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Reflective hyperbolic lattices

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Chapter 1. Introduction

Abstract

This thesis is devoted to classification of reflective hyperbolic lattices, which is an open problem since the 1970s. The dissertation was written under the supervision of Prof. E. B. Vinberg during my postgraduate study at the Department of Higher Algebra of the Faculty of Mechanics and Mathematics of the Moscow State University.

The thesis consists of five chapters. The first chapter is an introduction to the subject. It includes only some basic definitions along with a number of known facts and open problems as well as the formulations of the main results of the dissertation. Chapter 2 contains some auxiliary results, including a description of models of spaces of constant curvature, acute-angled polyhedra in them, discrete groups of reflections, and, finally, the fundamentals of the theory of reflective hyperbolic lattices and arithmetic hyperbolic reflection groups.

The main results of this dissertation are obtained in Chapters 3, 4, and 5. Chapter 3 gives a theoretical description of Vinberg's Algorithm and also the description of the project (our joint work with A. Yu. Perepechko) VinAl of a computer implementation of Vinberg's Algorithm for hyperbolic lattices over $\mathbb{Z}$. Finally, Chapters 4 and 5 contain the results of classification of stably reflective hyperbolic lattices of rank 4 over $\mathbb{Z}$ and $\mathbb{Z}[\sqrt{2}]$, respectively.

Discrete reflection groups

Let $\mathbb{X}^n$ be one of the three spaces of constant curvature, that is, either the Euclidean space $\mathbb{E}^n$, or the n-dimensional sphere $\mathbb{S}^n$, or the n-dimensional (hyperbolic) Lobachevsky space $\mathbb{H}^n$.

Consider a convex polytope P in the space $\mathbb{X}^n$. If we act on P by the group Γ generated by reflections in the hyperplanes of its faces it can occur that the images of this polyhedron corresponding to different elements of Γ will cover the entire space $\mathbb{X}^n$ and will not overlap with each other. In this case we say that Γ is a *discrete reflection group*, and the polytope P is *the fundamental polyhedron* for Γ. If the polytope P is *bounded* (or, equivalently, *compact*), then the group Γ is called *a cocompact reflection group*, and if the polytope P has a *finite volume*, then the group Γ is called *cofinite* or a discrete group of *finite covolume*.

Which properties characterize such polyhedra P? For example, any two hyperplanes H_i and H_j bounding P either do not intersect or form a dihedral angle equal to π/n_{ij}, where $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$.

Such polyhedra are called *Coxeter polyhedra*, since the discrete reflection groups of finite covolume (hence their finite volume fundamental polyhedra) for $\mathbb{X}^n = \mathbb{E}^n, \mathbb{S}^n$ were determined and found by H.S.M. Coxter in 1933 (see [19]).

In 1967 (see [37]), E. B. Vinberg developed his theory of discrete groups generated by reflections in the Lobachevsky spaces. He proposed new methods for studying hyperbolic reflection groups, in particular, a description of such groups in the form of the so-called Coxeter diagrams. He formulated and proved the arithmeticity criterion for hyperbolic reflection groups and constructed a number of various examples.

Arithmetic reflection groups and reflective hyperbolic lattices

Suppose $\mathbb{F}$ is a totally real number field with the ring of integers $A = \mathbb{O}_{\mathbb{F}}$. For convenience we will assume that it is a principal ideal domain.

Definition 1. *A free finitely generated A-module L with an inner product of signature $(n, 1)$ is said to be a hyperbolic lattice if, for each non-identity embedding $\sigma\colon \mathbb{F} \to \mathbb{R}$, the quadratic space $L \otimes_{\sigma(A)} \mathbb{R}$ is positive definite. (The inner product in L is associated with some admissible quadratic form.)*

Suppose that L is a hyperbolic lattice. Then the vector space $\mathbb{E}^{n,1} = L \otimes_{\mathrm{id}(A)} \mathbb{R}$ is identified with the $(n+1)$-dimensional real *Minkowski space*. The group $\Gamma = O'(L)$ of integer (that is, with coefficients in A) linear transformations preserving the lattice L and mapping each connected component of the cone

$$\mathfrak{C} = \{v \in \mathbb{E}^{n,1} \mid (v, v) < 0\} = \mathfrak{C}^{+} \cup \mathfrak{C}^{-}$$

into itself is a discrete group of motions of the *Lobachevsky space*. Here we mean the *vector model* $\mathbb{H}^n$ given as a set of points of the hyperboloid

$$\{v \in \mathbb{E}^{n,1} \mid (v, v) = -1\},$$

in the convex open cone $\mathfrak{C}^{+}$. The group of motions is $\mathrm{Isom}(\mathbb{H}^n) = O'(n, 1)$, which is the group of pseudoorthogonal transformations of the space $\mathbb{E}^{n,1}$ that leaves invariant the cone $\mathfrak{C}^{+}$.

It is known from the general theory of arithmetic discrete groups (A. Borel & Harish-Chandra [11] and G. Mostov & T. Tamagawa [24] in 1962) that if $\mathbb{F} = \mathbb{Q}$ and the lattice L is isotropic (that is, the quadratic form associated with it represents zero), then the quotient space $\mathbb{H}^n/\Gamma$ (the fundamental domain of Γ) is not compact, but is of finite volume (in this case we say that Γ is *a discrete subgroup of finite covolume*), and in all other cases it is compact. For $\mathbb{F} = \mathbb{Q}$, these assertions were first proved by B.A. Venkov in 1937 (see [16]).

Definition 2. *Two subgroups Γ_1 and Γ_2 of some group are said to be commensurable if the group $\Gamma_1 \cap \Gamma_2$ is a subgroup of finite index in each of them.*

Definition 3. *The groups Γ obtained in the above way and the subgroups of the group* $\mathrm{Isom}(\mathbb{H}^n)$ *that are commensurable with them are called arithmetic discrete groups of the simplest type. The field $\mathbb{F}$ is called the definition field (or the ground field) of the group Γ (and all subgroups commensurable with it).*

A primitive vector e of a quadratic lattice L is called a *root* or, more precisely, a *k-root*, where $k = (e, e) \in A_{>0}$ if $2(e, x) \in kA$ for all $x \in L$. Every root e defines an *orthogonal reflection* (called a *k-reflection* if $(e, e) = k$) in the space $L \otimes_{\mathrm{id}(A)} \mathbb{R}$

$$\mathcal{R}_e : x \mapsto x - \frac{2(e, x)}{(e, e)} e,$$

which preserves the lattice L. In the hyperbolic case, R_e determines the reflection of the space $\mathbb{H}^n$ with respect to the hyperplane

$$H_e = \{x \in \mathbb{H}^n \mid (x, e) = 0\},$$

called the *mirror* of the reflection $\mathcal{R}_e$.

We denote by $O_r(L)$ the subgroup of the group $O'(L)$ generated by all reflections contained in it.

Definition 4. *A hyperbolic lattice L is said to be reflective if the index $[O'(L) : O_r(L)]$ is finite.*

Theorem 1. *(Vinberg, 1967, see [37])*
A discrete reflection group of finite covolume is an arithmetic group with a ground field $\mathbb{F}$ (or an $\mathbb{F}$-arithmetic reflection group) if it is a subgroup of finite index in a group of the form $O'(L)$, where L is some (automatically reflective) hyperbolic lattice over a totally real number field $\mathbb{F}$.

Now we formulate some fundamental theorems on the existence of arithmetic reflection groups and cocompact reflection groups in the Lobachevsky spaces.

Theorem 2. *(Vinberg, 1984, see [41])*

1. *Compact Coxeter polyhedra do not exist in the Lobachevsky spaces $\mathbb{H}^n$ for $n \geq 30$.*
2. *Arithmetic reflection groups do not exist in the Lobachevsky spaces $\mathbb{H}^n$ for $n \geq 30$.*

The next important result belongs to several authors.

Theorem 3. *For each $n \geq 2$, up to scaling, there are only finitely many reflective hyperbolic lattices. Similarly, up to conjugacy, there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$.*

The proof of this theorem is divided into the following stages:

- 1980, 1981 – V. V. Nikulin proved that there are only finitely many maximal arithmetic reflection groups in the spaces $\mathbb{H}^n$ for $n \geq 10$, see [26, 28];
- 2005 – D. D. Long, C. Maclachlan and A. W. Reid proved the finiteness of maximal arithmetic reflection groups in dimension $n = 2$, see [21];
- 2005 — I. Agol proved the finiteness in dimension $n = 3$, see [1];
- 2007 — V. V. Nikulin proved by induction the finiteness in the remaining dimensions $4 \leq n \leq 9$, see [31];
- 2008 — I. Agol, M. Belolipetsky, P. Storm, and K. Whyte independently proved the finiteness theorem for all dimensions by their spectral method, see [2] (see also the recent survey [7] of M. Belolipetsky).

The above results give the hope that all reflective hyperbolic lattices, as well as maximal arithmetic hyperbolic reflection groups can be classified.

Open problems

Our discussion above leads us to the following fundamental open problems connected with the theory of discrete reflection groups and Coxeter polytopes in the Lobachevsky spaces $\mathbb{H}^n$.

Problem 1. *Which is the maximal dimension of the Lobachevsky space in which there exist compact Coxeter polytopes? A similar question is open for Coxeter polytopes of finite volume.*

Problem 2. *Classification of reflective hyperbolic lattices and maximal arithmetic hyperbolic reflection groups.*

Remark 1. *The problem of classification of reflective hyperbolic lattices was actually posed in cited work of Vinberg in 1967. Further results obtained in the 1970-80s (and also some recent results) definitely confirm that there is a hope to solve these problems.*

A very efficient tool for solving problems 1 and 2 is Vinberg's Algorithm (1972, see [38]) of constructing the fundamental polyhedron for a hyperbolic reflection group. Practically it is efficient for arithmetic reflection groups. It enables one given a lattice to determine if this lattice is reflective.

The record example of a compact Coxeter polyhedron was found by V.O. Bugaenko for $n = 8$ (see [15]), although the maximal possible dimension is bounded by the inequality $n < 30$ (see Theorem 2 above).

A record example of a Coxeter polyhedron of finite volume belongs to R. Borcherds in the dimension $n = 21$ (see [12]). It is known that the Coxeter polytopes of finite volume can exist only for $n < 996$ (see papers [33] of M. Prokhorov and [20] A. Khovanskii, 1986).

Both examples came from arithmetic reflection groups. Bugaenko's polyhedron is the fundamental polyhedron for some arithmetic reflection group over the field $\mathbb{Q}[\sqrt{2}]$ in the space $\mathbb{H}^8$, the Borcherds polyhedron is the fundamental polyhedron for some arithmetic group of reflections over a field $\mathbb{Q}$ in the space $\mathbb{H}^{21}$.

Moreover, D. Allcock, using an elegant and a simple doubling trick, has constructed infinite series (see [3]) of finite volume Coxeter polytopes in Lobachevsky spaces through dimension 19, and also of compact Coxeter polytopes through dimension 6. We also note that in dimensions 7 and 8 they can be taken to be either arithmetic or nonarithmetic.

As for the second problem, it is also far from being completely solved. An effective description of all discrete reflection groups in the spaces $\mathbb{H}^n$ is obtained only for $n = 2$ (H. Poincaré, 1882, see [32]) and for $n = 3$ (the famous theorems of E. M. Andreev, 1970, see [5] and [6]).

In the classification of arithmetic hyperbolic reflection groups a more significant success has been achieved. Over the definition field $\mathbb{Q}$, the reflective hyperbolic lattices are classified for $n = 2$ (V.V. Nikulin, 2000, see [30], and D. Allcock, 2011, see [4]), $n = 4$ (R. Sharlau and C. Walhorn, 1989–1993, see [35, 43]), $n = 5$ (I. Turkalj, 2017, see [36]) and in the noncompact (isotropic) case for $n = 3$ (R. Sharlau and C. Walhorn, 1989–1993, see [34, 35]).

A classification of reflective hyperbolic lattices of signature $(2, 1)$ with the definition field $\mathbb{Q}[\sqrt{2}]$ was obtained by A. Mark in 2015, see [22, 23].

In all other cases, Problem 2 remains open.

Chapter 2. Discrete reflection groups

Chapter 2 contains some auxiliary results, including a description of models of spaces of constant curvature, acute-angled polyhedra in them, discrete groups of reflections, and, finally, the fundamentals of the theory of reflective hyperbolic lattices and arithmetic reflection groups. Here we define the Coxeter diagrams and we also give a list of connected elliptic and parabolic Coxeter diagrams.

Chapter 3. Vinberg's Algorithm

Project VinAl: for hyperbolic lattices over $\mathbb{Z}$

This chapter is devoted to Vinberg's algorithm and the creation of a tool for solving problem 1 and 2. With the help of different computer implementations of Vinberg's algorithm, the reflectivity was investigated for dozens of hyperbolic lattices over $\mathbb{Z}$ and $\mathbb{Z}[\sqrt{2}]$. In this way, was obtained a large number of previously unknown arithmetic compact Coxeter polytopes in Lobachevsky spaces.

As mentioned above, Vinberg's algorithm is an efficient tool for constructing the fundamental polyhedra for arithmetic reflection groups. Some efforts to implement Vinberg's algorithm by using a computer have been made since the 1980s, but all of them dealt with particular lattices, usually with an orthogonal basis. Such programs are mentioned, e.g., in the papers of Bugaenko (1992, see [15]), Scharlau and Walhorn (1989–1993, see [35]), Nikulin (2000, see [30]), and Allcock (2011, see [4]). But the programs themselves have not been published; the only exception is Nikulin's paper, which contains a program code for lattices of several different special forms. The only known implementation published together with a detailed documentation is Guglielmetti's 2016 program[1], processing hyperbolic lattices with an orthogonal basis with square-free invariant factors over several ground fields. Guglielmetti used this program in his thesis to classify reflective hyperbolic lattices with an orthogonal basis with small inner squares of its elements. His program works fairly efficiently in all dimensions in which reflective lattices exist.

In this paper, we present our own implementation of Vinberg's algorithm for arbitrary integral (with the ground field $\mathbb{Q}$) hyperbolic lattices subject to no constraints. The project is written jointly with A.Yu. Perepechko in the Sage computer algebra system. It is available in the Internet (see [9]), and it was published with a detailed description (see [45]).

The program was tested on a large number of known examples of reflective hyperbolic lattices. We have also found a series of new reflective lattices.

Some results yielded by the program are presented in Table 1. In the table, $U = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ denotes the standard two-dimensional hyperbolic lattice and A_n denotes the Euclidean root lattice of type A_n. All lattices in this table, excepting $[-1] \oplus A_3$ and $[-4] \oplus A_3$, are new.

Moreover, we have proved the reflectivity of the lattices

$$[-2] \oplus A_2 \oplus \underbrace{[1] \oplus \ldots \oplus [1]}_{n-1}$$

for $n \leq 6$.

At present, the program works effectively for $2 \leq n \leq 5$. Thus, it turns out to be useful, e.g., for solving the open problem of classifying reflective lattices in the dimension $n = 3$; it has already been successfully applied by the author to obtain partial classification results. We plan to optimize the program so as to make it efficient for $n \leq 10$.

[1] see project AlVin https://rgugliel.github.io/AlVin

Table 1: Lattices of type $[-k] \oplus A_3$, $[-k] \oplus [1] \oplus A_2$ for some $k \leq 15$, and $U \oplus [36] \oplus [6]$.

L	# faces	t (sec)
$[-1] \oplus A_3$	4	0,7
$[-2] \oplus A_3$	5	1,9
$[-3] \oplus A_3$	5	1,0
$[-4] \oplus A_3$	4	0,66
$[-5] \oplus A_3$	6	1,56
$[-6] \oplus A_3$	6	1,5
$[-8] \oplus A_3$	7	1,72
$[-9] \oplus A_3$	9	79,5
$[-10] \oplus A_3$	12	1,72
$[-12] \oplus A_3$	5	1.02
$[-15] \oplus A_3$	12	28,7
$U \oplus [36] \oplus [6]$	15	56,6

L	# faces	t (sec)
$[-1] \oplus [1] \oplus A_2$	4	0,6
$[-2] \oplus [1] \oplus A_2$	6	0,8
$[-3] \oplus [1] \oplus A_2$	5	0,6
$[-4] \oplus [1] \oplus A_2$	5	1,02
$[-5] \oplus [1] \oplus A_2$	7	1,9
$[-6] \oplus [1] \oplus A_2$	8	1,2
$[-7] \oplus [1] \oplus A_2$	11	19,2
$[-8] \oplus [1] \oplus A_2$	6	1,02
$[-9] \oplus [1] \oplus A_2$	5	0,9
$[-10] \oplus [1] \oplus A_2$	11	11
$[-15] \oplus [1] \oplus A_2$	15	44
$[-30] \oplus [1] \oplus A_2$	20	36,6

Table 2: Unimodular lattices over $\mathbb{Q}[\sqrt{13}]$ и $\mathbb{Q}[\sqrt{17}]$.

L	n	# faces
$[-\frac{3+\sqrt{13}}{2}] \oplus [1] \oplus \ldots \oplus [1]$	2	4
$[-\frac{3+\sqrt{13}}{2}] \oplus [1] \oplus \ldots \oplus [1]$	3	9
$[-\frac{3+\sqrt{13}}{2}] \oplus [1] \oplus \ldots \oplus [1]$	4	40

L	n	# faces
$[-4-\sqrt{17}] \oplus [1] \oplus \ldots \oplus [1]$	2	4
$[-4-\sqrt{17}] \oplus [1] \oplus \ldots \oplus [1]$	3	6
$[-4-\sqrt{17}] \oplus [1] \oplus \ldots \oplus [1]$	4	20

Vinberg's Algorithm for hyperbolic lattices over $\mathbb{Z}[\sqrt{d}]$

Since we also investigate the reflectivity of lattices over $\mathbb{Z}[\sqrt{2}]$, the author decided to write a program for Vinberg's Algorithm over quadratic fields. At the moment the author has a program for lattices over $\mathbb{Z}[\sqrt{2}]$, which requires some minor editing for each new lattice. This program enables one to investigate a lattice without orthogonal bases.

For lattices with an orthogonal basis was used the program of Guglielmetti mentioned above. The work of author's program was partly verified on the lattices from Table 5. In the nearest future we plan to merge the author's programs for lattices over $\mathbb{Z}[\sqrt{2}]$ with the VinAl project. Further work on the project that implements Vinberg's algorithm for arbitrary lattices over the quadratic fields $\mathbb{Z}[\sqrt{d}]$ is being carried out jointly with A. Yu. Perepechko.

As the result of experiments with different programs were obtained some new series of reflective hyperbolic lattices of different ranks over different quadratic fields. Some of these results were obtained jointly with A. A. Kolpakov in 2017–2018.

The results obtained are presented in Tables 2–5. In these tables we indicate first the form of the lattice of signature $(n, 1)$, then we specify the dimension n of the corresponding Lobachevsky space, and then the number of faces for the fundamental Coxeter polytope of the corresponding reflection group.

Chapter 4. Stably reflective hyperbolic $\mathbb{Z}$-lattices of rank 4

Definition 5. *A number $k \in A$, $k > 0$ is said to be stable if $k \mid 2$ in the ring A.*

Table 3: Some lattices over $\mathbb{Q}[\sqrt{5}]$.

L	n	# faces	L	n	# faces
$[-1-\sqrt{5}]\oplus[1]\oplus\ldots\oplus[1]$	2	4	$[-1-\sqrt{5}]\oplus[2]\oplus\ldots\oplus[2]\oplus[1]$	2	4
$[-1-\sqrt{5}]\oplus[1]\oplus\ldots\oplus[1]$	3	5	$[-1-\sqrt{5}]\oplus[2]\oplus\ldots\oplus[2]\oplus[1]$	3	5
$[-1-\sqrt{5}]\oplus[1]\oplus\ldots\oplus[1]$	4	7	$[-1-\sqrt{5}]\oplus[2]\oplus\ldots\oplus[2]\oplus[1]$	4	7
$[-1-\sqrt{5}]\oplus[1]\oplus\ldots\oplus[1]$	5	13	$[-1-\sqrt{5}]\oplus[2]\oplus\ldots\oplus[2]\oplus[1]$	5	13
$[-1-\sqrt{5}]\oplus[1]\oplus\ldots\oplus[1]$	6	18	$[-1-\sqrt{5}]\oplus[2]\oplus\ldots\oplus[2]\oplus[1]$	6	18

Table 4: Some lattices over $\mathbb{Q}[\sqrt{2}]$.

L	n	# faces
$[-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]$	2	4
$[-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	3	6
$[-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	4	10
$[-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	5	31

L	n	# faces
$[-7-5\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	2	3
$[-7-5\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	3	5
$[-7-5\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	4	7
$[-7-5\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	5	11
$[-7-5\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	6	45

Table 5: Some lattices over $\mathbb{Q}[\sqrt{2}]$.

L	n	# faces	L	n	# faces
$[-\sqrt{2}]\oplus[1]\oplus[1]$	2	4	$[-1-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]$	2	4
$[-\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	3	6	$[-1-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	3	6
$[-\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	4	8	$[-1-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	4	8
$[-\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	5	27	$[-1-\sqrt{2}]\oplus[2+\sqrt{2}]\oplus[1]\oplus\ldots\oplus[1]$	5	27

For example, if $\mathbb{F} = \mathbb{Q}, A = \mathbb{Z}$, then the definition holds for numbers $k \leq 2$. For $A = \mathbb{Z}[\sqrt{2}]$ the stable numbers are 1, 2, and $2 + \sqrt{2}$.

Definition 6. *A reflection R_e is called stable if the number (e, e) is stable.*

Let L be a hyperbolic lattice over a ring of integers A. We denote by $S(L)$ the subgroup of $O'(L)$ generated by all stable reflections.

Definition 7. *A hyperbolic lattice L is said to be stably reflective if the index $[O(L) : S(L)]$ is finite.*

Remark 2. *In the articles [8], [44], and [10], stably reflective $\mathbb{Z}$-lattices are called $(1,2)$-reflective, since for $A = \mathbb{Z}$ only the numbers 1 and 2 are stable.*

Definition 8. *A hyperbolic $\mathbb{Z}$-lattice L is called 2-reflective if the group $O'(L)$ is up to finite index generated by 2-reflections.*

Remark 3. *All 2-reflective hyperbolic $\mathbb{Z}$-lattices are already classified: for rank $\neq 4$ this was done by V. V. Nikulin in 1979, 1981 and 1984, see [25, 27, 29], and for the rank 4 this was done by E. B. Vinberg in 1981–2007 (see [42]). Presumably, stably reflective lattices should form wider class of reflective lattices then 2-reflective.*

The main task in this chapter is a classification of stably reflective hyperbolic $\mathbb{Z}$-lattices of rank 4. The author hopes that the *method of the outermost edge* (which is a modification of the *method of narrow parts of polyhedra*, applied by V. V. Nikulin) will be applicable for classifying all reflective anisotropic hyperbolic lattices of rank 4.

Let P be an acute-angled compact polytope in $\mathbb{H}^3$ and let E be some edge of it. We denote by F_1 and F_2 the faces of the polytope P, containing the edge E. Let u_3 and u_4 be the unit external normals to the faces F_3 and F_4 containing the vertices of the edge E, but not the edge itself.

Definition 9. *The faces F_3 and F_4 are called the framing edges of the edge E, and the number $|(u_3, u_4)|$ is its width.*

We associate with the edge E the set $\bar{\alpha} = (\alpha_{12}, \alpha_{13}, \alpha_{23}, \alpha_{14}, \alpha_{24})$, where α_{ij} is the angle between the faces F_i and F_j.

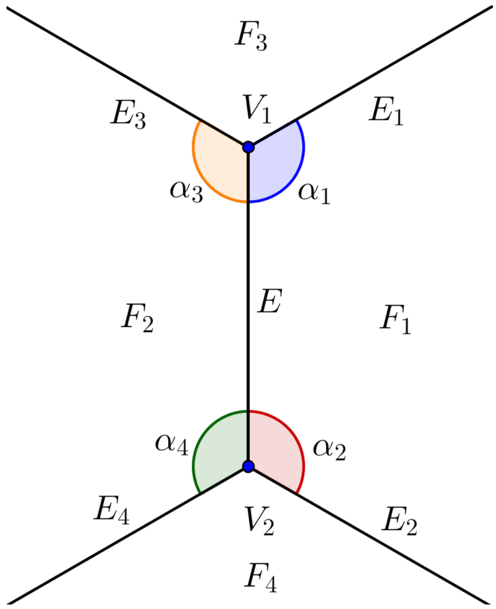
Theorem 4. *The fundamental polyhedron of every $\mathbb{Q}$-arithmetic cocompact group of reflections in $\mathbb{H}^3$ has an edge of width less than 4.14.*

In fact, a stronger result is obtained. Namely, it is proved that there is an edge of width $t_{\bar{\alpha}}$, where $t_{\bar{\alpha}} \leq 4.14$ is a number depending on the set $\bar{\alpha}$ of dihedral angles around this edge.

To obtain this result, the following method is used. Let P be the fundamental polytope of a $\mathbb{Q}$-arithmetic cocompact reflection group in $\mathbb{H}^3$. Following Nikulin, we consider a point O inside the polyhedron P. Let E be *the outermost*[2] *edge from it*. We denote the vertices of the edge E by V_1 and V_2, and the dihedral angles between the faces F_i and F_j will be denoted by α_{ij}.

Let E_1 and E_3 be the edges of the polytope P outgoing from the vertex V_1 and let E_2 and E_4 be the edges outgoing from V_2 such that the edges E_1 and E_2 lie in the face F_1. The length

Fig. 1. *The outermost edge*



of the edge E is denoted by a, and the plane angles between the edges E_j and E are denoted by α_j (see Figure 1).

The following result is true for an arbitrary compact acute-angled polytope in $\mathbb{H}^3$.

Theorem 5. *The length of the outermost edge satisfies the inequality*

$$a < \operatorname{arcsinh}\left(\frac{\tanh(\ln(\operatorname{ctg}(\frac{\alpha_{12}}{4})))}{\tan\left(\frac{\alpha_3}{2}\right)}\right) + \operatorname{arcsinh}\left(\frac{\tanh(\ln(\operatorname{ctg}(\frac{\alpha_{12}}{4})))}{\tan\left(\frac{\alpha_4}{2}\right)}\right).$$

Then it remains to estimate, by using a linear inequality, the width of the edge through its length. To do this, we use the fact that we initially considered the fundamental polyhedron of the $\mathbb{Q}$-arithmetic cocompact reflection group in $\mathbb{H}^3$. As we see, the estimates in Theorem 5 depend on the set of angles around this edge, therefore, the estimates for the width of the edge also depend on it.

To formulate the results of classification of stably reflective lattices we introduce some notation for hyperbolic lattices:

- $[C]$ is a quadratic lattice whose inner product in some basis is given by a symmetric matrix C,
- $d(L) := \det C$ is the discriminant of the lattice $L = [C]$,
- $L \oplus M$ is the orthogonal sum of the lattices L and M,
- $[k]L$ is the quadratic lattice obtained from L by multiplying all inner products by $k \in A$. *is the adjoint lattice.*

[2]In an acute-angled polyhedron the distance from an interior point to a face (of any dimension) is equal to the distance to the plane of this face.

Theorem 6. *Any stably reflective anisotropic hyperbolic lattice of rank* 4 *over* $\mathbb{Z}$ *is either isomorphic to* $[-7] \oplus [1] \oplus [1] \oplus [1]$ *or* $[-15] \oplus [1] \oplus [1] \oplus [1]$*, or to an even index* 2 *sublattice of one of them.*

Actually, these lattices are even 2-reflective (see [42]).

Chapter 5. Stably reflective $\mathbb{Z}[\sqrt{2}]$-lattices of rank 4

Theorem 7. *The fundamental polyhedron of any* $\mathbb{Q}[\sqrt{2}]$*-arithmetic group of reflections in* $\mathbb{H}^3$ *has an edge of width less than* 4.14.

As above, actually a stronger result is obtained. Namely, it is proved that there is an edge of width $t_{\bar{\alpha}}$, where $t_{\bar{\alpha}} \le 4.14$ is a number depending on the set $\bar{\alpha}$ of dihedral angles around this edge.

Theorem 8. *Any maximal stably reflective hyperbolic lattice of rank* 4 *over* $\mathbb{Z}[\sqrt{2}]$ *is isomorphic to one of the following seven lattices:*

№	*L*	*# faces*	*Discriminant*
1	$[-1-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	5	$-1-\sqrt{2}$
2	$[-1-2\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	6	$-1-2\sqrt{2}$
3	$[-5-4\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	5	$-5-4\sqrt{2}$
4	$[-11-8\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	17	$-11-8\sqrt{2}$
5	$[-\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	6	$-\sqrt{2}$
6	$\begin{bmatrix} 2 & -1 & -\sqrt{2} \\ -1 & 2 & \sqrt{2}-1 \\ -\sqrt{2} & \sqrt{2}-1 & 2-\sqrt{2} \end{bmatrix} \oplus [1]$	6	$-\sqrt{2}$
7	$[-7-5\sqrt{2}] \oplus [1] \oplus [1] \oplus [1]$	5	$-7-5\sqrt{2}$

Approbation of the work

The results of the thesis have been reported at the following meetings:

- the seminar "Lie groups and invariant theory", led by E.B. Vinberg, D.A. Timashev and I.V. Arzhantsev, the Faculty of Mechanics and Mathematics, Moscow State University, May 2016 and October 2017;
- the Sixth School-Conference "Lie Algebras, Algebraic Groups and Invariant Theory", MSU & IUM, Moscow, Russia, January–February 2017;
- S.P. Novikov's Seminar "Geometry, topology and mathematical physics", the Faculty of Mechanics and Mathematics, Moscow State University, March 2017;
- the international conference "Geometry and Topology" in honor of C. Bavard, Institute of Mathematics, Bordeaux, France, November 2017;
- the seminar "Hyperbolic geometry and combinatorial structures", Institute of Mathematics, Neuchatel, Switzerland, November 2017;

- the seminar “Automorphic forms and their applications”, led by V.A. Grytsenko, the Faculty of Mathematics, HSE, Moscow, Russia, February 2018;
- the international conference “Automorphic forms and algebraic geometry”, PDMI Steklov Institute of RAS, St. Petersburg, Russia, May 2018.

References

[1] Ian Agol, Finiteness of arithmetic Kleinian reflection groups. In Proceedings of the International Congress of Mathematicians: Madrid, August 22–30, 2006: invited lectures, pages 951–960, 2006.

[2] Ian Agol, Mikhail Belolipetsky, Peter Storm, and Kevin Whyte. Finiteness of arithmetic hyperbolic reflection groups. — Groups Geom. Dyn., 2008, Vol. 2(4), p. 481 — 498.

[3] D. Allcock. “Infinitely many hyperbolic Coxeter groups through dimension 19”, Geom. Topol. 10 (2006), 737–758.

[4] D. Allcock. The reflective Lorentzian lattices of rank 3. — Mem. Amer. Math. Soc. 220, no 1033., American Mathematical Society, 2012, p. 1 — 125.

[5] E. M. Andreev. Convex polyhedra in Lobachevski spaces. Mat. Sb, 1970, 81, p. 445–478.

[6] E. M. Andreev. Convex polyhedra of finite volume in Lobachevski space. Mat. Sb. , 1970, 83, p. 256–260.

[7] M. Belolipetsky. Arithmetic hyperbolic reflection groups. — Bulletin (New Series) of the Amer. Math. Soc., 2016, Vol. 53 (3), p. 437 — 475.

[8] N. V. Bogachev. Reflective anisotropic hyperbolic lattices of rank 4. ArXiv: https://arxiv.org/abs/1610.06148v1

[9] N. Bogachev, A. Perepechko, *Vinberg’s algorithm*, DOI:10.5281/zenodo.1098448, https://github.com/aperep/vinberg-algorithm, 2017.

[10] N. V. Bogachev, Classification of $(1,2)$-Reflective Anisotropic Hyperbolic Lattices of Rank 4 — Izvestiya Mathematics, 2019, Vol. 81:1, p. 3–24.

[11] Armand Borel and Harish-Chandra. Arithmetic subgroups of algebraic groups. Ann. of Math. (2), 75:485–535, 1962.

[12] R. Borcherds, Automorphism groups of Lorentzian lattices, J. Algebra 111 (1987), 133–153.

[13] V. O. Bugaenko. Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring $\mathbb{Z}[(\sqrt{5}+1)/2]$. Vestnik Moskov. Univ. Ser. I Mat. Mekh., (5):6–12, 1984.

[14] V. O. Bugaenko. On reflective unimodular hyperbolic quadratic forms. Selecta Math. Soviet., 9(3):263–271, 1990. Selected translations.

[15] V. O. Bugaenko. Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices. — Advances in Soviet Mathematics, 1992, Volume 8, p. 33 — 55.

[16] Б. А. Венков. Об арифметической группе автоморфизмов неопределенной квадратичной формы. — Изв. АН СССР, 1937, том 1, выпуск 2, стр. 139–170

[17] R. Guglielmetti. Hyperbolic isometries in (in-)finite dimensions and discrete reflection groups: theory and computations. — Switzerland, PhD Thesis, University of Fribourg, 2017.

[18] Frank Esselmann. Über die maximale Dimension von Lorentz-Gittern mit coendlicher Spiegelungsgruppe. — Journal of Number Theory, 1996, Vol. 61, p. 103 — 144.

[19] H. S. M. Coxeter. Discrete groups generated by reflections, — Ann. of Math. (2), 35:3 (1934), 588–621.

[20] A.G. Khovanskii, Hyperplane sections of polyhedra, toroidal manifolds, and discrete groups in Lobachevskii space, Functional Analysis and Its Applications 20 (1986), no. 1, p. 41–50.

[21] D.D. Long, C. Maclachlan, and A.W. Reid. Arithmetic fuchsian groups of genus zero. Pure and Applied Mathematics Quarterly, 2(2):569–599, 2006.

[22] A. Mark. The classification of rank 3 reflective hyperbolic lattices over $\mathbb{Z}[\sqrt{2}]$ — Mat. Proc. Camb. Phil. Soc. 12, 2016, p. 1–37.

[23] A. Mark. The classification of rank 3 reflective hyperbolic lattices over $\mathbb{Z}[\sqrt{2}]$, Ph.D. thesis, University of Texas at Austin, 2015.

[24] G. D. Mostow and T. Tamagawa. On the compactness of arithmetically defined homogeneous spaces. Ann. of Math, 1962, Vol.76, No. 3, pp. 446–463.

[25] V. V. Nikulin. Quotient-groups of groups of automorphisms of hyperbolic forms of subgroups generated by 2-reflections. Dokl. Akad. Nauk SSSR 1979, Vol. 248, N. 6, p. 1307–1309.

[26] V. V. Nikulin. On the arithmetic groups generated by reflections in Lobachevski spaces. Izv. Akad. Nauk SSSR Ser. Mat., 1980, Vol. 44, N 3, p. 637–669.

[27] V. V. Nikulin. Quotient-groups of groups of automorphisms of hyperbolic forms by subgroups generated by 2-reflections. Algebro-geometric applications. In Current problems in mathematics, Vol. 18, p. 3–114. Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Informatsii, Moscow, 1981

[28] V.V. Nikulin. On the classification of arithmetic groups generated by reflections in Lobachevski spaces. Izv. Akad. Nauk SSSR Ser. Mat., 45(1): p. 113–142, 240, 1981.

[29] V.V. Nikulin. K3 surfaces with a finite group of automorphisms and a Picard group of rank three. Trudy Mat. Inst. Steklov., 165:119–142, 1984. Algebraic geometry and its applications.

[30] V. V. Nikulin. On the classification of hyperbolic root systems of rank three. Tr. Mat. Inst. Steklova, 230:256, 2000.

[31] V. V. Nikulin. Finiteness of the number of arithmetic groups generated by reflections in Lobachevski spaces. Izv. Ross. Akad. Nauk Ser. Mat., 71(1): p. 55–60, 2007.

[32] H. Poincare . Theorie des groupes fuchsiennes.— Acta math., 1882, 1, p. 1—62.

[33] M. N. Prokhorov. Absence of discrete groups of reflections with a noncompact fundamental polyhedron of finite volume in a Lobachevski space of high dimension. Izv. Akad. Nauk SSSR Ser. Mat., 50(2): p. 413–424, 1986.

[34] Rudolf Scharlau. On the classification of arithmetic reflection groups on hyperbolic 3-space. — Preprint, Bielefeld, 1989.

[35] R. Scharlau, C. Walhorn. Integral lattices and hyperbolic reflection groups. — Asterisque. 1992, V.209, p. 279–291.

[36] Ivica Turkalj. Reflective Lorentzian Lattices of Signature $(5, 1)$. — Dissertation, 2017, Technische Universität Dortmund.

[37] E. B. Vinberg. Discrete groups generated by reflections in Lobachevski spaces. Mat. Sb., 1967, Vol. 72(114), N 3, p. 471 — 488.

[38] E. B. Vinberg. The groups of units of certain quadratic forms. Mat. Sb., 1972, Vol. 87, p. 18 — 36

[39] È. B. Vinberg, On unimodular integral quadratic forms, Funct. Anal. Appl., 6:2 (1972), p. 105–111

[40] E. B. Vinberg. Some arithmetical descrete groups in Lobachevskii spaces. — In: Proc. Int. Coll. on Discrete Subgroups of Lie Groups and Appl. to Moduli (Bombay, January 1973). — Oxford: University Press, 1975, p. 323 — 348.

[41] E. B. Vinberg. Absence of crystallographic groups of reflections in Lobachevski spaces of large dimension. — Transaction of Moscow Math. Soc., 1984, T. 47, p. 68 — 102.

[42] E. B. Vinberg. Classification of 2-reflective hyperbolic lattices of rank 4. Tr. Mosk. Mat. Obs., 2007, Vol.68, p. 44 – 76.

[43] Claudia Walhorn. Arithmetische Spiegelungsgruppen auf dem 4-dimensionalen hyperbolischen Raum. — PhD thesis, Univ. Bielefeld, 1993.

The dissertation related publications by the author in peer-reviewed journals indexed by the Web of Science, Scopus and MathSciNet

[44] N.V. Bogachev, Reflective Anisotropic Hyperbolic Lattices of Rank 4 — Uspekhi Matematicheskikh Nauk 72:1 (433),193–194 (2017) [Russian Mathematical Surveys 72:1,179–181 (2017)].

[45] N. V. Bogachev and A. Ju. Perepechko, Vinberg's Algorithm for Hyperbolic Lattice, Matematicheskie Zametki, 2018, V. 103:5, p.769–773 [English transl.: Mathematical Notes, 2018, Vol. 103:5, 836–840].