National Research University Higher School of Economics

Faculty of Mathematics

as a manuscript

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Del Pezzo fibrations

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

Academic supervisor:
doctor of sciences,
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Moscow – 2020

Introduction

The main objects of study in the field of algebraic geometry are algebraic varieties. They can be classified either up to biregular equivalence (that is, up to isomorphism) or up to a weaker equivalence relation called birational equivalence. To do such a classification in the latter case one needs to understand birational properties of the varieties. This is the goal of birational geometry.

It is well known that in dimension one birational equivalence coincides with the biregular one, hence this case is not very interesting from the point of view of birational geometry. On the other hand, the birational geometry of surfaces is a very rich theory. Initially, it was developed by the Italian school of algebraic geometry, in the works of Fano, Enriques, Castelnuovo, and others.

One of the main tools in higher-dimensional birational geometry is the Minimal Model Program (the MMP for short, see [Ma02], [KMM87]). It was developed in the works of Shigefumi Mori ([Mo82], [Mo88])and was a field of the active research in recent years ([BCHM09], [B10], [HM05]). The minimal category in which it works is the category $\mathcal{C}$ of the projective varieties with at worst terminal $\mathbb{Q}$-factorial singularities. The result of applying this program to a projective variety is either a minimal model, that is a variety $X \in \mathcal{C}$ whose canonical divisor class K_X is nef, or a Mori fiber space, that is a variety $X \in \mathcal{C}$ admitting a contraction morphism $\pi\colon X \longrightarrow B$ whose fibers are of positive dimension, the anti-canonical divisor class $-K_X$ is relatively ample, and the relative Picard number $\rho(X/B)$ is equal to 1.

In this work we mainly focus on the three-dimensional case. In this case, the base B of the Mori fiber space $\pi\colon X \longrightarrow B$ can be of dimension 0, 1 or 2. If $\dim B = 0$ then X is a (possibly singular) Fano variety. It is known that Fano varieties with restricted singularities and of fixed dimension lie in a finite number of algebraic families, see [B16]. However, they are classified only in the smooth case and in dimensions up to 3, see [Is77], [Is78], [MM83], and [IP99] for a survey. In the singular case for Fano threefolds there are partial classificational results, see e.g. [Pr16a]. There is a database [GRDB] of all possible numerical invariants of terminal $\mathbb{Q}$-Fano threefolds. See also [BKR12].

If $\dim B = 2$ then a general fiber of π is a smooth rational curve. In this case, $\pi\colon X \longrightarrow B$ is called a $\mathbb{Q}$-*conic fibration*, see [MP08-2]. It is known that in this case there exists a standard model, that is a $\mathbb{Q}$-conic fibration $\pi'\colon X' \longrightarrow B'$ where X' and B' are smooth, X' is fiberwise birationally equivalent to X, and $\rho(X'/B') = 1$ (see [Sa82]).

Finally, if $\dim B = 1$ then a general fiber of π is a smooth del Pezzo surface. In this situation the fibration $\pi\colon X \longrightarrow B$ is called a $\mathbb{Q}$-*del Pezzo fibration*. The main invariant of such fibrations is the degree $K^2_{X_\eta}$ of its general fiber. Since the general fiber is smooth, $1 \leq K^2_{X_\eta} \leq 9$. Slightly abusing notation, sometimes we will refer to π as *del Pezzo fibration* even in the case when X is not Gorenstein.

There are several main directions of our research.

1 Standard models

One can ask what can be done with the output of the MMP in the case of Mori fiber spaces. More precisely, can we obtain a "good" model of a given Mori fiber space? By a model we mean a Mori fiber space of the same type that is birational to a given one over the base of the fibration. The word "good" means that we allow our model to have only mild singularities (for example, Gorenstein singularities). In the literature such models are called standard. Standard models of threefold del Pezzo fibrations were considered in the work of A. Corti [Co96], see also [Ko97]. More precisely, Corti constructed a terminal Gorenstein model in the case $K^2_{X_\eta} \geq 3$ and terminal 2-Gorenstein model in the case $K^2_{X_\eta} = 2$. Also see [Kr18] for the questions of birational rigidity of del Pezzo fibrations.

For the applications to the problem of classification of the finite subgroups in the Cremona group (see, for example, [PrSh16]), as well as for birational classification of varieties over algebraically non-closed fields one should change the category $\mathcal{C}$. We will consider the varieties defined over an arbitrary field of characteristic 0 that admit an action of a finite group G. In this case, we can apply the G-equivariant Minimal Model Program (the G-MMP, see [KM-1998, 2.18, 2.19], [Mo88, 0.3.14]). Again, a final product of applying this program can be either a G-minimal model, or a G-Mori fiber space. For a three-dimensional G-Mori fiber space $\pi\colon X \longrightarrow B$, as in the "classical" situation, we have three possibilities:

- $G\mathbb{Q}$-Fano varieties. In general, this class is poorly understood. Partial results can be found in the works [Pr15], [Pr16], [Pr16a], [PrSh16].
- $G\mathbb{Q}$-conic fibrations. In this case existence of the standard model is proven in the work [Av14].
- $G\mathbb{Q}$-del Pezzo fibrations. We construct standard models of $G\mathbb{Q}$-del Pezzo fibrations of degree 1. It would be desirable to construct standard models for the fibrations of higher degree, but at the moment our methods do not allow us to do that.

The following theorems are the main results of this section.

Theorem A. *Let X be a projective three-dimensional G-variety and C be a G-curve. Let $\pi\colon X \longrightarrow C$ be a proper G-morphism whose generic fiber is a smooth degree 1 del Pezzo surface X_η, and $\operatorname{Pic}^G(X/C)$ is generated by $-K_X$ and G-components of fibers of π. Then there exists a* Gorenstein model*, that is*

(i) *a commutative diagram*

$$\begin{tikzcd} X \arrow[r, dashed, "\chi"] \arrow[d, "\pi"] & Y \arrow[d, "\sigma"'] \\ C \arrow[r, dashed] & C' \end{tikzcd}$$

where σ is a generalised G-del Pezzo fibration and χ is a birational G-equivariant map,

(ii) *Y has $\mathbb{Q}$-factorial canonical Gorenstein singularities,*

(iii) *C' is smooth, projective and birational to C,*

(iv) *χ induces an isomorphism between X_η and $Y_{\eta'}$ where $Y_{\eta'}$ is the generic fiber of σ,*

(v) *any fiber of σ is reduced and irreducible.*

Corollary B. *If $\pi : X \longrightarrow C$ is a $G\mathbb{Q}$-del Pezzo fibration of degree 1 then it has a model with at worst $\mathbb{Q}$-factorial canonical Gorenstein singularities, with irreducible fibers and with the same generic fiber as π.*

Theorem C. *Let $\sigma\colon Y \longrightarrow C$ be a generalised G-del Pezzo fibration of degree 1, and let Y have only Gorenstein canonical singularities. Then Y admits an embedding over C into the relative weighted projective space*

$$Y \hookrightarrow \mathbb{P}_C(1,1,2,3).$$

Fiberwise, this embedding coincides with the "anticanonical" embedding of the degree 1 del Pezzo surface into $\mathbb{P}(1,1,2,3)$.

Notice that the theorems A, B and C allow us to study the singular fibers which are Gorenstein del Pezzo surfaces.

We prove these results in several steps. First, we establish some rigidity properties for del Pezzo surfaces and del Pezzo fibrations of degree 1. Then, starting from a del Pezzo fibration of degree 1 as in Theorem A, we show that X is G-birational over C to a $G\mathbb{Q}$-del Pezzo fibration of degree 1. After that, we construct a canonical model of X, that is a fibration $\bar{\pi}\colon \bar{X} \longrightarrow C$ which is G-birational to X over C and such that the pair $(\bar{X}, |-K_{\bar{X}} + \bar{\pi}^*\bar{D}|)$ is canonical for some $\bar{D}$. Next, we construct a Gorenstein model, thus proving Theorem A. After that, we recall some facts on the anticanonical algebra of degree 1 del Pezzo surfaces. Finally, we embed a Gorenstein G-fibration Y into $\mathbb{P}_C(1,1,2,3)$ proving Theorem C.

2 Non-rational fibers of del Pezzo fibrations

The second direction of our research is the study of the singular fibers of del Pezzo fibrations (and more general Mori fiber spaces). Here we work over the field of complex numbers. The question is local with respect to the base, so we consider fibrations over curve germs. The singular fiber over the marked point of a germ is called special. We say that the special fiber is a degeneration of a smooth fiber. There are many works on degenerations of del Pezzo surface, see, for example, [HW81], [F95]. We consider two types of degenerations: non-rational and semistable. We start with formulating a more general rationality problem for the fibers of three-dimensional Mori fiber spaces.

The rationality problem for (singular) Fano threefolds is far from complete solution, although much is known in the smooth case, see [IP99, Chapter 12]. If π is a $\mathbb{Q}$-conic bundle then its fibers are trees of rational curves. In this case the rationality problem for the fibers of π is trivial.

We consider the case of del Pezzo fibrations. As mentioned above, the classical MMP works in the terminal category. In the three-dimensional case this implies that the singularities of the total space are isolated. Hence, in the case of del Pezzo fibrations the general fiber is a smooth del Pezzo surface. The geometry of smooth del Pezzo surfaces is well understood. It is well known that such surfaces are rational. But a special fiber can be non-rational. It is easy to show that such fiber is a surface which is birationally ruled over a curve C of genus $g(C) > 0$.

The simplest example of such degeneration is a cone over a plane elliptic curve in the three-dimensional projective space. One can easily see that a smooth cubic del Pezzo surface can degenerate into such a cone in a family whose total space is smooth. We investigate when a del Pezzo surface can degenerate into a non-rational surface in a "reasonably good" family. By such family we mean a del Pezzo fibration in the sense of the Minimal Model Program.

We show that the geometry of the non-rational special fiber (for example, the value of $g(C)$ above) depends on the degree of the generic fiber $K^2_{X_\eta}$ and on the singularities of the total space X. We prove that if X is smooth (respectively, terminal Gorenstein) then $K^2_{X_\eta} \leq 3$ (respectively, ≤ 4) and the non-rational fiber is a cone over an elliptic curve. This fact is rather elementary and follows from the classification of Gorenstein del Pezzo surfaces [HW81]. In the terminal Gorenstein case any fiber is reduced and irreducible, and moreover, a non-rational fiber is necessarily normal. On the other hand, in the non-Gorenstein terminal case, multiple fibers are possible. However, their multiplicity is bounded by 6 as shown in [MP08].

In Theorem D we use the base change construction to show that in the smooth case such del Pezzo fibrations with a non-rational fiber are in 1-to-1 correspondence with smooth μ_n-del Pezzo fibrations with certain properties. This shows that the non-rational fibers of terminal Gorenstein del Pezzo fibrations form a very restricted class. The precise statement is as follows.

Theorem D. *Let $\pi : X \longrightarrow B \ni o$ be a del Pezzo fibration such that X is smooth and the fiber $F = \pi^{-1}(o)$ is non-rational. Then there is 1-to-1 correspondence between such π and μ_n-del Pezzo fibrations $\pi_V : V \longrightarrow B \ni o$ with the following properties:*

- *the special fiber $E_V = \pi_V^{-1}(o)$ is a smooth μ_n-minimal del Pezzo surface of degree d,*
- *the locus of fixed points of μ_n on V is an elliptic curve $C \subset E_V$,*
- *the action of μ_n on $\mathbb{P}(N_{C/V})$ is trivial.*

There are only three possible cases (here $d = K_F^2$):

(i) $d = 3,\ n = 3,$
$E_V \simeq (w^3 = q_3(x, y, z)) \subset \mathbb{P}^3,$
$\mu_3 : w \mapsto \zeta_3 w,$
$F \simeq (0 = q_3(x, y, z)) \subset \mathbb{P}^3;$

(ii) $d = 2,\ n = 4,$
$E_V \simeq (w^2 = q_4(x, y) + z^4) \subset \mathbb{P}(1, 1, 1, 2),$
$\mu_4 : z \mapsto \sqrt{-1}z,$
$F \simeq (w^2 = q_4(x, y)) \subset \mathbb{P}(1, 1, 1, 2);$

(iii) $d = 1,\ n = 6,$
$E_V \simeq (w^2 = z^3 + \alpha x^4 z + \beta x^6 + y^6) \subset \mathbb{P}(1, 1, 2, 3),$
$\mu_6 : y \mapsto \zeta_6 y, ,\ \alpha, \beta \in \mathbb{C},$
$F \simeq (w^2 = z^3 + \alpha x^4 z + \beta x^6) \subset \mathbb{P}(1, 1, 2, 3).$

On the other hand, if we allow X to have log terminal singularities then the non-rational fibers are not bounded. We also give examples of the fibrations with terminal singularities such that the special fiber is birationally ruled over a curve C of genus $g(C) = 2, 3, 4$. It is not known whether one can achieve $g(C) > 4$ in this setting.

After that we consider fibrations with very mild singularities, the ordinary double points. Using the base change construction, we classify such fibrations with non-rational special fiber in terms of certain μ_n-del Pezzo fibrations. It appears that in this case $K_{X_\eta}^2 = 1$ or 4. The precise statement is as follows.

Theorem E. *Let $\pi : X \longrightarrow B \ni o$ be a del Pezzo fibration with at worst ordinary double points. Suppose that the fiber $F = \pi^{-1}(o)$ is non-rational and X has at least one singular point on F. Then there is 1-to-1 correspondence between such π and (weak and analytic in the case* (ii) *below) μ_n-del Pezzo fibrations $\pi_V : V \longrightarrow B \ni o$ with the following conditions:*

- *the special fiber* $E_V = \pi_V^{-1}(o)$ *is a smooth (weak in the case* (ii) *below) del Pezzo surface of degree* d *with* $\rho^{\mu_n}(E_V) = 2$,
- *one-dimensional locus of fixed points of* μ_n *on* V *is an elliptic curve* $C \subset E_V$,
- *the action of* μ_n *on* $\mathbb{P}(N_{C/V})$ *is trivial.*

There are only two possible cases (here $d = K_F^2$*):*

(i) $d = 4,\ n = 2,\ E_V$ *has two* μ_2*-conic bundle structures,*

(ii) $d = 1,\ n = 4,\ E_V$ *has one* μ_4*-invariant* (-1)*-curve. There exists one* μ_4*-invariant point.*

For other results on rationality in families see [KT17], [T16], [P17] and references therein.

3 Semistable degenerations of Fano varieities

Next, we consider semistable families of del Pezzo surfaces, and more generally, of Fano varieties. By a *semistable family* we mean a family of projective algebraic varieties over a curve germ with a smooth total space such that the special fiber is reduced and has simple normal crossings. The semistable reduction theorem [KKMS73] states that any family with a smooth generic fiber can be birationally transformed into a semistable one after a finite base change. We say that the special fiber of a semistable family is a *semistable degeneration* of its generic fiber.

The dual complex of the special fiber is an important invariant of a degeneration. Its topology in some sense reflects the geometry of the generic fiber. There are many results along these lines. For example, a theorem of Kulikov [Ku77] states that for a semistable degeneration of a K3 surfaces the dual complex can have exactly one of the three types, and the maximal degeneration (such that its special fiber has the dual complex of maximal possible dimension) has a triangulation of a 2-sphere as a dual complex. The three Kulikov's cases can be distinguished in terms of the monodromy around the special fiber. In particular, if the monodromy is trivial then every fiber of the family is smooth. We say that such family is *smooth*.

It is natural to ask about the semistable degenerations of del Pezzo surfaces. In [Fu90] Fujita obtained the classification of such degenerations. Later, Kachi in [Ka07] used deformation theory to prove that all the cases in Fujita's list can be realized. However, this classification does not contain information about the monodromy. Below we give an alternative proof of their theorem which shows that the monodromy is trivial in all cases. For more results on the degenerations of surfaces see, for example, [Per77]. There is also a related notion of a dual complex of a singularity, see, for example, [St08].

In higher dimensions, de Fernex, Kollár and Xu showed that if the generic fiber of a semistable family is rationally connected then the dual complex of the special fiber is contractible, see [dFKX12, Theorem 4]. The main theorem of this section is a more specific result in the case when the fibers of a semistable family are Fano varieties.

Theorem F. *Let $\pi : X \longrightarrow B \ni o$ be a semistable family of n-dimensional Fano varieties. Then the dual complex of its special fiber $F = \pi^{-1}(o)$ is a simplex Δ^k of dimension $k \leq n$. In dimension $n \leq 3$ the maximal degeneration (such that $k = n$) is unique and has trivial monodromy. Moreover, it can be obtained as the blow-up of a flag of subspaces*

$$\{\mathrm{pt}\} = \mathbb{P}^0 \subset \cdots \subset \mathbb{P}^{n-1}$$

in a fiber of a smooth family whose fibers are isomorphic to $\mathbb{P}^n$. In this case, each of the $n+1$ components of the special fiber for $n = 1, 2, 3$ is isomorphic to the blow-up of $\mathbb{P}^n$ in a flag of subspaces

$$\{\mathrm{pt}\} = \mathbb{P}^0 \subset \cdots \subset \mathbb{P}^{n-2}.$$

In [Hu06] it is shown for any $n \geq 1$ any $k \leq n$ can be realized for some degeneration of $\mathbb{P}^n$. For $1 \leq n \leq 3$ this construction coincides with the maximal degeneration described in Theorem F.

The special fiber of a semistable family satisfies the d-semistability condition introduced by Friedman in [Fr83]. We use this condition and the three-dimensional Minimal Model Program to reprove the result of [Fu90], that is, to obtain the classification of semistable degenerations of del Pezzo surfaces. This gives the case $n = 2$ of the theorem (the case $n = 1$ is trivial).

The canonical line bundle is defined for simple normal crossing varieties. For such a variety F we say that F is Fano if $-K_F$ is ample. This is equivalent to the following condition: each component F_j is log Fano with respect to the boundary D_j given by the intersection with the other components. In [Tz15] Tziolas showed that any d-semistable simple normal crossing Fano variety can be smoothed, that is, included as the special fiber in a semistable family. Hence, the classification of semistable degenerations of Fano varieties is equivalent to the classification of d-semistable simple normal crossing Fano varieties.

From this point of view it is important to study log Fano varieties. If we consider only smooth log Fano varieties with non-empty integral boundary then in dimension 1 the situation is trivial: the only log Fano curve is a projective line. By contrast, already in dimension 2 there are infinitely many non-isomorphic log Fano varieties (they are called log del Pezzo surfaces). For example, one can take any Hirzebruch surface with the negative section as the boundary. The classification of log del Pezzo surfaces and three-dimensional log Fano varieties is contained in [Ma83]. We use it to show that the maximal degeneration is unique in dimension 3.

4 Approbation of the results

The results of the thesis were presented at the following conferences:

- Siberian summer school Current developments in Geometry. September 2019, Novosibirsk. Talk: "Semistable degenerations of Fano varieties".
- Géométrie Algébrique en Liberté XXVII, June 2019, Bucharest. Poster: "Semistable degenerations of Fano varieties".
- Birational geometry, Kähler-Einstein metrics and degenerations, June 2019, Shanghai. Talk: "Semistable degenerations of Fano varieties".
- Workshop on birational geometry, October 2018, Moscow. Talk: "Non-rational fibers in del Pezzo fibrations".
- Siberian summer school Current developments in Geometry. September 2018, Novosibirsk. Talk: "Non-rational fibers in del Pezzo fibrations".
- Conference on Lie groups and Lie algebras. August 2018, Samara. Talk: "Non-rational fibers in del Pezzo fibrations".
- Workshop on birational geometry, November 2017, Moscow. Talk: "Standard models of degree 1 del Pezzo fibrations".
- VII Conference on algebraic geometry and complex analysis. August 2017, Koryazhma, Arkhangelsk region. Talk: "Standard models of del Pezzo fibrations".
- Inaugural conference for the Laboratory of Mirror Symmetry and Automorphic forms. July 2017, Saint Petersburg. Talk: "Standard models of del Pezzo fibrations".

The results of the thesis were presented at the following seminars:

- "Semistable degenerations of Fano varieties", University of Loughborough, Mathematical Physics and Geometry Seminar, October 2019.
- "Semistable degenerations of Fano varieties", University of Bristol, Algebra and Geometry Seminar, October 2019.
- "Semistable degenerations of Fano varieties", Edinburgh Geometry seminar (EDGE), October 2019.

- “On semistable degenerations of Fano varieties”, Laboratory of Algebraic Geometry and Homological Algebra, Moscow Institute of Physics and Technology, Dolgoprudny, Moscow region, September 2019.
- “On semistable degenerations of Fano varieties”, HSE Laboratory of Algebraic Geometry seminar, September 2019.
- “Snc degenerations of Fano varieties”, at Iskovskikh seminar (Steklov Mathematical Institute), April 2019.
- “On non-rational fibers of del Pezzo fibrations” at Iskovskikh seminar (Steklov Mathematical Institute), October 2018.
- “Standard model of degree 1 del Pezzo fibrations” at Iskovskikh seminar (Steklov Mathematical Institute), November 2017.

The results of the thesis are published in two papers and one preprint.

- Standard models of degree 1 del Pezzo fibrations.

  Moscow Mathematical Journal, 18 (4), 2018, 721–737.
- On non-rational fibers of del Pezzo fibrations over curves.

  Mathematical Notes, 106 (6), 2019, 881–893.
- On semistable degenerations of Fano varieties.

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