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As a manuscript

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ASYMPTOTIC ANALYSIS OF GAUSSIAN CHAOS PROCESSES

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics HSE

Academic supervisor:
Doctor of Science,
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Moscow – 2019

Work description

Work relevance.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. We are interested in an asymptotic behavior of the probabilities of high extrema for the *Gaussian chaos processes*

$$P_{\boldsymbol{\xi}}(u,T) := \mathbf{P}\left(\sup_{t\in[0,T]} g(\boldsymbol{\xi}(t)) > u\right),$$

as $u \to \infty$, where $\boldsymbol{\xi}(t) = (\xi_1(t), \dots, \xi_d(t))^\top \in \mathbb{R}^d,\ d \geq 2,\ t \in \mathbb{R}^+$, be an a.s. continuous Gaussian vector process on $(\Omega, \mathcal{F}, \mathbf{P})$ with values in a measurable space $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$, and $g(\cdot) : \mathbb{R}^d \to \mathbb{R}$ be a homogeneous function of a positive order $\beta > 0$, such as for some $\mathbf{x} \in \mathbb{R}^d :\ g(\mathbf{x}) > 0$. The homogenity of order $\beta > 0$ means, that for every $a \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^d :\ g(a\mathbf{x}) = |a|^\beta g(\mathbf{x})$. We suppose, that the function $g(\cdot)$ is a $\mathcal{B}(\mathbb{R}^d)|\mathcal{B}(\mathbb{R})-$measurable. In this case we call the random process $g(\boldsymbol{\xi}(t))$ the *Gaussian chaos process*. Traditionally this notion goes to Wiener [2], who was first to introduce polynomial chaos processes ($g(\cdot)$ is a homogeneous polynomial of a positive order $\beta > 0$). We follow to the extended notion of Gaussian chaos.

Special cases of the problem have been studied before. Specific characteristics of the vector process $\boldsymbol{\xi}(t)$ was important. For example, the behavior of its covariance matrix.

In V. I. Piterbarg [7], [8], V. I. Piterbarg and S. Stamatovich [9] the case of stationary χ^2-processes have been studied ($g(\cdot)$ is a positive-definite quadratic form). The case when $g(\cdot)$ is a difference of two independent stationary χ^2-processes have been studied in paper P. Albin, E. Hashorva, Lanpeng Ji and Chengxiu Ling [10] that was published during this research. In V. I. Piterbarg [13] the problem was solved for an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ with i.i.d. components.

The problem of an asymptotic behavior of the probabilities of high extrema for Gaussian processes was also considered by S. M. Berman [15], [16] and Albin [18]. In case when an a.s. continuous Gaussian vector process is a differentiable in square mean, the main method of the research is *the method of moments* or *Rice's method* [17], that was also considered in detail in V. I. Piterbarg [11].

We suppose, that the Gaussian vector process $\boldsymbol{\xi}(t)$ is an a.s. continuous, zero-mean and stationary, meaning that the covariance matrix of the vector process at points $t \geq 0$ and $s \geq 0$ depends on $(t - s)$

$$R(t,s) := E\boldsymbol{\xi}(t)\boldsymbol{\xi}(s)^\top = R(t-s).$$

We also suppose, that the covariance matrix $R(t)$ of an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ satisfies the following Pickands' condition

$$R(t) = R - |t|^\alpha C + |t|^\alpha o(1),\ t \to 0,\ \alpha \in (0,2], \tag{1}$$

for some positive-definite $d \times d$ matrices $R,\ C$. Here and after $o(1)$ is a $d \times d$ matrix, which components tend to zero as $t \to 0$. The main result in this area is a J. Pickands theorem [6], that was obtained in 1969. In that work J. Pickands studied the asymptotic behavior of the probabilities of high extrema for an a.s. continuous zero-mean Gaussian stationary process $\xi(t)$ with covariance function of Pickand's type, meaning that

$$r(t) := E\xi(t)\xi(0) = 1 - c|t|^\alpha + |t|^\alpha o(1),\ t \to 0,\ c > 0,\ \alpha \in (0,2]. \tag{2}$$

J. Pickands studied the asymptotic behavior of the probability

$$\mathbf{P}\left(\max_{t\in[0,T]}\xi(t)>u\right),$$

as $u\to\infty$. The method, that was developed in that work, consists of the finding the asymptotical behavior of the probabilities of high extrema on a small intervals with a common length that goes to zero as $u\to\infty$. After that we can go to the interval $[0,T]$ using local characteristics of Gaussian process near zero point and Bonferroni inequality. This method was later developed by V. I. Piterbarg [11], [12] and called *the double sum method*.

The starting point of our investigation is the Laplace method for an asymptotical analysis of Laplace integrals. Using this method we can find the asymptotic behavior of distribution tails of $g(\boldsymbol{\xi}(t_0))$ for a fixed $t_0\in[0,T]$.

An asymptotic behavior of distribution tails of Gaussian chaos variables

It is natural at first consider *Gaussian chaos variables* instead of *Gaussian chaos processes*, that is random variables $g(\boldsymbol{\xi})$, where $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_d)^\top$ is a standard Gaussian vector. For example, it is a big interest to know the asymptotic behavior of distribution tails of $\prod_{i=1}^d\xi_i$. It is known that for a standard Gaussian variable ξ with values in $(\mathbb{R},\mathcal{B}(\mathbb{R}))$, as $u\to\infty$

$$\mathbf{P}\big(\xi>u\big)=\frac{1}{\sqrt{2\pi}u}\exp\left(-\frac{u^2}{2}\right)\left(1+O\left(\frac{1}{u^2}\right)\right)=:\Psi(u)\left(1+O\left(\frac{1}{u^2}\right)\right).$$

In general case the calculation of the asymptotic behavior of such probabilities is based on the Laplace method for the estimation of Laplace integrals and some generalizations of this method, that was considered in detail in M. V. Fedoryuk [1] and R. J. Adler, J. E. Taylor [14]. Define $\boldsymbol{\varphi}=(\varphi_1,\ldots,\varphi_{d-1})\in\Pi_{d-1}:=[0,\pi)^{d-2}\times[0,2\pi)$. We will write $g(\boldsymbol{\varphi}):=g(\mathbf{x}/|\mathbf{x}|)$, where $\boldsymbol{\varphi}$ are the angular coordinates of $\mathbf{x}\in\mathbb{R}^d$ under spherical parametrization. Denote

$$g:=\sup_{\boldsymbol{\varphi}\in\Pi_{d-1}}g(\boldsymbol{\varphi})=\sup_{\mathbf{e}\in\mathbb{S}_{d-1}}g(\mathbf{e}),$$

and

$$\mathcal{M}_{\boldsymbol{\varphi}}:=\{\boldsymbol{\varphi}\in\Pi_{d-1}\colon g(\boldsymbol{\varphi})=g\},\qquad \mathcal{M}:=\{\mathbf{e}\in\mathbb{S}_{d-1}\colon g(\mathbf{e})=g\}.$$

We consider two different types of the set $\mathcal{M}$:

Case 1 *$\mathcal{M}$ consist of a finite number of points.*

Case 2 *$\mathcal{M}$ is a smooth m-dimensional manifold, $1\le m\le d-1$.*

Such conditions entail some restrictions on the smoothness of the function $g(\boldsymbol{\varphi})$. We assume that:

Condition 1 *There exists an $\varepsilon>0$ such that $g(\boldsymbol{\varphi})$ is a twice continuously differentiable in the neighbourhood*

$$\mathcal{M}_{\boldsymbol{\varphi}}(\varepsilon):=\{\boldsymbol{\varphi}\in\Pi_{d-1}\colon g(\boldsymbol{\varphi})>g-\varepsilon\}$$

of $\mathcal{M}_{\boldsymbol{\varphi}}$. In the case 1 we assume that $|\det g''(\boldsymbol{\varphi})|>0$ for every $\boldsymbol{\varphi}\in\mathcal{M}_{\boldsymbol{\varphi}}$, where

$$g''(\boldsymbol{\varphi}):=\left[\frac{\partial^2 g(\boldsymbol{\varphi})}{\partial\varphi_i\partial\varphi_j}\right]_{i,j=1,\ldots,d-1}$$

is a Hessian matrix of $g(\boldsymbol{\varphi})$ at point $\boldsymbol{\varphi}$. In the case 2 we assume that the rank of the matrix $g''(\boldsymbol{\varphi})$ equals $d-1-m$ for every $\boldsymbol{\varphi}\in\mathcal{M}_{\boldsymbol{\varphi}}$.

Denote by $g''_{d-1-m}(\boldsymbol{\varphi})$ any non-singular diagonal $(d-1-m)$-sub-matrix of $g''(\boldsymbol{\varphi})$. For every fixed $\boldsymbol{\varphi}$ determinants of all such sub-matrices are equal to each other. It is known from the paper D. A. Korshunov, V. I. Piterbarg and E. Hashorva [3], [4], [5].

Theorem 1 *Let $g(\boldsymbol{\varphi})$ satisfies **condition 1**. Let the probability density function $p_{g(\boldsymbol{\xi})}(x)$ of $g(\boldsymbol{\xi})$ exists. Then the following asymptotic relations takes place as $u \to \infty$:*

$$p_{g(\boldsymbol{\xi})}(u) = \frac{h_0}{\beta g}\left(\frac{u}{g}\right)^{\frac{m+1}{\beta}-1} \exp\left(-\frac{u^{2/\beta}}{2g^{2/\beta}}\right)(1+o(1)),$$

$$\mathbf{P}(g(\boldsymbol{\xi}) > u) = h_0\left(\frac{u}{g}\right)^{\frac{m-1}{\beta}} \exp\left(-\frac{u^{2/\beta}}{2g^{2/\beta}}\right)(1+o(1)).$$

Here

$$h_0 := \frac{1}{\sqrt{2\pi}}(\beta g)^{\frac{d-1}{2}} \sum_{\boldsymbol{\varphi}\in\mathcal{M}_{\boldsymbol{\varphi}}} \frac{J(1,\boldsymbol{\varphi})}{\sqrt{|\det g''(\boldsymbol{\varphi})|}},$$

for the case 1, and

$$h_0 := \frac{1}{(2\pi)^{(m+1)/2}}(\beta g)^{\frac{d-1-m}{2}} \int_{\mathcal{M}_{\boldsymbol{\varphi}}} \frac{J(1,\boldsymbol{\varphi})}{\sqrt{|\det g''_{d-1-m}(\boldsymbol{\varphi})|}} \mathrm{d}V_{\boldsymbol{\varphi}},$$

for the case 2, where $\mathrm{d}V_{\boldsymbol{\varphi}}$ is an elementary volume in $\mathcal{M}_{\boldsymbol{\varphi}} \subset \Pi_{d-1}$, and $J(r,\boldsymbol{\varphi})$ is a Jacobian of the mapping from euclidean coordinates in $\mathbb{R}^d$ to spherical coordinates.

This result as well as all other introduced notions will be used in our study. Because of it we consider only such homogeneous functions $g(\cdot)$ that satisfy the **condition 1**.

Purpose of the work.

We study the asymptotic behavior of the probabilities of high extrema for the *Gaussian chaos processes* for an a.s. continuous zero-mean Gaussian stationary processes $\boldsymbol{\xi}(t)$ which covariance matrix satisfies Pickands' type condition (1) near zero point. In particular, we assume that components of the vector process can be dependent.

Work structure.

Thesis consists of an introduction, where we formulate the problem, give some important results, that have been obtained earlier, and describe basic methods of research; three chapters, a conclusion and bibliography of 30 titles. Thesis consists of 82 pages.

Scientific originality.

Results presented in the work are new and obtained by the author. The motivation of work is given above.

Main results submitted for the defense.

1. The exact asymptotic for the probability of high extrema of the product of two a.s. continuous zero-mean Gaussian stationary processes with a Pickands' type covariance functions (2) near zero point with a different exponents $\alpha_1,\ \alpha_2$ have been studied (the probability of high extrema of $\xi_1(t)\xi_2(t)$). We assume that these processes can be dependent.

2. The exact asymptotic for the probability of high extrema of the quadratic form from an a.s. continuous zero-mean Gaussian stationary vector process with a Pickands' type covariance matrix (1) near zero point have been studied (the probability of high extrema of $\langle \boldsymbol{\xi}(t), A\boldsymbol{\xi}(t)\rangle$). We assume that the maximum eigenvalue of A is positive and has an order 1 and components of the vector process can be dependent.

3. The exact asymptotic for the probability of high extrema of *Gaussian chaos processes* have been studied. It means that we consider the homogeneous function $g(\cdot) : \mathbb{R}^d \to \mathbb{R}$ of a positive order and an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ with a Pickands' type covariance matrix (1) near zero point (the probability of high extrema of $g(\boldsymbol{\xi}(t))$). We assume that the function $g(\cdot)$ satisfies **condition 1**, in particular it is twice continuously differentiable in some neigbourhood of the maximum point set on a unit sphere. Components of the vector process can be dependent.

Research methods.

In our work we use both - classic results of the probability theory and the theory of random processes, such as discrete approximation method and weak convergence technique - and special methods from the theory of Gaussian processes and Calculus, such as *the double sum method*, that is based on a Pickands' theorem, Borel lemma, the method of asymptotic estimation of Laplace integrals and asymptotic estimations from above for the probabilities of high extrema of Gaussian fields with a locally-stationary structure.

Theoretical and practical value.

The work is theoretical. Results can be applied in the theory of reliability, financial field and other applications of probabilistic methods.

Approbation of work.

Main results was presented during the following scientific conferences:

1. *Scientific conference "Lomonosov readings".* High extrema of Gaussian chaos processes. Discrete time approximation approach. Moscow, April 2017.

2. *4-th International Workshop "Analysis, Geometry and Probability".* High extrema of Gaussian chaos processes. Moscow, September-October 2016.

3. *XXIII International scientific conference of students and young scientists "Lomonosov".* High extrema probabilities of Gaussian chaos processes in case of dependent components. Moscow, April 2016.

The author also presented results of the thesis work at:

1. *"Principal seminar of the department of probability theory, Lomonosov Moscow State University"*, seminar chairman: member of the Russian Academy of Science A. N. Shiryaev. Asymptotic analysis of Gaussian chaos processes. Moscow, 21 March 2018.

2. *Research seminar "Probability theory. Analytical and economic applications", HSE.* Asymptotic analysis of Gaussian chaos processes. Moscow, 31 January 2019.

3. *"Principal seminar of the department of probability theory, Lomonosov Moscow State University"*, seminar chairman: member of the Russian Academy of Science A. N. Shiryaev. High extrema of the product of two Gaussian stationary processes. Moscow, 4 March 2015.

Publications.

Main results of the work were published in 3 articles (see [19], [20], [21], 2 of them were published with V. I. Piterbarg) in a peer-reviewed journals that included in the Web of Science, Scopus. Main results of the work were also presented in an abstracts of 2 international conferences. Lists of these articles are given at the end of the text.

Acknowledgment.

This work was supervised by Doctor of Science, professor Piterbarg Vladimir I. The author express sincere gratitude to the scientific supervisor for the problem formulation, significant support and numerous discussions.

SUMMARY OF WORK

The **introduction** provides a brief overview of research the problem of asymptotic behavior of high extrema probabilities for *Gaussian chaos process* with references to corresponding scientific papers. We also describe some well known methods and results, that use in our research. In particular, we give a detailed description of result about asymptotic behavior of distribution tails of *Gaussian chaos variables* (random variables $g(\boldsymbol{\xi})$, where $g(\cdot) : \mathbb{R}^d \to \mathbb{R}$ is a homogeneous function of a positive order $\beta > 0$ and $\boldsymbol{\xi}$ is a standard Gaussian vector in $\mathbb{R}^d$).

In the **first chapter** we consider two a.s. continuous zero-mean Gaussian stationary processes $\boldsymbol{\xi}(t) = (\xi_1(t), \xi_2(t))^\top \in \mathbb{R}^2,\ t \in \mathbb{R}^+$, with covariance functions of Pickands' type:

E1 $r_i(t) = 1 - c_i|t|^{\alpha_i} + o(|t|^{\alpha_i}),\ t \to 0,\ c_i > 0,\ \alpha_i \in (0, 2],\ i = 1, 2.$

E2 $|r_i(t)| < 1,\ i = 1, 2$ for every $t \neq 0$.

Let $r(t, s) := E\xi_1(t)\xi_2(s)$. Using stationarity we write $r(t)$, that is equal to $r(t, 0)$. Denote $\alpha = \min\{\alpha_1, \alpha_2\}$ and without loss of generality assume that $\alpha_1 \le \alpha_2$. Suppose, that the joint covariance function of $\xi_1(t),\ \xi_2(s)$ satisfies the following condition:

E3 $r(t) = r - c|t|^\alpha + o(|t|^\alpha),\ t \to 0$, for some real-valued $r,\ c$.

Define Pickands' constants:

$$H_\alpha(\Lambda) := E \exp\left(\max_{t\in\Lambda}(\sqrt{2}B_{\alpha/2}(t) - |t|^\alpha)\right), \quad H_\alpha(T) := H_\alpha([0, T]), \quad H_\alpha = \lim_{T\to\infty} \frac{H_\alpha(T)}{T},$$

where Λ is an arbitrary closed set and $B_{\alpha/2}(t)$ is a fractional Brownian motion with a Hurst parameter $\alpha/2$, meaning that it is an a.s. continuous zero-mean Gaussian process, started at zero point, such as

$$E(B_{\alpha/2}(t) - B_{\alpha/2}(s))^2 = |t - s|^\alpha.$$

It is known that $H_\alpha \in (0, \infty)$. We study the asymptotic behavior of the following probability as $u \to \infty$

$$\mathbf{P}\left(\max_{t\in[0,p]} \xi_1(t)\xi_2(t) > u\right),$$

for every fixed $p > 0$. We prove the following lemma for the probability of high extrema of random process $\xi_1(t)\xi_2(t)$ on a small interval.

Lemma 1 *Under conditions* **E1 – E3** *on covariance functions of* $\xi_i(t),\ i = 1, 2,$ *in case* $r \in (-1, 1)$*:*

1. *if* $\alpha_1 < \alpha_2$, *then for every* $T > 0$ *as* $u \to \infty$

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-1/\alpha}]} \xi_1(t)\xi_2(t) > u\right) = H_\alpha\left(\frac{c_1^{1/\alpha}}{(1+r)^{2/\alpha}}T\right)\frac{1+r}{\sqrt{2\pi u}}\exp\left(-\frac{u}{1+r}\right)(1+o(1));$$

2. *if* $\alpha_1 = \alpha_2$, *then in case* $c_1 + c_2 + 2c > 0$ *for every* $T > 0$ *as* $u \to \infty$

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-1/\alpha}]} \xi_1(t)\xi_2(t) > u\right) = H_\alpha\left(\frac{(c_1+c_2+2c)^{1/\alpha}}{(1+r)^{2/\alpha}}T\right)\frac{1+r}{\sqrt{2\pi u}}\exp\left(-\frac{u}{1+r}\right)(1+o(1)).$$

Using double sum method, that was described in detail in the first chapter, we derive the main result of the **first chapter**.

Theorem 2 *Under conditions* **E1** $-$ **E3** *on covariance functions of* $\xi_i(t),\ i=1,2,$ *in case* $r\in(-1,1)$*:*

1. *if* $\alpha_1<\alpha_2$*, then for every* $p>0$ *as* $u\to\infty$

$$\mathbf{P}\left(\max_{t\in[0,p]}\xi_1(t)\xi_2(t)>u\right)=\frac{c_1^{1/\alpha}}{\sqrt{2\pi}(1+r)^{2/\alpha-1}}pH_\alpha u^{1/\alpha-1/2}\exp\left(-\frac{u}{1+r}\right)(1+o(1));$$

2. *if* $\alpha_1=\alpha_2$*, then in case* $c_1+c_2+2c>0$ *for every* $p>0$ *as* $u\to\infty$

$$\mathbf{P}\left(\max_{t\in[0,p]}\xi_1(t)\xi_2(t)>u\right)=\frac{(c_1+c_2+2c)^{1/\alpha}}{\sqrt{2\pi}(1+r)^{2/\alpha-1}}pH_\alpha u^{1/\alpha-1/2}\exp\left(-\frac{u}{1+r}\right)(1+o(1)).$$

In the **second chapter** we consider symmetric $d\times d$ matrix A and an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)=(\xi_1(t),\dots,\xi_d(t))^\top,\ d\geq 2,\ t\in\mathbb{R}^+$, with covariance matrix of Pickands' type:

E1 For some positive-definite $d\times d$ matrices R, $C=[c_{ij}]_{i,j=1,\dots,d}$,

$$R(t):=E\boldsymbol{\xi}(t)\boldsymbol{\xi}(0)^\top=R-|t|^\alpha C+|t|^\alpha o(1),\ t\to 0,\ \alpha\in(0,2],\tag{3}$$

where $o(1)$ is a $d\times d$ matrix, which components tend to zero as $t\to 0$.

E2 For every $t\neq 0$ we have $R\ \pm\ R(t)\succ 0$ in the matrix sense, meaning that for all nonzero $\mathbf{x}\in\mathbb{R}^d$ and every $t\neq 0$

$$\langle\mathbf{x},(R\ \pm\ R(t))\mathbf{x}\rangle>0.\tag{4}$$

We study the asymptotic behavior of the following probability as $u\to\infty$

$$\mathbf{P}\left(\max_{t\in[0,p]}\langle\boldsymbol{\xi}(t),A\boldsymbol{\xi}(t)\rangle>u\right),$$

for every fixed $p>0$. We can reduce our probability to the probability

$$\mathbf{P}\left(\max_{t\in[0,p]}\langle\boldsymbol{\eta}(t),D\boldsymbol{\eta}(t)\rangle>u\right)$$

with some diagonal $d\times d$ matrix D and an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\eta}(t)$, with covariance matrix of Pickands' type:

E1′ For some positive-definite $d\times d$ matrix C_0

$$R_{\boldsymbol{\eta}}(t):=E\boldsymbol{\eta}(t)\boldsymbol{\eta}(0)^\top=I_d-|t|^\alpha C_0+|t|^\alpha o(1),\ t\to 0,\ \alpha\in(0,2],\tag{5}$$

where $o(1)$ is a $d\times d$ matrix, which components tend to zero as $t\to 0$, and I_d is an identity $d\times d$ matrix.

E2′ For every $t\neq 0$ we have $I_d\ \pm\ R_{\boldsymbol{\eta}}(t)\succ 0$ in the matrix sense, meaning that for all nonzero $\mathbf{x}\in\mathbb{R}^d$ and every $t\neq 0$

$$\langle\mathbf{x},(I_d\ \pm\ R_{\boldsymbol{\eta}}(t))\mathbf{x}\rangle>0.\tag{6}$$

Indeed, let us define an a.s. continuous zero-mean Gaussian stationary vector process by equation $\boldsymbol{\eta}(t) = Q^\top R^{-1/2}\boldsymbol{\xi}(t)$, where Q is an orthogonal matrix, that reduce symmetric matrix $R^{1/2}AR^{1/2}$ to diagonal D. Then

$$R_{\boldsymbol{\eta}}(t) := E\boldsymbol{\eta}(t)\boldsymbol{\eta}(0)^\top = I_d - |t|^\alpha C_0 + |t|^\alpha o(1),\ t \to 0,\ \alpha \in (0, 2],$$

where $d \times d$ matrix $C_0 = (R^{-1/2}Q)^\top C(R^{-1/2}Q)$ is a positive-definite and

$$\mathbf{P}\left(\max_{t\in[0,p]}\langle\boldsymbol{\xi}(t), A\boldsymbol{\xi}(t)\rangle > u\right) = \mathbf{P}\left(\max_{t\in[0,p]}\langle\boldsymbol{\eta}(t), D\boldsymbol{\eta}(t)\rangle > u\right).$$

Thus, here and after we assume that the process $\boldsymbol{\xi}(t)$ satisfies $\mathbf{E1'}$, $\mathbf{E2'}$ with some constant positive-definite matrix $C \succ 0$ and covariance matrix $R(t)$.

Let $\lambda_1, \dots, \lambda_d$ be diagonal elements of D. Without loss of generality we assume that all λ_i, $i = 1, \dots, d$, are nonzero (otherwise we can reduce the dimension), and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d$ (otherwise we can swap $\boldsymbol{\xi}(t)$ components). It is natural to assume that $\lambda_1 > 0$. We consider the case when

$$\lambda_1 > \lambda_2 \geq \dots \geq \lambda_d, \tag{7}$$

meaning that we assume that the restriction of $g(\mathbf{x}) = \langle\mathbf{x}, D\mathbf{x}\rangle$ on a unit sphere is reached on the manifold of 0-dimension: at points $(\pm 1, 0, \dots, 0) \in \mathbb{S}_{d-1}$. We use this restriction because it is a restriction of technique, that was developed for the case $g(\mathbf{x}) = x_1x_2$ in the first chapter. The case of arbitrary relations between eigenvalues of matrix $R^{1/2}AR^{1/2}$ (or, it is the same, matrix RA), in particular, the case when the maximum eigenvalue of $R^{1/2}AR^{1/2}$ has an order greater than 1, is considered as a part of a general problem in the third chapter. The following local lemma takes place.

Lemma 2 *Let an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ satisfies $\mathbf{E1'}$, $\mathbf{E2'}$ with some positive-definite matrix C. Then under condition (7) for eigenvalues of diagonal matrix D for every $T > 0$ as $u \to \infty$*

$$\mathbf{P}\left(\max_{t\in[0,Tu^{-1/\alpha}]}\langle\boldsymbol{\xi}(t), D\boldsymbol{\xi}(t)\rangle > u\right) = \frac{2}{\prod_{j=2}^d\sqrt{1-\lambda_j/\lambda_1}}H_\alpha\left(\left(\frac{c_{11}}{\lambda_1}\right)^{1/\alpha}T\right)\Psi\left(\sqrt{\frac{u}{\lambda_1}}\right)(1+o(1)).$$

Using double sum method we derive the main result of the **second chapter**.

Theorem 3 *Let an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ satisfies $\mathbf{E1'}$, $\mathbf{E2'}$ with some positive-definite matrix C. Then under condition (7) for eigenvalues of diagonal matrix D for every $p > 0$ as $u \to \infty$*

$$\mathbf{P}\left(\max_{t\in[0,p]}\langle\boldsymbol{\xi}(t), D\boldsymbol{\xi}(t)\rangle > u\right) = \left(\frac{c_{11}}{\lambda_1}\right)^{1/\alpha}\frac{2}{\prod_{j=2}^d\sqrt{1-\lambda_j/\lambda_1}}pH_\alpha u^{1/\alpha}\Psi\left(\sqrt{\frac{u}{\lambda_1}}\right)(1+o(1)).$$

In the **third chapter** we consider a general case of *Gaussian chaos processes*. Let $g(\cdot) : \mathbb{R}^d \to \mathbb{R}$, $d \geq 2$, be a homogeneous function of a positive order $\beta > 0$, such as for some $\mathbf{x} \in \mathbf{R}^d$ we have $g(\mathbf{x}) > 0$. The homogenity of a positive order $\beta > 0$ means, that for every $a \in \mathbb{R}$ and all $\mathbf{x} \in \mathbb{R}^d$ we have $g(a\mathbf{x}) = |a|^\beta g(\mathbf{x})$. Let $\boldsymbol{\xi}(t) = (\xi_1(t), \dots, \xi_d(t))^\top$, $d \geq 2$, $t \in \mathbb{R}^+$, be an a.s. continuous zero-mean Gaussian stationary vector process with covariance matrix of Pickands' type ($\mathbf{E1}, \mathbf{E2}$). We study the asymptotic behavior of the following probability as $u \to \infty$

$$P_{\boldsymbol{\xi}}(u, p) := \mathbf{P}\left(\max_{t\in[0,p]} g(\boldsymbol{\xi}(t)) > u\right), \quad p > 0.$$

Changing the function $g(\cdot)$, but keeping the homogenity order, let us simplify the Gaussian vector process. Reduce the matrix R to an identity matrix I_d, multiplying the process $\boldsymbol{\xi}(t)$ on a matrix $R^{-1/2}$. After this, using orthogonal transformation Q we reduce matrix $R^{-1/2}CR^{-1/2}$ to diagonal (with positive diagonal elements) C_0. It gives us, in particular, independence of the components of Gaussian vector process in every fixed moment. Then we need to find the asymptotic behavior of the following probability, where $\boldsymbol{\eta}(t) = Q^\top R^{-1/2}\boldsymbol{\xi}(t)$,

$$\mathbf{P}\left(\max_{t\in[0,p]} g(\boldsymbol{\xi}(t)) > u\right) = \mathbf{P}\left(\max_{t\in[0,p]} h(\boldsymbol{\eta}(t)) > u\right),$$

as $u \to \infty$, where function $h(\mathbf{x}) = g(R^{1/2}Q\mathbf{x})$ is a homogeneous function of a positive order $\beta > 0$. Covariance matrix of an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\eta}(t)$ satisfy Pickands' conditions:

E1$''$ For some positive-definite $d \times d$ matrix C_0

$$R_{\boldsymbol{\eta}}(t) := E\boldsymbol{\eta}(t)\boldsymbol{\eta}(0)^\top = I_d - |t|^\alpha C_0 + |t|^\alpha o(1),\ t \to 0,\ \alpha \in (0, 2], \tag{8}$$

where $o(1)$ is a $d \times d$ matrix, which components tend to zero as $t \to 0$.

E2$''$ For every $t \neq 0$ we have $I_d \ \pm\ R_{\boldsymbol{\eta}}(t) \succ 0$ in the matrix sense, meaning that for all nonzero $\mathbf{x} \in \mathbb{R}^d$ and every $t \neq 0$

$$\langle \mathbf{x}, (I_d \ \pm\ R_{\boldsymbol{\eta}}(t))\mathbf{x}\rangle > 0. \tag{9}$$

Here and after we assume that we study processes and functions after the described transformation. We use the same notion - $g(\mathbf{x})$ for a homogeneous function, $\boldsymbol{\xi}(t)$ for a Gaussian vector process, $R(t)$ for covariance matrix of this process and C for matrix, that appears in Pickands' conditions (we can assume that it is diagonal).

Let us introduce some other objects. Under **condition 1** let

$$j(\boldsymbol{\varphi}) = \frac{J(1, \boldsymbol{\varphi})|\det g''(\boldsymbol{\varphi})_{d-1-m}|^{-1/2}}{\int_{\mathcal{M}_{\boldsymbol{\varphi}}} J(1, \boldsymbol{\varphi})|\det g''(\boldsymbol{\varphi})_{d-1-m}|^{-1/2}\mathrm{d}V_{\boldsymbol{\varphi}}},$$

where, recall, that $\mathcal{M}_{\boldsymbol{\varphi}}$ is a maximum point set of $g(\boldsymbol{\varphi})$ on a unit sphere, m is a dimension of $\mathcal{M}_{\boldsymbol{\varphi}}$, and $\mathrm{d}V_{\boldsymbol{\varphi}}$ is an elementary volume in $\mathcal{M}_{\boldsymbol{\varphi}} \subset \Pi_{d-1}$. Let us define the function $\mathbf{v}(\boldsymbol{\varphi})$ on Π_{d-1} by $g(\mathbf{v}(\boldsymbol{\varphi})) = 1$, if $g(\boldsymbol{\varphi}) > 0$; and $\mathbf{v}(\boldsymbol{\varphi}) = 0$, if $g(\boldsymbol{\varphi}) \leq 0$. Correspondingly we define $\mathbf{v}(\mathbf{e})$ on $\mathbb{S}_{d-1}$. Since $g(a\mathbf{e})$ increases as $a > 0$ if $g(\mathbf{e}) > 0$, this definition if correct. And since by assumptions there exsist $\mathbf{e}$ with $g(\mathbf{e}) > 0$, $\mathbf{v}$ is not equal to zero. Denote

$$H_\alpha(T; \mathcal{M}_{\boldsymbol{\varphi}}) := \int_{\mathcal{M}_{\boldsymbol{\varphi}}} j(\boldsymbol{\varphi})H_\alpha\big(T\langle \mathbf{v}(\boldsymbol{\varphi}), C\mathbf{v}(\boldsymbol{\varphi})\rangle^{1/\alpha}\big)\mathrm{d}V_{\boldsymbol{\varphi}}.$$

Here and after if $m = 0$ (in the case 1) the integral over $\mathcal{M}_{\boldsymbol{\varphi}}$ means the sum (finite) over the points $\boldsymbol{\varphi} \in \mathcal{M}_{\boldsymbol{\varphi}}$. Since $j(\boldsymbol{\varphi})$ is a positive probability density (probability distribution in the case 1) on Π_{d-1}, there exists limit

$$H_\alpha(\mathcal{M}_{\boldsymbol{\varphi}}) := \lim_{T\to\infty} \frac{H_\alpha(T; \mathcal{M}_{\boldsymbol{\varphi}})}{T} = H_\alpha \int_{\mathcal{M}_{\boldsymbol{\varphi}}} j(\boldsymbol{\varphi})\langle \mathbf{v}(\boldsymbol{\varphi}), C\mathbf{v}(\boldsymbol{\varphi})\rangle^{1/\alpha}\mathrm{d}V_{\boldsymbol{\varphi}},$$

that is also positive and finite. Let us formulate the local lemma.

Lemma 3 *Let an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ satisfies* **E1**$''$, **E2**$''$. *Let the homogeneous function $g(\cdot)$ of a positive order $\beta > 0$ and the set of maximum points $\mathcal{M}_{\boldsymbol{\varphi}}$ on a unit sphere satisfies **condition 1**. Then for every $T > 0$*

$$\lim_{u\to\infty} \frac{\mathbf{P}\left(\max_{t\in[0,Tu^{-\frac{2}{\alpha\beta}}]} g(\boldsymbol{\xi}(t)) > u\right)}{u^{(m-1)/\beta}\exp\left(-\frac{1}{2}(u/g)^{2/\beta}\right)} = h_0 g^{-(m-1)/\beta} H_\alpha(T;\mathcal{M}_{\boldsymbol{\varphi}}),$$

where h_0 was defined in Theorem 1.

To follow from the small intervals (of the length $Tu^{-\frac{2}{\alpha\beta}}$) to the interval $[0,p]$ we use the discrete approximation approach. Consider the following grids on $\mathbb{R}$

$$\mathcal{R}(u,\delta) = \{k\delta u^{-\frac{2}{\alpha\beta}},\quad k\in\mathbb{Z}\},$$

and the corresponding probability

$$P_{\boldsymbol{\xi}}(\delta;u,p) := \mathbf{P}\left(\max_{t\in[0,p]\cap\mathcal{R}(u,\delta)} g(\boldsymbol{\xi}(t)) > u\right),\quad p>0.$$

Let us introduce Pickands' type constants in discrete time:

$$H_{\alpha,\delta}(T) = E\exp\left(\max_{t\in[0,T]\cap\mathcal{R}(\delta)}(\sqrt{2}B_{\alpha/2}(t) - |t|^\alpha)\right),\quad H_{\alpha,\delta} = \lim_{T\to\infty} H_{\alpha,\delta}(T)/T \in (0,\infty),$$

where

$$\mathcal{R}(\delta) = \{k\delta, k\in\mathbb{Z}\}.$$

Denote

$$H_{\alpha,\delta}(T;\mathcal{M}_{\boldsymbol{\varphi}}) = \int_{\mathcal{M}_{\boldsymbol{\varphi}}} j(\boldsymbol{\varphi}) H_{\alpha,\delta}\big(T\langle\mathbf{v}(\boldsymbol{\varphi}), C\mathbf{v}(\boldsymbol{\varphi})\rangle^{1/\alpha}\big)\mathrm{d}V_{\boldsymbol{\varphi}}.$$

Since the limit $\lim_{T\to\infty} H_{\alpha,\delta}(T)/T$ exists, it is positive and finite, going to the limit under integral, we derive that the limit

$$H_{\alpha,\delta}(\mathcal{M}_{\boldsymbol{\varphi}}) = \lim_{T\to\infty}\frac{H_{\alpha,\delta}(T;\mathcal{M}_{\boldsymbol{\varphi}})}{T}. \tag{10}$$

also exists and it is positive and finite. Using the same technic as in the continuous case, we can prove the following discrete local lemma.

Lemma 4 *Under conditions of Lemma 3 for every $T > 0$*

$$\lim_{u\to\infty}\frac{P_{\boldsymbol{\xi}}(\delta;u,Tu^{-\frac{2}{\alpha\beta}})}{u^{(m-1)/\beta}\exp\left(-\frac{1}{2}(u/g)^{2/\beta}\right)} = h_0 g^{-(m-1)/\beta} H_{\alpha,\delta}(T;\mathcal{M}_{\boldsymbol{\varphi}}).$$

Using Lemma 4 and Bonferroni inequality for a discrete grid we derive the main theorem on a discrete grid.

Theorem 4 *Let an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ satisfies* **E1**$''$, **E2**$''$. *Let the homogeneous function $g(\cdot)$ of a positive order $\beta > 0$ and the set of maximum points $\mathcal{M}_{\boldsymbol{\varphi}}$ on a unit sphere satisfies **condition 1**. Then for every $p > 0$ as $u\to\infty$*

$$P_{\boldsymbol{\xi}}(\delta;u,p) = ph_0 H_{\alpha,\delta}(\mathcal{M}_{\boldsymbol{\varphi}}) g^{(1-m)/\beta} u^{(m-1+2/\alpha)/\beta} e^{-\frac{1}{2}(u/g)^{2/\beta}}(1+o(1)).$$

Using the fact that the fractional Brownian motion is an a.s. continuous we derive the following lemma.

Lemma 5 *Under conditions of Lemma 3 for any $\varepsilon > 0$ one can find $\delta > 0$ such as for every $T > 0$*

$$\lim_{u\to\infty} \frac{P_{\boldsymbol{\xi}}(u, Tu^{-\frac{2}{\alpha\beta}}) - P_{\boldsymbol{\xi}}(\delta; u, Tu^{-\frac{2}{\alpha\beta}})}{u^{(m-1)/\beta} \exp\left(-\frac{1}{2}(u/g)^{2/\beta}\right)} \leq \varepsilon.$$

Using Theorem 4 and Lemma 5 we derive the main result of the **third chapter**.

Theorem 5 *Let an a.s. continuous zero-mean Gaussian stationary vector process $\boldsymbol{\xi}(t)$ satisfies* **E1″**, **E2″**. *Let the homogeneous function $g(\cdot)$ of a positive order $\beta > 0$ and the set of maximum points $\mathcal{M}_{\boldsymbol{\varphi}}$ on a unit sphere satisfies* ***condition 1***. *Then for every $p > 0$ as $u \to \infty$*

$$\mathbf{P}\left(\max_{t\in[0,p]} g(\boldsymbol{\xi}(t)) > u\right) = ph_0 H_\alpha(\mathcal{M}_{\boldsymbol{\varphi}}) g^{\frac{1-m}{\beta}} u^{\frac{m-1+2/\alpha}{\beta}} e^{-\frac{1}{2}(u/g)^{2/\beta}}(1 + o(1)),$$

where h_0 was defined in Theorem 1.

In **conclusion** we give a review of survey. In particular, we present the main results of work:

1. The exact asymptotic for the probability of high extrema of the **product of two** a.s. continuous zero-mean Gaussian stationary processes with Pickands' type covariance functions near zero with different exponents have been studied. We assume that the processes under consideration can be dependent and joint covariance function of two processes satisfies Pickands' type condition near zero.

2. The exact asymptotic for the probability of high extrema of the **quadratic form** from an a.s. continuous zero-mean Gaussian stationary vector process with Pickands' type covariance matrix near zero have been studied. We assume that the components of the process can be dependent and the maximum eigenvalue of the matrix, that defines quadratic form (or some transformation of this matrix in case of dependent components), is positive and has an order 1.

3. The exact asymptotic for the probability of high extrema of **Gaussian chaos processes** have been studied. It means that we consider the homogeneous function of a positive order from an a.s. continuous zero-mean Gaussian stationary vector process with Pickands' type covariance matrix near zero point. We assume that the components of the process can be dependent. We assume that the homogeneous function satisfies **condition 1**, in particular, it is a twice continuously differentiable in some neighborhood of the set of maximum points on a unit sphere. In particular, we **prove** that the asymptotic behavior of the considered probability depends on the maximum value of homogeneous function on a unit sphere, the dimension of the corresponding manifold, the structure of manifold, homogenity order and **behavior of covariance matrix** of an a.s. continuous zero-mean Gaussian stationary vector process near zero.

In **conclusion** we also give recommendations for the further survey:

1. It is necessary to derive the asymptotic behavior for the probability of high extrema of *Gaussian chaos processes* with **covariance matrix of a more general form**. In particular, one can start from the case, when the components of an a.s. continuous

zero-mean Gaussian stationary vector process satisfy Pickands' condition near zero, but with different exponents. We studied it only in the first chapter (for the product of two processes).

2. It is necessary to derive the asymptotic behavior for the probability of high extrema of *Gaussian chaos processes* for an a.s. continuous zero-mean Gaussian **nonstationary** vector processes. We need to formulate the condition on the behavior of the matrix $E\boldsymbol{\xi}(t)\boldsymbol{\xi}(t)^\top$ near its "points of maximum" in a multidimensional case.

3. It is necessary to study the asymptotic behavior of the probability of the event that Gaussian vector process $\boldsymbol{\xi}(t) \in \mathbb{R}^d,\ t \in \mathbb{R}^+$, on a finite interval $[0, p],\ p > 0$, falls into an infinitely expanding area $A_u \subset \mathbb{R}^d,\ u \to \infty$, that doesn't contain zero. It is necessary to consider the problem for stationary and nonstationary Gaussian vector processes, starting with the simplest areas A_u.

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