National Research University
Higher school of economics

Zasypko Veronika Vladimirovna

Development of a numerical-analytical method and an algorithm for solving the optimal control problem (illustrated by an example of three-sector investment economic model)

SUMMARY of DISSERTATION
for academic degree of Candidate of Sciences
in Applied Mathematics HSE

Academic superviser:
Candidate of Physical-Mathematical
Sciences, Associate Professor
Shnurkov Petr Viktorovich

Moscow - 2018

Statement of the problem

We consider the so-called classical optimal control problem, where the time interval is fixed and the boundary conditions are of the type of the fixed left and free right ends. This statement of the problem is encountered in many specific control problems in economical and technical systems.

The classical optimal control problem is posed as

$$B(x(\cdot), u(\cdot)) = \int_{t_0}^{t_1} f(t, x(t), u(t))dt + l(x(t_1)) \to \min;$$

$$\dot{x} - \varphi(t, x(t), u(t)) = 0 \quad \forall t \in T, \tag{1}$$

$$u(t) \in U \quad \forall t \in [t_0, t_1], \quad x(t_0) = x_0,$$

where the state function $x(\cdot) \in PC^1([t_0, t_1], \mathbf{R^n})$ is the set of piecewise continuously differentiable vector functions defined on $[t_0, t_1]$ and ranging in R^n, the control $u(\cdot) \in PC([t_0, t_1], \mathbf{R^r})$ is the set of piecewise continuous vector functions defined on $[t_0, t_1]$ and ranging in a given subset $U \subset \mathbf{R^r}$ (the set of admissible controls), and $T \subset [t_0, t_1]$ is the set of points of continuity of the control $u(\cdot)$.

The main assertion on necessary conditions for extremum in this optimal control problem, which is called the Pontryagin maximum principle, is formulated as follows.

An auxiliary function, called the Pontryagin function, is introduced:

$$H(t, x, u, p, \lambda_0) = (p, \varphi(t, x, u)) - \lambda_0 f(t, x, u).$$

Theorem. Assume that $(x_*(\cdot), u_*(\cdot))$ is an optimal controlled process in the optimal control problem (1) ($(x_*, u_*) \in \mathrm{strlocmin} P$), the functions f, φ are continuous in a neighborhood of the Cartesian product of the set $\Gamma_{x_*} = \{(t, x_*(t)) | t \in [t_0, t_1]\}$ by the set U, the partial (Frechet) derivatives f_x, φ_x are defined on this set and continuous at the points of the set $\Gamma_{x_*, u_*} = \{(t, x_*(t), u_*(t)) | t \in [t_0, t_1]\}$, and the function l is (Frechet) differentiable at the point $x_*(t_1)$, ($l \in D(x_*(t_1))$).

The the following conditions are satisfied:

$$\max_{u \in U} H(t, x_*(t), u, p(t), \lambda_0) = H(t, x_*(t), u_*(t), p(t), \lambda_0), t \in T, \tag{2}$$

or in other form,

$$(p(t), \varphi(t, x_*(t), u_*(t)) - f(t, x_*(t), u_*(t)) \geq (p(t), \varphi(t, x_*(t), u) - f(t, x_*(t), u), \quad t \in T, \forall u \in U, \tag{3}$$

and the Lagrange multiplier λ_0 can be set equal to 1. The function $p(t)$, which is called the conjugate variable, is the unique solution of the Cauchy problem consisting of the differential equation

$$\dot{p}(t) = -\varphi_x^*(t, x_*(t), u_*(t))p + f_x(t, x_*(t), u_*(t)) \tag{4}$$

and the boundary condition

$$p(t_1) = -l'(x_*(t_1)). \tag{5}$$

Remarks on the theorem.

The optimality condition of the form (2) or (3) is called the maximum condition, and the name of this fundamental assertion on necessary conditions for extremum originates from this maximum condition. Relation (4), which, analytically, is a system of differential equations for the vector function $p(t)$, is called the conjugate equation. The boundary conditions to this equation are called the transversality condition. In this problem, the transversality condition is meaningful only at the point $t = t_1$, and the corresponding condition at the point $t = t_0$ is not informative and not included in the system of necessary conditions for extremum.

The maximum condition for the Pontryagin function (2) plays the key role in the system of necessary conditions for extremum and determines the general structure of the optimal control when the problem is solved. In the most widely used standard version of the problem, the Pontryagin function linearly depends on the control $u \in U$. Let $Q(t)$ denote the coefficient of $u \in U$ in the formula for the Pontryagin function. In this case, the function $Q(t)$ explicitly depends on the conjugate variable $p(t)$. If one assumes that the set of admissible controls is the interval $U = [u_0, u_1] \subset R$, then the maximum condition implies that the optimal control $u_*(t)$ has the following structure:

$$u_*(t) = \begin{cases} u_1 & \text{if } Q(t) > 0, \\ u^{(0)}(t), & \text{if } Q(t) = 0, \\ u_0 & \text{if } Q(t) < 0. \end{cases} \tag{6}$$

The function $u^{(0)}(t)$ contained in (6) is called a singular control, which arises when the function $Q(t)$ takes the zero value; in this case, the Pontryagin function is explicitly independent of the control $u \in U$. The singular control cannot be determined from the maximum conditions, and there are special methods for determining such a control.

Further, $Q(t)$ is the function determining the control. It follows from relation (6) that the behavior of this function mainly determines the form of optimal control.

The main problem encountered when necessary conditions for extremum are used in the maximum principle form is that it is required to analyze a complex system of relations consisting of several systems of mutually related differential equations and boundary conditions. In particular, the conjugate system of differential equations can depend on the controls $u(t)$ and states $x(t)$, and the differential constraint determining the variation in the states $x(t)$ depends on the controls $u(t)$. Such a system can be solved analytically only in special cases. Thus, it is necessary to develop methods for numerical analysis of systems consisting necessary conditions for extremum and the constraints of the original problem. In the present work, one of such methods is proposed.

Degree of results availability

In the 1950s, the needs of applied disciplines (technology, economics, and others) stimulated the statement and consideration of a new class of problems, which were called the optimal control problems. A necessary condition for extremum in problems of this class is the "maximum principle" formulated by L.S. Pontryagin in 1956, which was proved and further developed by him and his pupils and colleagues.

The analytical difficulties related to applications of the maximum principle have been known since the time of construction of a method for solving the optimal control problems based on the maximum principle, and this results in a lot of scientific research aimed at obtaining numerical solutions of various optimal control problems. Most of them used numerical methods for solving systems of differential equations contained in necessary conditions and the constraints of the original problem. In the present work, the optimal control problem is analyzed in the whole with regard to the structure of possible controls.

Objectives and goals of the research

Objective the research.

To develop a numerical-analytical method and an algorithm for studying the system of necessary conditions in the classical optimal control problem based on the maximum principle.

Goals of the research:

1. To analyze the problem of optimal control of investments in a dynamical model of three-sector economics based on the maximum principle and to reveal the general structure of the functions determining the optimal control.

2. To obtain analytic representations of the functions describing the state of the

system under study and of the conjugate variables in optimal control.

3. To construct an algorithm and a program which permit numerically analyzing the system of relations comprising necessary conditions and the constraints of the original problem and to obtain a numerical solution of this system, i.e., to obtain admissible extremals.

Relevance of the dissertation

The most famous result of the mathematical theory of optimal control underlying such studies is the Pontryagin maximum principle. A method based on this theoretical assertion was used in the classical works of mathematical economics and in several contemporary investigations.

Unfortunately, the method based on the use of the maximum principle very seldom results in obtaining analytic solutions of the optimal control problems. The direct application of this method leads to several mutually connected systems of relations (necessary conditions for extremum and the constraints of the original problem), and most often, it is impossible to obtain an analytic solution of such a system. Therefore, it is extremely important to develop new numerical-analytical and numerical methods for analyzing such systems of relations and determining admissible extremals and optimal controlled processes. This dissertation research deals precisely with this problem.

Personal contribution of the author to the development of the problem

Applicant Zasypko V.V. personally participated in the process of obtaining all basic results of the dissertation and in preparing all publications on the topic of the dissertation.

The results of this research were presented by Zasypko V.V. as reports at the following scientific conferences, symposia, and seminars:

1. Scientific-Technical Conferences for students, post-graduate students, and young scientists at MIEM and MIEM NRU HSE in 2011, 2012, 2013, and 2014;

2. All-Russia Symposium on Applied and Industrial Mathematics. Autumn Open Session (Sochi, September 2011);

3. All-Russia Conference "Applied Probability Theory and Theoretical Informatics" (Moscow, April 2012);

4. International Conference " Probability Theory and Its Applications" dedicated to B.V. Gnedenko on the occasion of his 100th birthday (Moscow, June 2012);

5. Seminar "Approximation Theory and Theory of Extremum Problems", Department of General Control Problems, Faculty of Mechanics and Mathematics, MSU; the seminar

leader – Prof. Tikhomirov V.M. (Moscow, October 2012);

6. XXXI International Seminar on Stability Problems for Stochastic Models (Moscow, April 2013);

7. Scientific-Research Seminal "Optimal Control: Mathematical Theory and Applied Problems" Department of Optimal control, Faculty of Computational Mathematics and Cybernetics, MSU; the seminar leader – Corresponding member of RAS, Prof. Aseev S.M. (Moscow, February 2014).

8. Seminar "Analysis of Investment Projects"; Federal Research Center "Informatics and Control", Institute for System Analysis RAS; the seminar leader – doctor of economic sciences, Livshits V.N. (Moscow, June 2016).

9. Seminar "Approximation Theory and Theory of Extremum Problems", Department of General Control Problems, Faculty of Mechanics and Mathematics, MSU; the seminar leader – Prof. Tikhomirov V.M. (Moscow, October 2016).

Description of the research methodology

The following mathematical methods are used to solve the problem posed above.

1. General methods of the optimal control theory .

2. Methods of the theory of differential equations.

3. Analytical methods of the classical mathematical analysis.

4. Methods of numerical analysis.

The developed numerical-analytical method was realized as an example of analysis of the optimal control problem in a closed dynamical model of three-sector economics. In this model, the states are functions of funds (specific capital) in the sectors of the model, the control parameter is a function which is the investment fraction of the major fund lending sector in the total volume of investments. Mathematically, this problem is a classical optimal control problem on a prescribed finite time interval with mixed criterion functional, differential constrains, the fixed left end of the trajectory, and several constraints on the control. The process of solving this problem is based on the use of the Pontryagin maximum principle. The maximum condition is used to determine the general structure of optimal controls. For the control functions of a prescribed structure with arbitrarily many switchings, the analytic representation is obtained for the functions of state and the so-called conjugate variables which, by their theoretical meaning, are Lagrange multipliers in the original extremum problem with constraints. The obtained analytical results permit constructing an algorithm for determining the control functions and the corresponding state functions which satisfy the general system of relations consisting of necessary conditions for extremum and the constraints of the original optimal control problem. The constructed algorithm is realized as a program package. This program product, using the prescribed initial parameters of the model, allows one numerically to analyze a rather wide class of theoretically possible control functions and to determine the controlled processes satisfying necessary conditions and constraints.

Basic results presented to be defended

1. Development of a general numerical-analytical method for analysis of optimal control problems and its realization as an example of the posed control problem.

2. Statement of a new mathematical optimal control problem in the framework of a dynamical model of three-sector economics.

3. Development of the numerical algorithm for solving systems comprising necessary conditions and the constraints of the original optimal control problem.

4. Development of a program realizing this numerical algorithm.

Scientific novelty of the work

The following scientific results are obtained in the work:

1. A new numerical-analytical method for solving the optimal control problem is developed. This method permits analyzing the system of relations comprising necessary conditions and the constraints of the original problem and numerically determining the admissible extremals.

2. A new meaningful mathematical optimal control problem is posed and justified in the framework of the dynamical model of three-sector economics. This problem is solved by a method based on the Pontryagin maximum principle.

3. The system comprising necessary conditions and the constraints of the original optimal control problem is analytically analyzed. Explicit analytic representations of the state functions and conjugate variables are obtained for a class of control functions whose structure is determined by methods based on the maximum principle.

4. An algorithm is constructed for determining controlled processes which are admissible extremals in the optimal control problem under study. This algorithm can be used in a wide class of control problems, where the conjugate equations can, in general, depend on the state functions of the model.

5. A program realizing this algorithm is developed. For a given set of input parameters of the model, this program permits determining numerical and graphical representations of controlled processes which are admissible extremals in the original optimal control problem.

General research conclusions

This dissertation work is a study of a specific fundamental problem of mathematical economics, namely, of the optimal control problem in a macroeconomic dynamical model. The study itself has a profound complex character and consists of two stages. At the first stage, the analytic methods related to the contemporary mathematical theory of optimal control are used. The main point at the second stage is the numerical algorithm which permits determining one or several solutions of a very complicated system of equations comprising necessary conditions for extremum and the constraints of the original optimal control problem. This algorithm is completely original. Moreover, it is rather general and can be used to solve various optimal control problems, not only in economics. The developed algorithm is realized as the program package which, for prescribed initial parameters of the model, permits determining specific controlled processes suspected to be optimal, i.e., admissible extremals in the original optimal control problem.

Acknowledgments

The authors especially expresses her thanks to her scientific adviser, cand. phys.-math. sci., associate professor Shnurkov P.V. for active participation in her research and the help in solving the scientific organization processes, and for his constant attention and highly qualified guidance during her work on the dissertation.

List of publications in the topic of the dissertation

The main points of the dissertation are presented in the author's publications in the leading reviewed scientific journals recommended by the Higher Attestation Commission of the Russian Federation Ministry of Education and Science:

1. Zasypko V.V., Optimal control of investments in a closed dynamical model of three-sector economics: mathematical statement of the problem and general analysis based on the maximum principle. // Vestnik MGTU im. N.E. Bauman. Ser. Estestv. Nauki. – 2014. – № 2, pp. 101–115, 0.71 a.sh. (with Shnurkov P.V.; author's personal contribution – 0.35 a.sh.).

2. Zasypko V.V., Analytical study of problems of investment optimal control in a closed dynamical model of three-sector economics. // Vestnik MGTU im. N.E. Bauman. Ser. Estestv. Nauki. – 2014. – № 4, pp. 101–120., 0.81 a.sh. (with Shnurkov P.V.; author's personal contribution – 0.4 a.sh.).

3. Zasypko V.V., Development of an algorithm for solving the problem of investment optimal control in a closed dynamical model of three-sector economics. // Informatika i ee Primeneniya, – 2016. – 10, № 1, pp. 82–95. (with Shnurkov P.V., Belousov V.V., and

Gorshenin A.K.).

and in other publications of the author:

4. Pisarenko V.V., Control of investments of the fund lending sector in the dynamical model of three-sector economics. // Survey of Applied and Industrial Mathematics. Vol. 18. no. 4. Scientific reports. XII All-Russia Symposium on Applied and Industrial Mathematics. 2011. Sochi. pp. 654–655, 0.12 a.sh.

5. Pisarenko V.V., Control of investments of the fund lending sector in the dynamical model of three-sector economics. // Book of abstracts. All-Russia Conference "Applied Probability Theory and Theoretical Informatics". – Moscow: IPI RAN, 2012. – pp. 88–90, 0.14 a.sh. (with Shnurkov P.V.; author's personal contribution – 0.07 a.sh.).

6. Pisarenko V.V., Control of investments of the fund lending sector in the dynamical model of three-sector economics. // Book of abstracts. International Conference "Probability Theory and Its Applications"dedicated to B.V. Gnedenko on the occasion of his 100th birthday, – Moscow: LENAND, 2012. – pp. 269–270, 0.12 a.sh. (with Shnurkov P.V.; author's personal contribution – 0.06 a.sh.).

7. Zasypko V., Trajectory analysis of control process for optimal control of investments in the model of a three-sector economy. // Book of abstracts. XXXI International Seminar on Stability Problems for Stochastic Models and VII International Workshop "Applied Problems in Theory of Probabilities and Mathematical Statistics Related to Modeling of Information Systems"and International Workshop "Applied Probability Theory and Theoretical Informatics". – Moscow: IPI RAN, 2013. – pp. 111–113, 0.1 a.sh. (with Shnurkov P.V.; author's personal contribution – 0.05 a.sh.).