec{P}) = (S(a_1, \vec{P}), ..., S(a_m, \vec{P}))$. If voter i's preferences are $a_1 P_i a_2 P_i a_3 ... a_{m-1} P_i a_m$, then alternative a_j gets s_j scores from voter i. Then voter i changes his preferences to P_i',

$$a_{\sigma(1)} P_i' a_{\sigma(2)} P_i' a_{\sigma(3)} ... a_{\sigma(m-1)} P_i' a_{\sigma(m)}.$$

Since voters know the rule, then the new vector of scores is also known to voter i. For each alternative $a_j \in X$ the new number of scores is $S'(a_j, \vec{P}) - s_j + s_{\sigma^{-1}(j)}$. Thus, voter i can compute the winner for any $P_i' \in L(X)$.

4 Manipulability

4.1 Theoretical results

In this Section we answer the question of how often there exists a voter which has an incentive to misrepresent his preferences. First, we provide some theoretical results and then compare the manipulability of rules using different PIFs by conducting computational experiments.

It is of interest to know how incomplete information influences manipulability: does it reduce the probability of manipulation compared to complete information case? The first result shows under which PIFs the degree of manipulability does not differ from manipulability under Profile-PIF.

Theorem 1. If SCR C_F is strongly computable from π-images, then $I_1(m,n,\pi,F)=$ $=I_1(m,n,Profile,F)$.

Proof. The truthful vote of voter i is P_i. Voter i receives information $\pi(\vec{P})$. Since C_F is strongly computable from π-images, for all untruthful votes $\tilde{P}_i \in (L(X)\setminus\{P_i\})$ for every two preference profiles $\vec{P}'_{-i},\vec{P}''_{-i} \in W_i^{\pi(\vec{P})}$, $C_F(\tilde{P}_i,\vec{P}'_{-i}) = C_F(\tilde{P}_i,\vec{P}''_{-i})$ and the result $C_F(\tilde{P}_i,\vec{P}'_{-i})$ is known for voter i. Thus, given an untruthful vote $\tilde{P}_i \in L(X)\setminus\{P_i\}$, either $C_F(\tilde{P}_i,\vec{P}'_{-i})EP_iC_F(\vec{P})$ is true for all $\vec{P}'_{-i} \in W_i^{\pi(\vec{P})}$, or it is not true also for all $\vec{P}'_{-i} \in W_i^{\pi(\vec{P})}$.

If $\exists \tilde{P} \in L(X)\setminus\{P_i\}$ s.t. $\forall \vec{P}'_{-i} \in W_i^{\pi(\vec{P})}$, $C_F(\tilde{P}_i,\vec{P}'_{-i})EP_iC_F(\vec{P}) = C_F(P_i,\vec{P}'_{-i})$, then in all preference profiles $\vec{P}$ where there is a voter who has an incentive to π-manipulate, there is also a voter (the same one) who has an incentive to Profile-manipulate, and vice versa. And other direction, if a voter has an incentive to Profile-manipulate in $\vec{P}$, then it has an incentive to π-manipulate. Consequently, the number of manipulated preference profiles will be the same under π-PIF and Profile-PIF, and $I_1(m,n,\pi,F) = I_1(m,n,Profile,F)$. Q.E.D.

An especially interesting case is the vulnerability of SCF to manipulation under Winner-PIF, the least informative non-zero information function. The following result reveals the asymptotic susceptibility of the most popular social choice rule, the plurality rule to Winner-manipulation.

Theorem 2. $\lim_{n\to\infty} I_1(m,n,Winners,Plurality)=1$ under Leximin and Leximax PEMs. $\lim_{n\to\infty} I_1(m,n,Winner,Plurality)=1$ with alphabetic tie-breaking.

Proof. Let $\pi(\vec{P}) = C_F(\vec{P})$. Suppose, the winner is alternative d, and for voter i alternative d is the least preferable. Try to find a strategy $\tilde{P}_i$ for voter i which is not worse (in terms of the voting result) than his sincere preferences $aP_ibP_i...P_id$.

Since the plurality rule counts only the first alternatives in preferences, we could simply say that sincerely voter i votes for a. Any other preference order with a on the top does not differ from his sincere preference. Moreover, there is no strategy for i to make a win, so, a could not be on the top of $\tilde{P}_i$.

Then, let voter i vote for his second-best alternative, $b\tilde{P}_i...\tilde{P}_id$. $S(d,\vec{P}) > S(b,\vec{P}) = S(b,\vec{P}_{-i})$. $S(b,(\tilde{P}_i,\vec{P}_{-i})) = S(b,\vec{P}_{-i})+1$. $\forall \vec{P}_{-i} \in W_i^{\pi(\vec{P})}$, $S(d,(\tilde{P}_i,\vec{P}_{-i})) \geq S(b,(\tilde{P}_i,\vec{P}_{-i}))$, and $\exists \vec{P}_{-i} \in W_i^{\pi(\vec{P})}$ s.t.

$S(d,(\tilde{P}_i,\vec{P}_{-i}))=S(b,(\tilde{P}_i,\vec{P}_{-i}))$. Consequently, if voter i votes for b, then either $C_F(\tilde{P}_i,\vec{P}'_{-i})=\{d\}$ or $C_F(\tilde{P}_i,\vec{P}'_{-i})=\{b,d\}$. According to Leximin and Leximax, for voter i outcome $\{b,d\}$ is better than $\{d\}$. Thus, voter i has an incentive to Winner-manipulate by voting for his second-best alternative.

If ties are broken alphabetically and π is 1Winner-PIF, then outcome $T(C_F(\vec{P}))=\{d\}$ may be obtained two possible ways. Either $C_F(\vec{P})=\{d\}$, or $|C_F(\vec{P})|>1$ and $\forall c\in C_F(\vec{P})\setminus\{d\}$, dP_Tc. Both cases are possible and voter i cannot distinguish between them. If bP_Td, then voting for b will make b the winner in case of tie $C_F(\tilde{P}_i,\vec{P}'_{-i})=\{b,d\}$. If dP_Tb, then there is a possibility that $b\in C_F(\vec{P})$ and voting for b will make b the winner $C_F(\tilde{P}_i,\vec{P}'_{-i})=\{b\}$. Thus, for a voter having the winning alternative on the last position in preferences voting for his second-best alternative is never worse than voting sincerely for a.

Now consider the set of preference profiles with a single-valued outcome. The proportion of such profiles tends to 1 as n goes to infinity.[3] If ties are broken, then the winner is always unique. Our aim is to find the share of profiles where at least one voter has the winning alternative as their worst preference, because all these profiles will be manipulable.

Let us consider the set of preference profiles with n voters, where alternative a_1 has p_1% of votes from $\hat{n}_1=np_1$ voters, a_2 has p_2% from $\hat{n}_2=np_2$ voters, etc. (thus, $n=\hat{n}_1+...+\hat{n}_m$).

The number of such preference profiles is

$$D=\frac{n!}{\hat{n}_1!\hat{n}_2!...\hat{n}_m!}((m-1)!)^n.$$

The number of preference profiles where alternative a_1 does not take the last position in any preference order is

$$D_1=\frac{n!}{\hat{n}_1!\hat{n}_2!...\hat{n}_m!}((m-1)!)^{\hat{n}_1}((m-1)!-(m-2)!)^{\hat{n}_2}...((m-1)!-(m-2)!)^{\hat{n}_m}.$$

The share of profiles where a_1 cannot take the last position

[3] As shown in [Gehrlein and Fishburn, 1981], the probability of a tie between any pair of alternatives with plurality rule tends to 0 as the number of voters goes to infinity (by Central Limit Theorem). As a consequence, the probability of any tie goes to zero and the probability of a single-valued outcome tends to 1.

$$\frac{D_1}{D}=\frac{((m-1)!)^{\hat{n}_1}((m-1)!-(m-2)!)^{\hat{n}_2}...((m-1)!-(m-2)!)^{\hat{n}_m}}{((m-1)!)^{\hat{n}_1}((m-1)!)^{\hat{n}_2}...((m-1)!)^{\hat{n}_m}}=$$

$$=\left(1-\frac{1}{m-1}\right)^{\hat{n}_2}...\left(1-\frac{1}{m-1}\right)^{\hat{n}_m}=\left(1-\frac{1}{m-1}\right)^{1-\hat{n}_1}.$$

Finally,

$$\lim_{n\to\infty}\left(1-\left(1-\frac{1}{m-1}\right)^{1-\hat{n}_1}\right)=\lim_{n\to\infty}\left(1-\left(1-\frac{1}{m-1}\right)^{n(1-p_1)}\right)=1.$$

Thus, in a set of preference profiles with a given distribution of votes $(p_1,...,p_m)$ the share of profiles with at least one voter having a particular alternative as their worst preference tends to 1 as n goes to infinity. Consequently, in the union of profile sets for different vote distributions (the set of profiles with a unique winner), this proportion also tends to 1 as n goes to infinity. Therefore, $\lim_{n\to\infty} I_1(m,n,Winners,Plurality)=1$ under Leximin and Leximax PEMs and $\lim_{n\to\infty} I_1(m,n,Winner,Plurality)=1$ with alphabetic tie-breaking. Q.E.D.

Theorem 2 shows that less information does not always lead to less manipulability and sometimes manipulation almost always occurs. Moreover, there is a case when manipulation takes place in 100% of preference profiles.

Theorem 3. Under Leximin PEM, $I_1(3,3,MG,Borda)=1$.

Proof. Since $n=3$ and $m=3$, there are two possible structures of majority graphs. [1] $\exists\sigma\in Sym(n)$ (a permutation of the symmetric group on N), s.t. $MG(\vec{P})_{\sigma(1),\sigma(2)}=1$, $MG(\vec{P})_{\sigma(2),\sigma(3)}=1$, $MG(\vec{P})_{\sigma(1),\sigma(3)}=1$ (the majority relation is transitive). [2] $\exists\sigma\in Sym(n)$, s.t. $MG(\vec{P})_{\sigma(1),\sigma(2)}=1$, $MG(\vec{P})_{\sigma(2),\sigma(3)}=1$, $MG(\vec{P})_{\sigma(3),\sigma(1)}=1$ (a cyclic majority relation).

Let us consider the first case. Without loss of generality, assume that $\sigma(1)=1$, $\sigma(2)=2$, $\sigma(3)=3$. In any preference profile corresponding to this majority graph, $\vec{P}$, $|\{i\in N: a_1P_ia_2\}|\geq 2$, suppose, those are voters i_1 and i_2. At the same time, $|\{i\in N: a_2P_ia_3\}|\geq 2$, and these could be voters i_1, i_3, or i_2, i_3, or i_1, i_2. Thus, in any case, there is at least one voter, i, with preferences $a_1P_ia_2$, $a_2P_ia_3$. By transitivity, $a_1P_ia_3$. For this voter, all preference profiles $\vec{P}_{-i}\in L(X)^{N\setminus\{i\}}$, s.t. $MG(P_i,\vec{P}_{-i})$ is of type [1] constitute voter i's information set $W_i^{MG(\vec{P})}$.

For this majority graph, a weighted majority graph satisfies the following inequalities:

$3 \geq WMG_{1,2}(\vec{P}) \geq 2$, $3 \geq WMG_{1,3}(\vec{P}) \geq 2$, $1 \geq WMG_{2,1}(\vec{P}) \geq 0$,

$3 \geq WMG_{2,3}(\vec{P}) \geq 2$, $1 \geq WMG_{3,1}(\vec{P}) \geq 0$, $1 \geq WMG_{3,2}(\vec{P}) \geq 0$.

Since $S_B(a_j, \vec{P}) = \sum_{a_l \in A} WMG_{j,l}(\vec{P})$, then the Borda score of a_1 is a value from $\{4,5,6\}$, for a_2 - $\{2,3,4\}$, and for a_3 - $\{0,1,2\}$. Consequently, in all preference profiles of $W_i^{MG(\vec{P})}$ the outcome of the Borda rule is either $\{a_1\}$, or $\{a_1, a_2\}$. If voter i changes his preferences to P_i', s.t. $a_1 P_i' a_3 P_i' a_2$, then a_2 loses the chance to win, but a_3 still does not have enough scores to compete with a_1. Thus, in all preference profiles with the majority graph [1] there is at least one voter, for whom voting insincerely is more preferable.

There could also be two types of cyclic majority graph: $MG(\vec{P})_{1,2} = 1$, $MG(\vec{P})_{2,3} = 1$, $MG(\vec{P})_{3,1} = 1$, or $MG(\vec{P})_{1,3} = 1$, $MG(\vec{P})_{3,2} = 1$, $MG(\vec{P})_{2,1} = 1$. In all 12 profiles producing a cyclic majority relation the outcome of the Borda rule is $\{a_1, a_2, a_3\}$, and every voter i having preferences $a_{j_1} P_i a_{j_2} P_i a_{j_3}$ will benefit from misrepresenting preferences $a_{j_2} P_i a_{j_1} P_i a_{j_3}$, since the outcome $\{a_{j_2}\}$ is better for i according to Leximin. Q.E.D.

Finally, we provide an immunity result for the Copeland rule.

Theorem 4. Under Leximax PEM, $I_1(3, n, Winner, Copeland) = 0$ for an odd number of voters.

Proof. Since the number of voters is odd, the result of the Copeland rule may consist of either one alternative or all three alternatives $\{a, b, c\}$ (the Condorcet paradox).

If $\pi(\vec{P}) = C_F(\vec{P}) = \{a\}$, then there are two possible majority graphs: [a] $aP_M b$, $aP_M c$, $bP_M c$ and [b] $aP_M b$, $aP_M c$, $cP_M b$.

Obviously, voters having preferences $aPbPc$ and $aPcPb$ would not manipulate.

For a voter having a on the second place in their preferences, $bPaPc$ ($cPaPb$), the only achievable outcome is $\{a, b, c\}$. This outcome can be produced by majority relation [i] $aP_M b$, $cP_M a$, $bP_M c$ or [ii] $bP_M a$, $aP_M c$, $cP_M b$. If it is case a (b), then the majority relation [i] (majority relation [ii]) can be obtained, if the voter tries to manipulate by voting $bPcPa$ ($cPbPa$). He either changes the majority relation to $aP_M b$, $cP_M a$, $bP_M c$ ($bP_M a$, $aP_M c$, $cP_M b$) or does not change anything. But if it is case b (a) then voting $bPcPa$ ($cPbPa$) may lead to $aP_M b$, $cP_M a$, $cP_M b$ ($bP_M a$, $aP_M c$, $bP_M c$) and the winner would be c (b) which is the least preferred alternative for the voter. The majority relation [ii] ([i]) cannot be obtained, since the voter cannot change majority relation from $aP_M b$ to $bP_M a$. Thus, voters with a on the second place in their preferences do not have an incentive to Winner-manipulate.

For voters having a on the last place any other outcome is better than $\{a\}$. But neither of these outcomes is achievable: if a wins despite that voters having preferences $bPcPa$ or $cPbPa$ vote sincerely, then no other vote can change majority relation to $bP_M a$ or $cP_M a$. Therefore, there is no voter having an incentive to π-manipulate if the winner is unique.

Consider the case when the outcome is $\{a,b,c\}$. Suppose, the favourite alternative of a voter is a, and suppose $bP_M a$. For the voter the only outcome which is better than $\{a,b,c\}$ by Leximax is $\{a\}$. However, the voter cannot do anything to change majority relation $bP_M a$.

Consequently, if there are an odd number of alternatives, then the Copeland rule is immune to Winner-manipulation under Leximax PEM. Q.E.D.

4.2 Computational experiments

This section presents the results of computational experiments on the susceptibility of SCFs to π-manipulation. We study the case of multiple choice with Leximin and Leximax PEMs and the case when ties are broken alphabetically. The last case includes consideration of the 1Winner-PIF. Since all the SCFs we consider are anonymous, we do not calculate indices for Ballot-PIF. Moreover, by Theorem 1, we do not calculate I_1 of scoring rules for Profile-PIF and Position-PIF, since they will be the same as for Score-PIF. The same argument works for I_1 of the Copeland rule with WMG-PIF.

In order to check whether a preference profile is susceptible to π-manipulation we need to have all profiles from the relevant information set. This makes a statistical test with randomly chosen preference profiles inconvenient. For this reason we conduct experiments using the whole set of preference profiles, but reduce enumeration largely with the help of IAC culture. Instead of preference profiles, we generate the set of all ballots (anonymous preference profiles), construct a representative preference profile for each ballot and further use the set of representatives, Π.

Obviously, all preference profiles generated from a given ballot are equally susceptible to π-manipulation (all susceptible or none). Consequently, we need to check only representatives. We check each voter in a representative profile. For voter i in profile $\vec{P}$ we are looking for profiles $\vec{P}'$ in Π, s.t. $\pi(\vec{P}) = \pi(\vec{P}')$ and $\exists i'(\vec{P}') \in N$ s.t. $P_i = P'_{i'(\vec{P}')}$. Denote these profiles by $\tilde{W}_i^{\pi(\vec{P})} \subseteq \Pi$. This is not an information set of voter i, because a voter i' might not be i, but profiles of $W_i^{\pi(\vec{P})}$ could be obtained from profiles of $\tilde{W}_i^{\pi(\vec{P})}$ by permuting voters. Then we are looking for a manipulation strategy s.t. $\forall \vec{P}' \in \tilde{W}_i^{\pi(\vec{P})}$ voter $i'(\vec{P}')$ does not make the result worse for himself and there is at least one $\vec{P}' \in \tilde{W}_i^{\pi(\vec{P})}$ where voter $i'(\vec{P}')$ makes the result better by using this strategy. If there is such a strategy, then the voter has an incentive to π-manipulate. We mark this preference profile $\vec{P}$ and all profiles of $\tilde{W}_i^{\pi(\vec{P})}$ as susceptible to π-manipulation and we do not need to check other voters in these profiles. If a representative profile of a ballot $\vec{p} = (p_1, ..., p_{m!})$ is susceptible to π-manipulation, then all other $p_1! p_2! ..., p_{m!}! / n!$ profiles with the same ballot are also susceptible to π-manipulation.

The algorithm was implemented in MatLab, and computational experiments were done for 3 alternatives and the number of voters up to 15. Although this number is not large it turns out to be sufficient for our analysis. Results are illustrated by Figs. 6–23, Fig. 5 is the legend.

The first observation is that manipulability does not decrease when we consider less informative PIFs. Quite the contrary, although there is no clear monotonic dependence between informativeness and I_1, we could easily see that

$$I_1(3, n, Winner, F) \geq I_1(3, n, Rank, F) \geq I_1(3, n, Profile, F)$$

for plurality and Borda, and

$$I_1(3, n, Winner, F) \geq I_1(3, n, Rank, F) \geq I_1(3, n, Score, F) \geq I_1(3, n, Profile, F)$$

for Runoff and STV almost for all n considered (both for Leximin and Leximax). For an alphabetic tie-breaking rule the same inequalities hold, and

$$I_1(3, n, 1Winner, F) \geq I_1(3, n, Winner, F),$$

i.e. maximal manipulability corresponds to the 1Winner-PIF, the least informative PIF which allows us to compute the result. Moreover, I_1 for the Winner-PIF with Leximin and Leximax and 1Winner-PIF with an alphabetic tie-breaking gets closer and closer to 1 as the number of voters grows for plurality (here experiments illustrate the result of Theorem 2), Borda, Runoff and STV. Manipulability is weakly increasing along the chain of PIFs (Fig. 4) for the Copeland rule with Leximin PEM. With Leximax and with alphabetic tie-breaking

$$I_1(3, n, Rank, Copeland) \geq I_1(3, n, Score, Copeland) \geq I_1(3, n, MG, Copeland) \geq$$

$$\geq I_1(3, n, WMG, Copeland) \geq I_1(3, n, Profile, Copeland)$$

for all n from 3 to 15. A curious effect observed is a stable periodicity with high amplitude of I_1 for the Copeland rule. While the peaks of $I_1(3, n, Winner, Copeland)$ and $I_1(3, n, 1Winner, Copeland)$ approach 1, its lowest value is almost constant and minimum over all PIFs. With Leximax, the Copeland rule is immune to Winner-manipulation for an odd number of voters (Theorem 4). The figures for the veto rule look different. With Leximin/Leximax, PEM I_1 for Winner-PIF is lower than Profile-PIF and Rank-PIF. The highest values of I_1 correspond to Rank-PIF. With alphabetic tie-breaking, the veto rule is immune to 1Winner-manipulation, which illustrates the result of Theorem 6 in [Reijngoud and Endriss, 2012] (the corresponding line is not shown on Fig. 20, since it is always zero-valued).

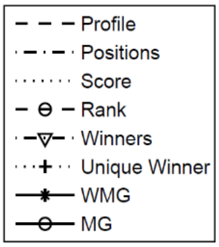


Fig. 5. The legend for Figs. 6-23

Other immunity results are obtained for the cases where SCR is not computable from π-images. Manipulability index I_1 turns to zero for plurality, veto, and STV for MG-PIF in all three series. On the contrary, I_1 for the MG-manipulation is very high for the Borda rule (with a partial case of 100%-manipulation for $m = 3$, $n = 3$ and Leximin PEM).

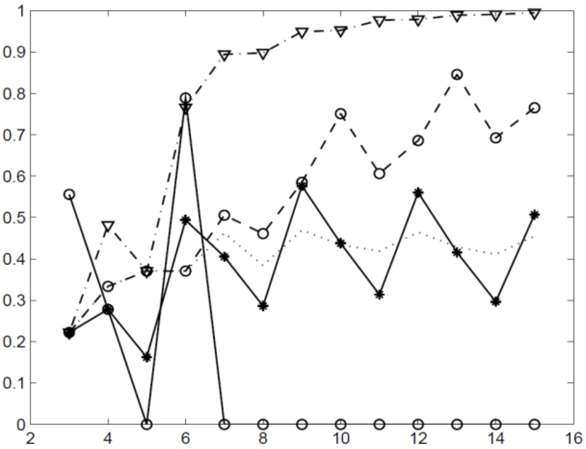


Fig. 6. I_1 for plurality rule, Leximin

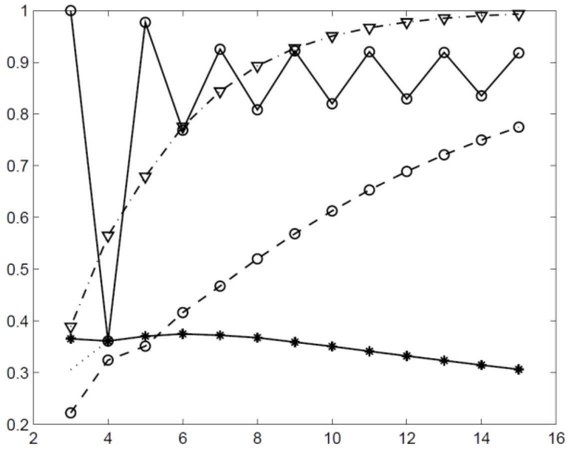


Fig. 7. I_1 for Borda rule, Leximin

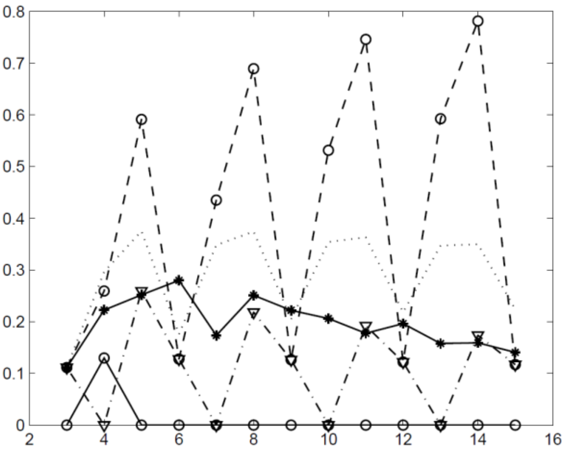


Fig. 8. I_1 for veto rule, Leximin

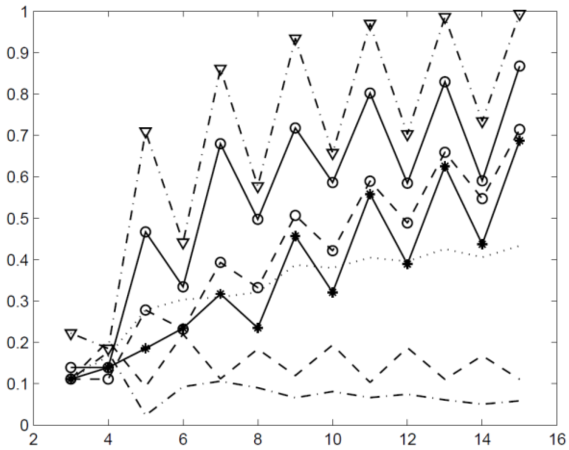


Fig. 9. I_1 for Runoff, Leximin

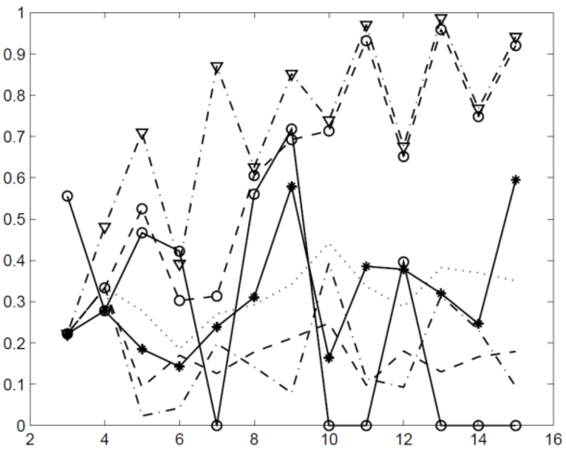


Fig. 10. I_1 for STV, Leximin

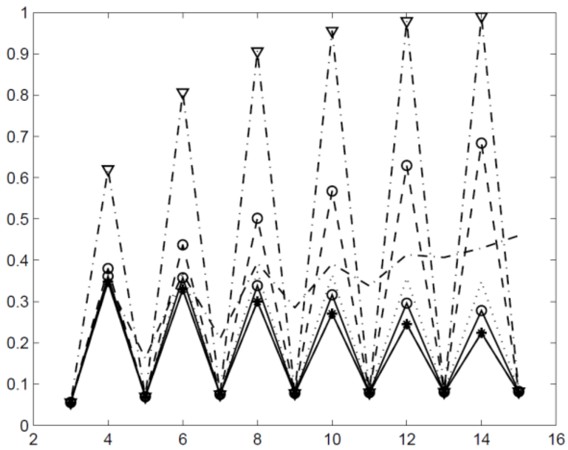


Fig. 11. I_1 for Copeland rule, Leximin

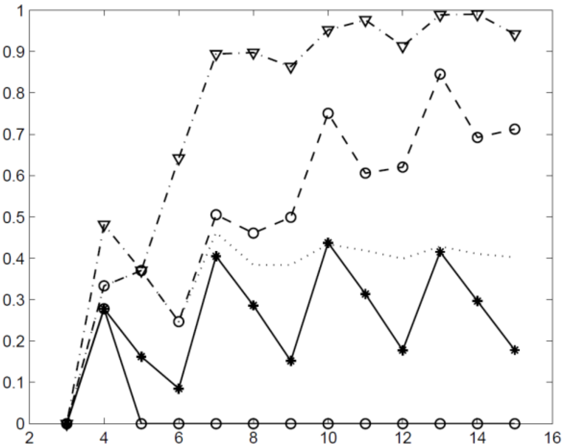


Fig. 12. I_1 for plurality rule, Leximax

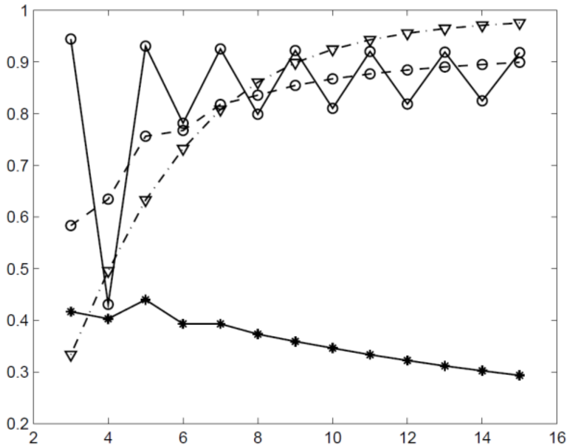


Fig. 13. I_1 for Borda rule, Leximax

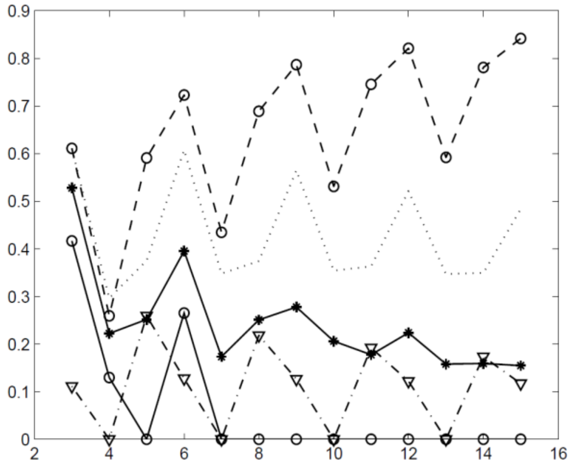


Fig. 14. I_1 for veto rule, Leximax

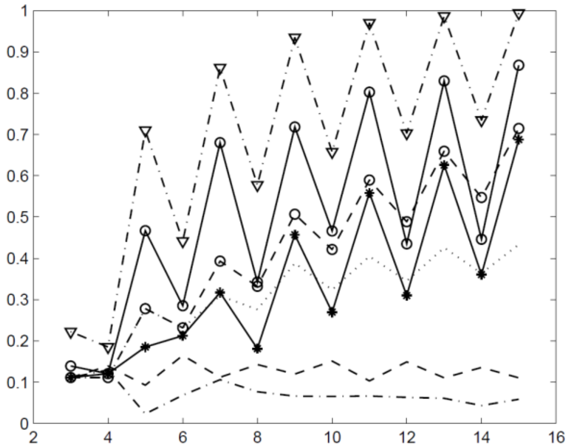


Fig. 15. I_1 for Runoff, Leximax

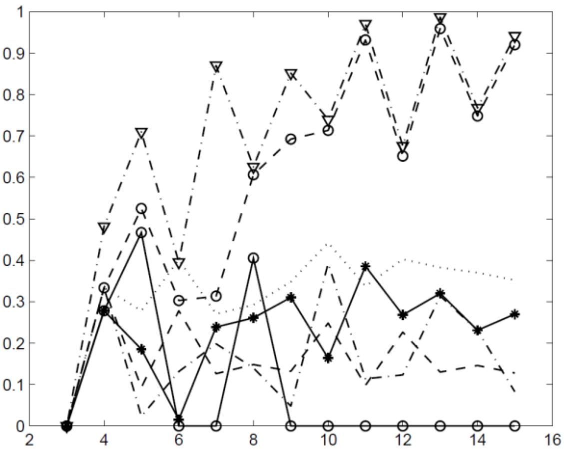


Fig. 16. I_1 for STV, Leximax

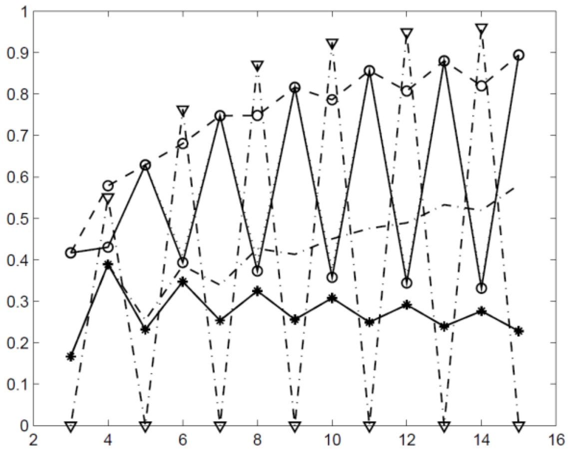
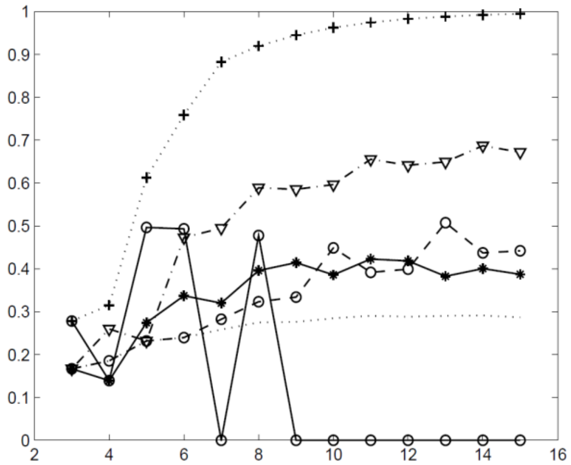


Fig. 17. I_1 for Copeland rule, Leximax



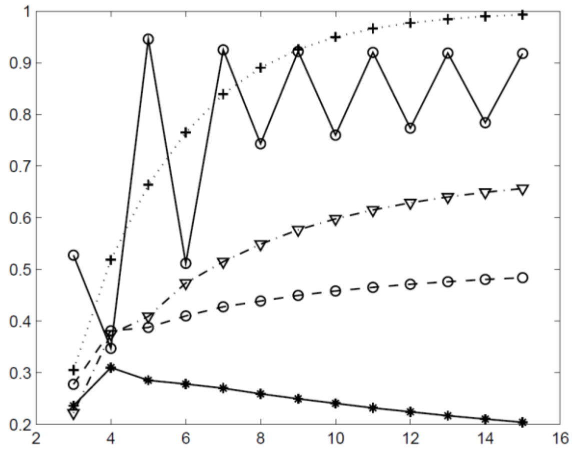
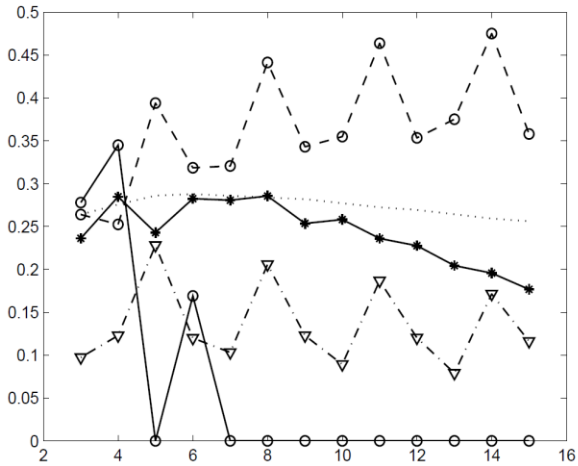


Fig. 18. I_1 for plurality rule, alphabetic tie-breaking

Fig. 19. I_1 for Borda rule, alphabetic tie-breaking



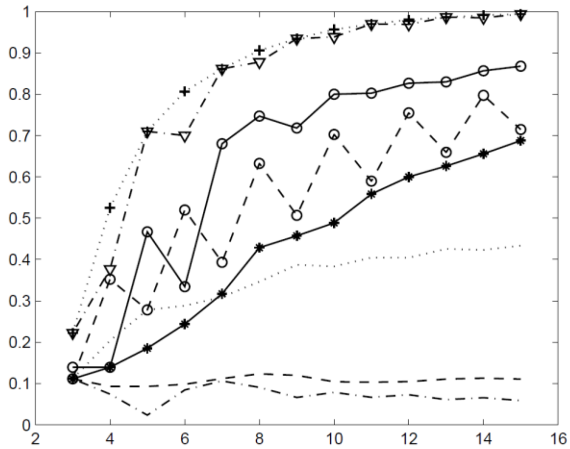
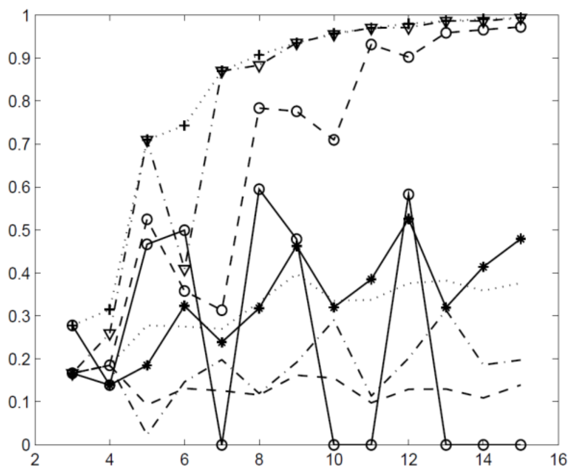


Fig. 20. I_1 for veto rule, alphabetic tie-breaking

Fig. 21. I_1 for Runoff, alphabetic tie-breaking



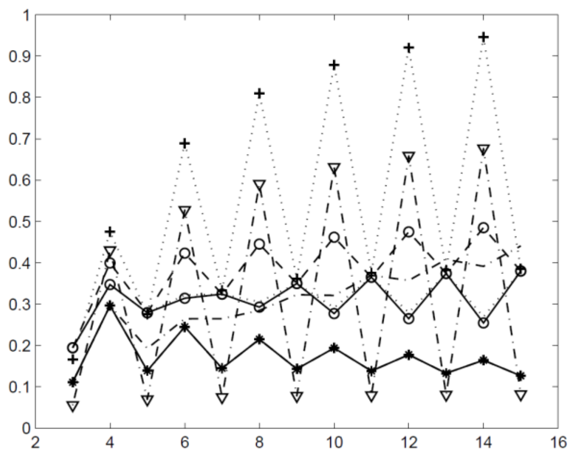


Fig. 22. I_1 for STV, alphabetic tie-breaking

Fig. 23. I_1 for Copeland rule, alphabetic tie-breaking

To sum up, the susceptibility of SCRs to π-manipulation is not sufficient: when a rule is computable from π-images, then it is almost always susceptible to π-manipulation. Then we studied the proportion of preference profiles with at least one voter who has an incentive to π-manipulate. This gave us an interesting result: the manipulability measured this way in most cases increased compared to the complete information case.

5 Manipulation success and stimulus to manipulation

Having the results of the previous Section, we try to extend the analysis of manipulability under incomplete information by introducing two other measures. As we have seen, the probability that at least one voter would deviate could be high enough, but not in all situations this leads to success (achieving the goal of manipulation). Thus, the second index measures the probability that in a preference profile there is a manipulating voter and his manipulation is successful in this preference profile.

Let $I_2(m,n,\pi,F)$ be the probability that in a preference profile randomly chosen from $L(X)^N$ there is at least one voter who has an incentive to π-manipulate under SWF F and his manipulation is successful in this preference profile. Formally,

$$I_2(m,n,\pi,F) = \left|\{\vec{P} \in L(X)^N : \exists i \in N \text{ s.t. } \vec{P}'_{-i} \in WS_i^{\pi(\vec{P})}\}\right| / (m!)^n ,$$

where

$$WS_i^{\pi(\vec{P})} = \{\vec{P}'_{-i} \in W_i^{\pi(\vec{P})} : \exists \tilde{P}_i \text{ s.t. } C_F(\tilde{P}_i, \vec{P}'_{-i}) E P_i C_F(\vec{P})\} .$$

Another observation is that the cardinality of the voter's information set could be quite large, while the share of profiles where manipulation is successful could tend to 0. This explains the high values of I_1 under incomplete information. In order to take this into account we suggest measuring the stimulus to manipulation for each voter. This function would reflect the level of willingness to manipulate, which is assumed to be proportional to the probability of success. Let us denote this by $stimulus(i,\vec{P},\pi(\vec{P}))$ and define as follows

$$stimulus(i,\vec{P},\pi(\vec{P})) = \begin{cases} \dfrac{|WS_i^{\pi(\vec{P})}|}{|W_i^{\pi(\vec{P})}|}, & \text{if voter } i \text{ has an incentive to } \pi\text{-maniulate,} \\ 0, & \text{otherwise.} \end{cases}$$

Instead of a binary measure of manipulability for a preference profile as in I_1, we set a number from the interval $[0,1]$, calculated as the maximum stimulus among all voters having an incentive to π-manipulate in this preference profile. Index I_3 is a summation of such manipulation measures over all preference profiles normalized by $(m!)^n$.

$I_3(m,n,\pi,F)$ is an aggregated voter stimulus to manipulation, calculated as follows

$$I_3(m,n,\pi,F) = \sum_{\vec{P}\in L(X)^N} \max_{i\in N}(stimulus(i,\vec{P},\pi(\vec{P}))) / (m!)^n .$$

By definition, $I_2(m,n,\pi,F) \le I_1(m,n,\pi,F)$ and $I_3(m,n,\pi,F) \le I_1(m,n,\pi,F)$. In the case of complete information, $\pi(\vec{P}) = \vec{P}$, all three indices are the same

$$I_1(m,n,Profile,F) = I_2(m,n,Profile,F) = I_3(m,n,Profile,F) .$$

Theorem 5. If SWR F is strongly computable from π-images, then $I_1(m,n,\pi,F) = I_2(m,n,\pi,F) = I_3(m,n,\pi,F)$.

Proof. Suppose, voter i has an incentive to π-manipulate in $\vec{P}$ by voting $\tilde{P}_i$. Since F is strongly computable from π-images, then $\forall \vec{P}'_{-i}, \vec{P}''_{-i} \in W_i^{\pi(\vec{P})}$ $C_F(\tilde{P}_i, \vec{P}'_{-i}) = C_F(\tilde{P}_i, \vec{P}''_{-i})$ and $C_F(\tilde{P}_i, \vec{P}'_{-i}) EP_i C_F(\vec{P})$.

Consequently, $WS_i^{\pi(\vec{P})} = W_i^{\pi(\vec{P})}$ and $stimulus(i,\vec{P},\pi(\vec{P})) = 1$. Obviously, if there is no $i \in N$, s.t. i has an incentive to π-manipulate, then $\max_{i\in N}(stimulus(i,\vec{P},\pi(\vec{P}))) = 0$. Thus, $I_3(m,n,\pi,F) = I_1(m,n,\pi,F)$ and by Theorem 1, $I_1(m,n,\pi,F) = I_1(m,n,Profile,F)$. By the argument provided above, if there is a voter having an incentive to π-manipulate in $\vec{P}$, then his manipulation is successful in all preference profiles of his information set $W_i^{\pi(\vec{P})}$, including $\vec{P}$. Therefore, $I_2(m,n,\pi,F) = I_1(m,n,\pi,F) = I_1(m,n,Profile,F)$. Q.E.D.

Further we consider indices I_2 and I_3 just for the plurality rule as the most typical example of a social choice rule. First, we find the limit of the aggregate stimulus measure for the Winner-PIF and 1Winner-PIF to compare it with the result of Theorem 2.

Theorem 6. $\lim_{n\to\infty} I_3(m,n,Winner,Plurality) = 0$ under Leximin and Leximax PEMs. $\lim_{n\to\infty} I_3(m,n,1Winner,Plurality) = 0$ with alphabetic tie-breaking rule.

Proof. Suppose that voter i has preferences $a_1 P_i a_2 P_i a_3 ... a_m$. With both Leximin and Leximax PEMs voter i has an incentive to π-manipulate in two cases: [1] a unique winner is an alternative from $\{a_3, a_4, ..., a_m\}$; [2] $|C_F(\vec{P})| > 1$ & $a_1 \notin C_F(\vec{P})$. For Leximin there is a third one: [3] $|C_F(\vec{P})| > 1$ & $\exists a_j \in C_F(\vec{P})$, $a_j \neq a_m$.

1) Consider case [1] and take alternative a_3 as a winner. In $W_i^{C_F(\vec{P})}$ there is a profile $\vec{P}'_{-i}$, s.t. the number of scores of the second best alternative a_2 is one score less than of a_3, $S(a_2, (P_i, \vec{P}'_{-i})) = S(a_3, (P_i, \vec{P}'_{-i})) - 1$. Then voting for a_2 either does not change anything, or leads to a tie $\{a_2, a_3\}$, which is more preferable than $\{a_3\}$ for voter i by both Leximin and Leximax. The stimulus of voter i for this manipulation, $stimulus(i, \vec{P}, \pi(\vec{P}))$, is the share of preference profiles $\vec{P}'_{-i} \in L(X)^{N \setminus \{i\}}$, s.t. $S(a_2, (P_i, \vec{P}'_{-i})) = S(a_3, (P_i, \vec{P}'_{-i})) - 1$ in the set of preference profiles s.t. $C_F(P_i, \vec{P}'_{-i}) = \{a_3\}$.

First, we consider sets of ballots $(n_1, ..., n_m)$ s.t. $n_1 + ... + n_m = n - 1$, where n_j is the number of voters voting for the j-th alternative.

$$B = \{(n_1, ..., n_m) \in \mathbb{Z}_+^m : n_1 + ... + n_m = n - 1, \forall j \in \{1, ...m\} \setminus \{3\}\ n_3 > n_j\},$$

$$A_1 = \{(n_1, ..., n_m) \in \mathbb{Z}_+^m : n_1 + ... + n_m = n - 1, \forall j \in \{1, ...m\} \setminus \{3\}\ n_3 > n_j, n_2 = n_3 - 1\}.$$

And then sum up the number of preference profiles for all ballots in the sets using function ϕ.

$$\phi(A) = \sum_{(n_1, ..., n_m) \in A} \frac{(n_1 + ... + n_m)!}{n_1! ... n_m!} ((m-1)!)^{n-1},$$

$$stimulus(i, \vec{P}, \pi(\vec{P})) = \phi(A_1) / \phi(B).$$

Consider the following set:

$$A_1 = \{(n_1, ..., n_m) \in \mathbb{Z}_+^m : n_1 + ... + n_m = n - 1,\ \exists i, j, \text{ s.t. } n_i = n_j\}.$$

The total number of preference profiles is $(m!)^n$. Thus, the share of preference profiles where plurality rule produces a tie is $\phi(A_2) / (m!)^n$. Since the probability of any tie goes to zero [Gehrlein and Fishburn, 1981], $\lim_{n \to \infty} \phi(A_2) / (m!)^n = 0$.

Take a point from A_1, $\vec{n} = (n_1, n_2, n_2 + 1, ..., n_m)$, and map it to the point form A_2, $\vec{n}' = (n_1 + 1, n_2, n_2, ..., n_m)$. Therefore, for any point from A_1 there is only one point from A_2.

$$\frac{\phi(\vec{n}')}{\phi(\vec{n})} = \frac{n!}{(n_1+1)!n_2!n_2!n_4!...n_m!} \Bigg/ \frac{n!}{n_1!n_2!(n_2+1)!n_4!...n_m!} = \frac{n_2+1}{n_1+1} \leq 0,$$

i.e. $\phi(\vec{n}') \geq \phi(\vec{n})$, and thus, $\lim_{n\to\infty} \phi(A_1)/(m!)^n = 0$. Since the probability of a tie between winners also tends to zero, $\lim_{n\to\infty} \phi(B)/(m!)^n = 1/m$ and

$$\lim_{n\to\infty} \frac{\phi(A_1)/(m!)^n}{\phi(A_1)/(m!)^n} = \lim_{n\to\infty} \frac{\phi(A_1)}{\phi(A_1)} = 0.$$

The share of preference profiles with a unique plurality winner where at least one voter has an incentive to Winner-manipulate tends to 1 (as shown in Theorem 3), but the stimulus to manipulation of this type tends to 0.

2) Cases [2] and [3] assume a tie between some alternatives, a manipulating voter should vote for the best alternative in $C_F(\vec{P})$ in case [2] and for an alternative which is in $C_F(\vec{P})$ and is better than the worst alternative in $C_F(\vec{P})$. Although the stimulus to vote insincerely is 1 in these two cases, the share of preference profiles producing ties tends to 0.

Finally, we get that $\lim_{n\to\infty} I_3(m, n, Winner, Plurality) = 0$ under Leximin and Leximax PEM.

3) If alphabetic tie-breaking is used and the winner is a_3, then either [i] $C_F(\vec{P}) = \{a_3\}$ or [ii] $C_F(\vec{P}) > 1$ and $\forall c \in C_F(\vec{P}) \setminus \{a_3\}$ $a_3 P_T c$. Voting for a_2 again is a dominating strategy for voter i. In case [i] if $a_3 P_T a_2$, then the voter does not change anything, if $C_F(\vec{P}_{-i}, P_i') = \{a_3, a_2\}$ and $a_2 P_T a_3$, then voter makes a_2 a winner. In case [ii] the voter makes a_2 a winner only if $a_2 \in C_F(\vec{P})$. Thus, the share of preference profiles where manipulation is successful is even less than under Leximin and Leximax PEM, and consequently, it also tends to 0. Q.E.D.

5.1 Computational experiments

To illustrate the difference between manipulability indices I_1, I_2, and I_3, we conduct several computational experiments for plurality rule with 3 alternatives and the number of voters from 3 to 20. Fig. 24 is the legend, results are presented in Figs. 25–33. Since the plurality rule is strongly computable from π-images for Ballot-PIF, Positions-PIF, and Score-PIF, I_2 and I_3 for these PIFs are equal to $I_1(m, n, Profile, Plurality)$, the probability of manipulation under complete information (by Theorem 5). We do not calculate I_2 and I_3 for MG-PIF, because $I_1(m, n, MG, Plurality)$ is almost always zero-valued.

The index of manipulation success, I_2, calculated for Rank-PIF and Winner-PIF turns out to be equal to $I_1(m, n, Profile, Plurality)$. This means that if for voter i there are preference profiles $\vec{P}'_{-i} \in W_i^{\pi(\vec{P})}$, where manipulation is successful ($WS_i^{\pi(\vec{P})} \neq \varnothing$), then there is no such $\vec{P}'_{-i}$, where the same manipulation makes the voter worse off. Indeed, the plurality rule is computable from π-images for these PIFs and monotonic, which means that the voter knows who wins and has opportunity to achieve a better outcome without the possibility of being worse off. Of course, this does not hold for WMG-PIF, $I_2(m, n, WMG, Plurality) \leq I_1(m, n, Profile, Plurality)$, because the winner cannot be computed from a weighted majority graph and misrepresenting preferences could lead both to a better and a worse outcome.

Indices I_2 and I_3 are almost always strictly lower than I_1. Moreover, for Rank-PIF, Winner-PIF, and 1Winner-PIF the difference between I_1 and two other indices seems to become greater as the number of voters grows, since I_1 gets closer to 1, while I_2 and I_3 decrease.

In most cases the index of stimulus to manipulation I_3 is lower for Winner-PIF than for Rank-PIF (and in case of alphabetic tie-breaking I_3 for 1Winner-PIF is the lowest). This shows that the lack of information does not increase the manipulability measure, which takes into account the willingness of voters to manipulate. The most significant difference between I_1 and I_3 is for 1Winner-PIF with alphabetic tie-breaking: more than 0.75 for $n \geq 10$. Finally, if a social choice rule is not computable from pπ-images, as in the case of WMG-PIF, then the indices of manipulation success and stimulus to manipulation are very low, and with an increasing number of voters can be regarded as negligible.

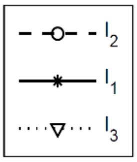


Fig. 24. The legend for Figs. 25-33

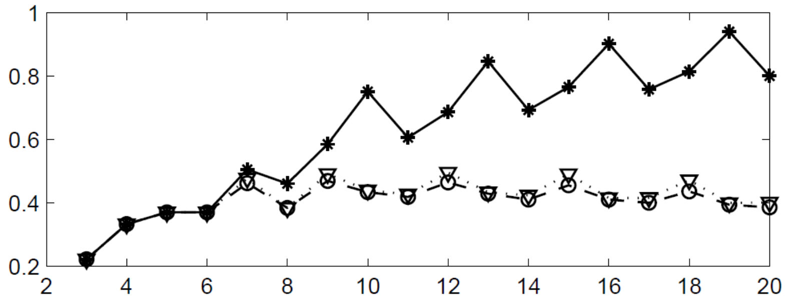


Fig. 25. Leximin, Rank-PIF

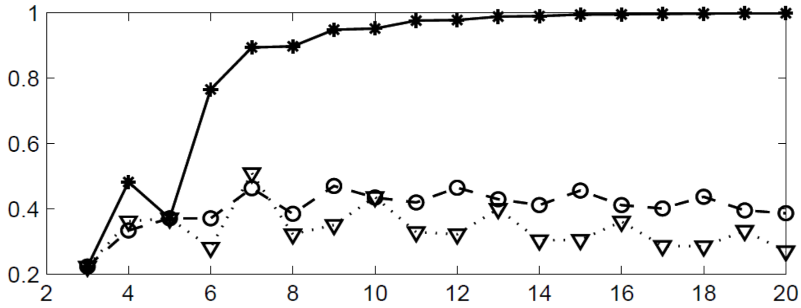


Fig. 26. Leximin, Winner-PIF

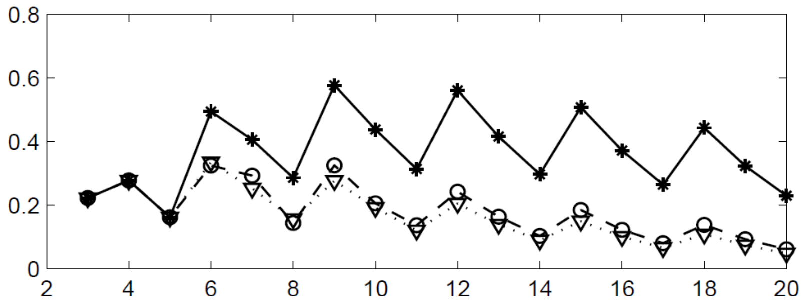


Fig. 27. Leximin, WMG-PIF

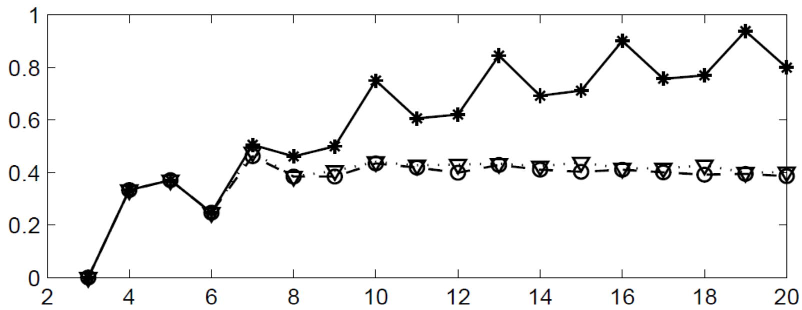


Fig. 28. Leximax, Rank-PIF

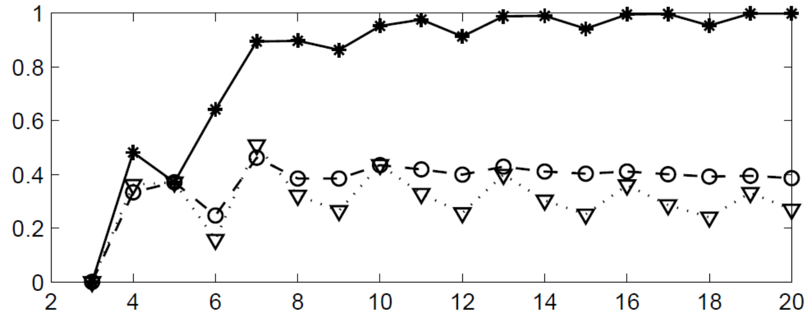


Fig. 29. Leximax, Winner-PIF

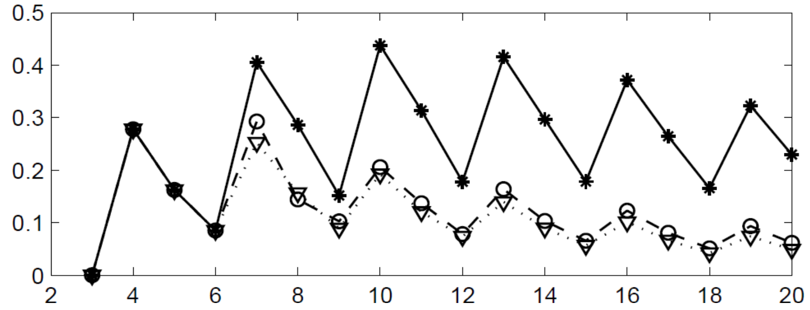


Fig. 30. Leximax, WMG-PIF

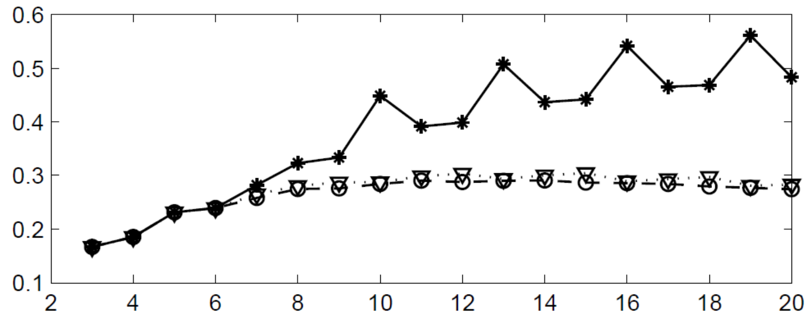


Fig. 31. Alphabetic tie-breaking, Rank-PIF

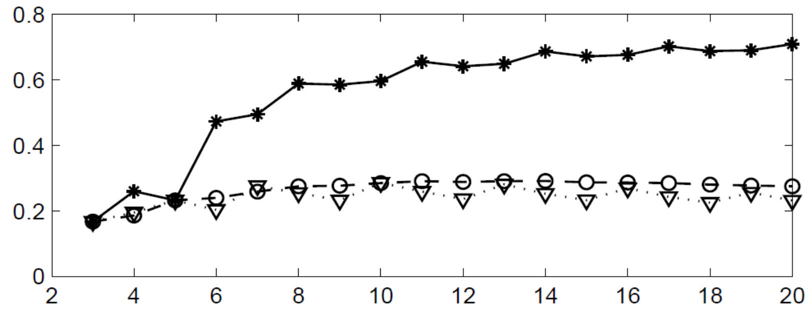


Fig. 32. Alphabetic tie-breaking, Winner-PIF

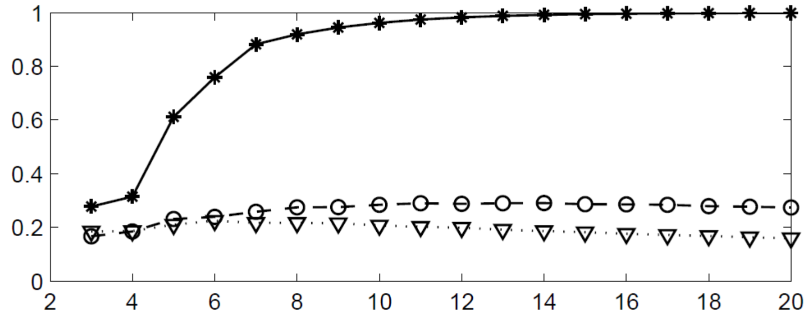


Fig. 33. Alphabetic tie-breaking, 1Winner-PIF

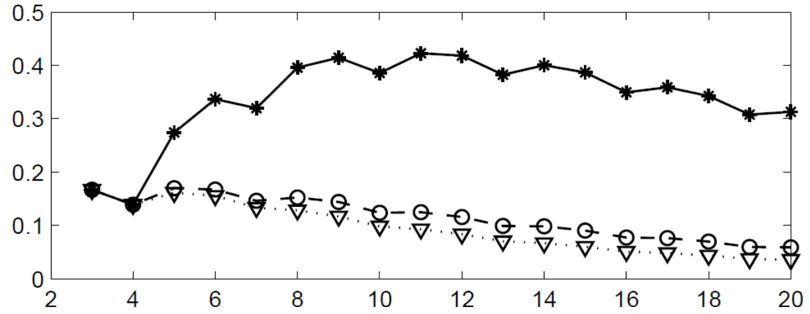


Fig. 34. Alphabetic tie-breaking, WMG-PIF

Conclusion

We studied the vulnerability of social choice rules to voter manipulations under incomplete information using the model of opinion polls. We realized that the fact of susceptibility of SCRs to π-manipulation is not enough and considered the probability of such manipulation to compare SCRs and the influence of different PIFs on manipulability.

It turned out that many rules are susceptible to π-manipulation not only when they are strongly computable from π-images, but even when they are not computable from π-images. For example, information about a weighted majority graph allows for manipulation in about 40% of preference profiles. The values of the first manipulability measure with information about the winner grow very fast and approach 100% in some cases. Another interesting observation is that less information leads to greater manipulability for many rules.

Thus, we also could not be satisfied with the analysis of the probability of π-manipulation, because it does not show what is behind the high values of the manipulability measure. The second manipulability measure we considered counts only preference profiles where manipulation is successful. For all parameters used in experiments we revealed that incomplete information does not influence the probability of manipulation success for the plurality rule when it is computable from π-images.

The third manipulability index measures the stimulus of voters to manipulate. For the plurality rule, the greater the growth of the manipulation probability, the less the aggregate stimulus to manipulation. In asymptotics, for an infinite number of voters, while the probability of the manipulation of the plurality rules is 1, the willingness of voters to manipulate is zero. For the PIF which does not allow to compute the winner, this index gives the least values among other PIFs.

The analysis of manipulation under incomplete information made in this study allows us to view the problem from different perspectives. Importantly, it shows that a formal approach like the calculation of manipulation probability has its drawbacks: the values are high, but we do not see what kind of manipulation takes place. Another, more subjective, criterion may be needed which shows that manipulation is not such a dramatic problem as seemed earlier.

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