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Minimax hedging of European option in incomplete market

SUMMARY of DISSERTATION
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Introduction

The PhD thesis is on theory of risky asset's portfolio management in incomplete multidimensional markets without transaction costs with discrete time and final horizon. In the PhD thesis this theory was applied to solve some calculation problems for European options in incomplete multidimensional markets of risky assets.

To outline the approach used in the PhD thesis let us give necessary definitions of option's theory. Risky assets are objects with price evolving as adapted random sequences (stocks, for example). Multidimensional market is a set of risky assets. It is possible to describe fully multidimensional market by probability distribution of these random sequences. Multidimensional predictable random sequences (with the same dimensions as markets) are called portfolio. Only markets without transaction costs are considered, i.e. one doesn't have to pay when transfer asset of one type to another.

European option is a contract, under which the Seller of assets (the Issuer) sells and the Buyer has right (but not obliged) to buy at fixed in advance price at the future moment stated in the contract (called execution moment). Herewith to obtain this right the Buyer has to pay to the Issuer a fee (money, for example) at the contract conclusion moment. The fee is called option premium or option value. When option is executed the Issuer has to deliver the assets to the Buyer, i.e. at the moment of execution the Issuer's obligation appears. The Issuer has to fulfill the obligation. This obligation is called payoff. The payoff is a measurable function, possible, depending on all risky asset's prices up to the execution moment. So, to fulfill the obligation the Issuer has to construct a portfolio with capital not less than the value of obligation with given probability. Here a capital of portfolio at any moment is a sum of products of quantities of risky assets and their prices, i.e. the value of the portfolio.

Note, there are arbitrage and arbitrage-free markets. As it is defined, in an arbitrage markets there is positive probability to make a profit with zero investments. Otherwise market is called arbitrage-free. Simple criterion for arbitrage-free markets is known[1]: a market is arbitrage-free then and only then risky asset's prices do not change in mean while evolve. This means, that random sequences describing risky asset's price evolution are martingales[2]. Appropriate probability measures are called martingale or risk-neutral measures. Arbitrage-free markets encompass complete and incomplete markets. Complete markets are defined by the fact that in such market any payoff might be fulfilled almost surely. This means, that there is a portfolio of risky assets with capital equal to value of the Issuer's obligation. Criterion for market completeness is known[1]: a market is complete if and only if there is unique martingale measure. A complete market is

[1]Shiryaev A.N. Essentials of Stochastic Finance. Vol. 2. Theory. Moscow: Fazis. 1998. - 1056 p. (in Russian)
[2]Shiryaev A.N. Probability. Moscow: Nauka. 1980. - 576 p. (in Russian).

idealization, which, as a rule, doesn't have place, i.e. real multidimensional markets are incomplete. This means, that probability measure defining incomplete market is not unique. That is why the Issuer to fulfill the obligation has to: 1) chose probability measure with respect to which he or she will calculate European option; 2) construct portfolio of risky assets assuring fulfillment of the obligation with given probability; 3) generate option's value.

Relevance of the topic

To calculate European option in incomplete market fair value principle is generally used[1,3]. In contrast to above stated works, in the PhD thesis minimax principle has been used. This principle was chose for the following reasons. The Issuer don't know probability distribution of risky asset's prices a priori. Suppose, that Issuer's risk function is exponential and depends on his profit. The Issuer minimizes expected value of exponential risk. This might be gained by portfolio, which enforces the Issuer to parry any unfavorable for him probability distribution of risky asset's prices. To implement this principle one need to justify applicability for stochastic version of dynamic programming in the case than adapted sequence is observed and objective functional is multiplicative. So, there is minimax problem of optimal stochastic portfolio management. This problem has not been considered in scientific literature yet.

Within the approach it has been managed to establish new existence conditions for: 1) optimal portfolios being predictable random sequences and are invariant with respect to any equivalent probability measure; 2) uniform Doob decomposition with respect to any equivalent probability measure for measurable bounded functionals set on trajectories of adapted random sequences; 3) extreme measures delivering maximum value to expected risk and to find properties of these measures, to prove (for the first time), that initial incomplete market is complete with respect to extreme measure.

Above stated results allow to calculate constructively European option in incomplete market.

This justifies the topic and the results of the PhD thesis.

Purposes of research are to find:

(1) minimax value of issuer's expected exponential risk,

(2) constructive existence condition of hedging (superhedging, quantile hedging, quantile superhedging) portfolio for European option in incomplete market without transaction costs.

As a rule in theory of European option's calculation static problem is considered. There in the PhD thesis the problem is considered in dynamics. So the **scientific novelty** of the PhD thesis is related to the following:

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[3] Bertsekas D., Shreve S. Stochastic Optimal Control. Moscow: Nauka. 1985. - 280 p. (in Russian).

(1) it is the first time when the applicability of the dynamic programming method for non Markov systems with multiplicative risk function has been justified for the case of discreet time. This allows to find out that evolution of upper guaranteed value for expected exponential issuer's risk is submitted to Bellman's type recurrent relation even if risk asset's prices are presented by semimartingales;

(2) new conditions for existence of uniform Doob decomposition have been obtained;

(3) conditions have been obtained for existence of superhedging, quantile superhedging portfolios of European options in incomplete multidimensional markets without transaction costs with respect to any equivalent probability measure;

(4) criterion have been constructed for existence of extreme probability measure delivering the maximum of expected exponential issuer's value, characteristics of the extreme measure have been examined.

The PhD thesis is a theoretical one. It's results belongs to the field of optimal stochastic control. It is possible to use them in stochastic theory of optimal control as well as in stochastic financial mathematics. **Theoretical significance** of the results is justified by the following:

(1) conditions have been obtained under which evolution of upper guaranteed value for expected exponential issuer's risk is submitted to Bellman's type recurrent relation when risk asset's prices are presented by semimartingales,

(2) it is proved that any bounded payoff allows uniform Doob decomposition with respect to any probability measure from the set of equivalent probability measures,

(3) criterion have been constructed for existence of extreme probability measure and portfolio delivering minimax value for expected exponential issuer's risk, moreover it has been proved that with respect to the measure initial incomplete market is a complete one,

(4) it has been proved that for the case of incomplete multidimensional markets without transaction costs in discrete time it is possible to reduce problem of quantile hedging (quantile superhedging) to two problems of perfect hedging (superhedging).

In the PhD thesis methods of functional analysis, probability theory, theory of stochastic processes and stochastic analysis were used.

The practical significance of the results obtained is as follows:

(1) since the sequences of risk asset's prices, as a rule, are semimartingales, obtained statements might be used to select the minimax portfolio management;

(2) the criterion has been established for the existence of extreme probability measure with respect to which: (a) the initial market is complete; (b) the upper bound of the option's value has been found, (c) a hedging portfolio has been constructed,

(3) for incomplete markets without transaction costs the results obtained allow to construct a quantile hedging portfolio.

Results to defend:

(1) Bellman's type recurrent relation for the sequence of upper guaranteed values for expected exponential issuer's risk in incomplete multidimensional market without transaction costs;

(2) existence conditions for superhedging, quantile superhedging portfolio of European option in incomplete multidimensional market without transaction costs with respect to any measure form the set of equivalent probability measures;

(3) existence criterion for probability measure (the worst-case measure) delivering essential supremum for expected exponential issuer's risk, characteristics of the measure;

(4) existence conditions for minimax and quantile minimax portfolios.

The degree of development for the research problem. There are a lot of works dealing with European option's calculation theory in one dimensional complete market without transaction costs in discreet time. These are works by Cox J.C., Ross R.A., Rubinstein M.[4], Harrison J.M., Kreps D.[5], Shiryaev A. N., Kabanov Yu. M., Kramkov O. D. and Mel'nikov A. V.[6], Föllmer H., Schied A.[7]. There in the works they established uniqueness of martingale measure and obtained it's explicit form. It is proved that payoff allows S-representation. These results gave an opportunity to obtain option's value and to construct perfect hedging portfolio. There are also works of some authors on the theory of quantile hedging for European option in one-dimensional complete market. For example, there is in the article by Novikov A.A.[8]for the case of one-dimensional complete market method is justified for calculation of option's value and of hedging strategy. In H. Föllmer's and P. Leukert's article[9] they consider static problem to minimize option's value under given probability of payoff being fulfilled. They claim, that the solution of this problem coincides with the solution for the problem of European option's calculation with some modified payoff. To construct

[4]Cox J. C., Ross R.A., Rubinstein M. Option pricing: a simplified approach. / Journal of Financial Economics. - 1979. - v.7. - №3. - p.229-263

[5]Harrison, J.M., Kreps, D. Martingales and arbitrage in multiperiod security markets. / Journal of Economic Theory. - 1979. - v.20. - p.381-408

[6]Shiryaev A.N., KabanovYu.M., Kramkov O.D., Mel'nikov A.V. Toward the Theory of Pricing of Options of Both European and American Types. I. Discrete time / Theory of Probability and its Applications. – 1994. – 39. – 1. – p.23-79 (in Russian)

[7]Föllmer H., Schied A. Stochastic Finance. An Introduction in Discrete Time. Moscow: MTsNMO. 2008. - 496 p. (in Russian) (in Russian)

[8]Novikov A.A. Hedging of options with given probability / Theory of Probability and its Applications. 1998. – 43. –1. – p.152-161 (in Russian)

[9]Fëllmer H., Leukert, P. Quantile hedging. / Finance and Stochastics. - 1999. - v.3. - 3. - p.251-273

the latter they use the Neyman-Pearson lemma. In the article by P.V. Grigor'ev, Yu.S. Kan[10]two-step optimal control problem is considered with two types of assets and with quantile criterion under assumption that risk asset's yield is uniformly distributed. Based on Yu.S. Kan's article[11]analytical solution was constructed for the problem to manage portfolio of securities then strategy belongs to the set of Markov strategies. In the article by A.I. Kibzun, A.V. Naumov and V.I. Norkin[12]for one-dimensional market with time horizon equal to one they establish conditions when it is possible to reduce quantile hedging problem to a partially integer programming problem.

Theory of European option's calculation in incomplete market without transaction costs in discrete time was considered in some works. For example, in articles by V. Naik[13], Delbaen F. and Schachermayer W.[14]for a semimartingale model of a market with limited number of assets and bounded from below payoff *f* it is proved that upper value of the optionC_0^{sup} allowsrepresentation

$$C_0^{sup} = \sup_{Q \in M(S)} E^Q f$$

where$M(S)$is the set of equivalent locally martingale measures specified on trajectories of risk assets's prices. In the article by F. Delbaen and W. Schachermayer[15] the validity of the above formula is established, where supremum is taken over the set of σ-martingale probability measures. In the works by Shiryaev A.N.[1], Föllmer H. and Schied A.[7] formula for upper value of option is derived in the case when payoff is a nonnegative bounded function. More over they establish existence conditions for superhedging portfolio in the case of equivalent martingale measures. In the article by A. Bizid, E. Jouni[16] in semimartingale market model where short selling is forbidden and payoff is bounded formula for upper value of an option is obtained. In L. Ruschendorf's article[17]formulas are derived allowing bottom and upper estimations for option's value. In the

[10]Grigor'ev P.V., Kan Yu. S. Optimal Control of the Investment Portfolio with Respect to the Quantile Criterion / Automation and Remote Control. – 2004. – 2. – p. 179-197 (in Russian)
[11]Kan Yu. S. Control Optimization by the Quantile Criterion / Automation and Remote Control. – 2001. – 5. – p.77-88 (in Russian)
[12]Kibzun A.I., Naumov A.V., Norkin V.I. On reducing a quantile optimization problem with discrete distribution to a mixed integer programming problem / Automation and Remote Control. – 2013. – 6. – p.66-86 (in Russian)
[13]Naik V., Uppal R. Leverage constraints and the optimal hedging of stock and bond options / Journal of Financial and Quantitative Analysis. - 1994. - v.29. -№2. - p.199-222
[14]Delbaen F., Schachermayer W. The no-arbitrage property under a change of numeraire / Stochastics and Stochastic Reports. - 1995. - v.53. - p.213-266
[15]Delbaen F., Schachermayer W. The Fundamental Theorem of Asset Pricing for Undounded Stochastic Processes / Mathematische Annalen. - 1998. - v.312. - №2. - p.215-250
[16]Bizid A., Jouini E. Incomplete markets and short-sales constraints: an equilibrium approach. / Int. J. of Theoretical and Applied Finance. - 2001. - v.4. - №2. - p.211-243
[17]Rüschendorf L. On Upper and Lower Prices in Discrete-Time Models / Tr. Mat. Inst. Steklova. - 2002. - V.237. - p.143-148

article by A.A. Gushchin and E. Mordecki[18]in one-dimensional semimartingale model of (*B*,*S*)-market conditions are established under which upper and lower option's values are attainable. In the work by E. Eberlein, A. Papapantoleon, A. N. Shiryaev[19] for one-dimensional semimartingale market model when risk assets' prices are specified by process with independent increments they established existence conditions for call-put parity for options of European, American and Asian types. R.V. Khasanov's in his PhD thesis[20] consider static European option's calculation problem in multidimensional market. Assuming that risk assets' prices are specified by semimartingales, author derives formula for option's upper value

$$C_0^{sup} = \sup_{Z\in\mu^l} Ef\, Z_T = \sup_{Z\in\mu^\sigma} EfZ_T,$$

where $\mu^l\mu^\sigma$are sets of locally martingale and σ-martingale densities respectively. It was shown that separating measure is finitely additive. The problem to calculate European option with quantile criterion in incomplete market without transaction costs was considered in articles of some authors: Föllmer H., Schied A., Leukert P.[21], Karatzas I.[22], Cvitanic J.[23], Leung T., Song Q. and Yang J.[24] They considered static problem to minimaze option's value under given probability of payoff's fulfillment. It is stated that solution of the problem coincides with solution of European option's calculation problem with some modified payoff equal to product of initial payoff f and indicator of some set. In the work by Azanov V.M. and Kan Yu. S.[25] they consider maximization problem for probability to achieve a given capital value under fixed initial capital. Relations are established for optimal strategy.

Note that in most works European option's calculation problem in incomplete market without transaction costs is considered as a static problem. This made it possible to derive formulas for upper (lower) option's value or it's estimation. But this approach can't give formula for hedging formula and correspondent capital.

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[18]Gushchin A.A., Mordecki E. Bounds for options' values for semimartingale market models / Proceedings of the Steklov Institute of Mathematics. – 2002. - 237, 80-122. (in Russian).

[19]Eberlein E., Papapantoleon A., Shiryaev A. N. On the duality principle in option pricing: semimartingale setting. / Finance and Stochastics. - 2008. - v.12. - 2. - p.265-292

[20]Khasanov R.V. Maximization of utility with random contribution and hedging of payoffs, thesis to obtain degree of candidate of physical and mathematical sciences. Moscow. 2013. 91p. (in Russian).

[21]Föllmer H., Leukert P. Efficient hedging: Cost versus shortfall risk. / Finance and Stochastics. - 2000. - v. 4. - 2. - p.117-146

[22]Cvitanić J., Karatzas I. On dynamic measures of risk. / Finance and Stochastics. - 1999. - v.3. - 4. - p.451-482

[23]Cvitanić J. Minimizing expected loss of hedging in incomplete and constraint markets. / SIAM Journal on Control and Optimization. - 2000. - v.38 - 4. - p.1050-1066.

[24]Leung T., Song Q., Yang J. Outperformance portfolio optimization via the equivalence of pure and randomized hypothesis testing. / Finance and Stochastics. - 2013. - v.17. - 4. - p.839-870

[25]Azanov V.M., Kan Yu. S. Bilateral Estimation of the Bellman Function in the Problems of Optimal Stochastic Control of Discrete Systems by the Probabilistic Performance Criterion / Automation and Remote Control. - 2018. - 2. - p.3-18 (in Russian)

Personal contribution of the author in problem's development: the results of the articles have been obtained by the dissertator personally, Khametov V.M. contributed by problem statement and by general guidance.

The list of publications on the theme of the PhD thesis

The results of the PhD thesis are published in peer-reviewed scientific editions from the list by Higher Attestation Commission and the list of high level by National Research University «High School of Economics».

The articles from the list of leading peer-reviewed scientific editions from the list by Higher Attestation Commission of Ministry of science and education of Russian Federation:

1. Zverev O.V. On conditions on fairness of optional decomposition. / Zverev O.V., Khametov V.M. // 2009. Surveys on Applied and Industrial Mathematics. vol. 16, issue 6. p. 1067-1068. (in Russian) (personal contribution of the author 0,04 pp).
2. Zverev O.V. Quantile hedging of European typo options in incomplete markets without transaction costa. Part 1. Superhrdging. / Zverev O.V., Khametov V.M. // 2014. Control problems.vol. 6. p. 31-44. (in Russian) (personal contribution of the author 0,7pp).
3. Zverev O.V. Quantile hedging of European typo options in incomplete markets without transaction costa. Part 2. Minimax hedging. / Zverev O.V., Khametov V.M. // 2015. Control problems. vol. 1, p. 47-52. (in Russian) (personal contribution of the author 0,3pp).

Other publications by the author:

4. Zverev O.V. Minimax hedging for European type options in incomplete markets (Discreet time). /Zverev O.V., Khametov V.M. // 2011. Surveys on Applied and Industrial Mathematics. vol. 18.issue 1. p. 26-54. (in Russian) (personal contribution of the author 0,8pp).
5. Zverev O.V. Minimax hedging for European type options in compact (1,S)-market. / Zverev O.V., Khametov V.M. // 2011. Surveys on Applied and Industrial Mathematics. vol. 18.issue 1. p. 121-122. (in Russian) (personal contribution of the author 0,06pp).
6. Zverev O.V. Minimax hedging for European type options in incomplete markets (Discreet time). / Zverev O.V., Khametov V.M. // 2011. Surveys on Applied and Industrial Mathematics. vol. 18.issue 2. p. 193-204. (in Russian) (personal contribution of the author 0,5pp).

The results of the research published in proceedings of scientific conferences:

1. Zverev O.V. «Quantile hedging of European option in multidimensional incomplete market without transaction costs (discrete time)»VIII Moscow international conference on operation

research, 2016, Moscow: MAKS Press. Proceedings of the conference. Vol. 1, p. 109-112. (in Russian).

2. Zverev O.V. «Quantile hedging of European option in complete market without transaction costs (discreet time)». Conference «Young economics: economic science by eyes of junior researches». 2016. Moscow: CEMI RAS. Proceedings of the conference, p. 16-17. (in Russian).

3. Zverev O.V. «Construction of the set of successful hedging for European option in incomplete multidimensional market without transaction costs». 2017. Moscow: CEMI RAS. Proceedings of the conference. p. 32-34. (in Russian).