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# R-matrix formalism in differential geometry of quantum groups and in integrable models of mathematical physics

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# Introduction

The dissertation is devoted to applications of the R-matrix formalism to the theory of quantum matrix algebras, the construction of q-analogues of differential geometric constructions for quantum linear groups, and the study of integrable stochastic processes and models of quantum mechanics. By R-matrix formalism we mean a technique based on the use of R-matrix representations of the braid group, and, more specifically, Hecke algebras, in the construction and investigation of various structures on quantum manifolds and in the consideration of models of mathematical physics.

Let us present briefly the contents of the dissertation.

In the first two sections of this work we introduce the family of quantum matrix algebras and derive their structure properties. In Section 1, we describe briefly the R-matrix technique used below and present a definition of a quantum matrix (QM-) algebra of type $GL(n)$. In Section 2, we introduce a commutative characteristic subalgebra of the QM-algebra, determine two sets of its generating elements, and present the Newton's relations connecting them. Then, the main structure theorem on $GL(n)$ type QM-algebras is formulated — a q-analogue of the Cayley-Hamilton theorem.

In sections 3 and 4 we study the Heisenberg double over a quantum linear group. From a geometric point of view this double is a quantization of the algebra of differential operators on a group. Algebraically, it is a special smash product algebra of two remarkable QM-algebras: the RTT-algebra and the reflection equation algebra. In Section 3 we give the definitions of the Heisenberg doubles of $GL(n)$ and $SL(n)$ types. Then, using the Cayley-Hamilton theorem, we construct their spectral extensions. As a byproduct, we obtain two series of the dynamical R-matrices of linear type. Section 4 is devoted to the study of a series of Heisenberg double automorphisms. This series was interpreted by L. Faddeev and A. Antonov as the discrete time evolution of the q-deformed Euler top. We compute the evolution operator for this model.

Sections 5 and 6 are devoted to the construction of differential calculus over linear quantum groups. The differential calculus algebra is generated by the components of four matrices: the RTT-algebra matrix generates quantized functions on a group; a pair of the reflection equation matrices plays a role of the basic left- and right-invariant Lie derivatives; a matrix of basic right-invariant 1-forms generates the quantum external algebra of differential forms. In Section 5 we define differential calculi algebras of the types $GL(n)$ and $SL(n)$. We construct the exterior derivative map for the calculi of the both types. In Section 6, we construct the spectral extension of the differential calculi. We introduce three series of automorphisms generalizing the series considered in Sec.4. Using these automorphisms we construct a unitary type anti-involution over the differential calculi algebras.

In Section 7, the R-matrix formalism is used to construct factorized expressions for two series of solutions of the quantum Knizhnik-Zamolodchikov equation. These solutions are relevant for the integrable stochastic model called Raise and Peel. This model introduced by J. de Geer, B. Nienhuis, P. Piers and V. Rittenberg gives physical interpretation to a wonderful combinatorial properties observed by A. Razumov and Yu. Stroganov in a non-physical mode of the XXZ spin chain.

Brief historical notes are made in the introductions to each thematic block of the dissertation. In the beginning of Section 1 we describe history of the origin of the quantum matrix algebras. In Section 3 we mention the works where the Heisenberg double was introduced. In the preface to the fifth section, we discuss problems in the construction of the differential calculus over quantum groups. The introduction to Section 7 describes the story of the quantum Knizhnik-Zamolodchikov equations and explains their application to the Raise and Peel model. Our historical notes by no means can be viewed as exhaustive reviews of the subjects. We mainly mention the results which are closely related and have influenced our work.

In Sctions 8 and 9 we present summary of results of the dissertation, and give a list of research papers submitted for defense.

In the concluding Section, we desribe recent advances in lines of the dissertation researches and give our estimates on possible directions for further investigations.

# 1 Quantum matrix algebras

Since an invention by the Leningrad school of mathematical physics of the quantum inverse scattering method (QISM), quantum groups were implicitly present in its formalism. Their discovery is due to V. Drinfeld [D1].

The R-matrix formalism in the theory of quantum groups has been intensively developed since the end of the 80s as an effective tool for applications to quantum integrable models and for studies of the non-commutative geometry. Originally this formalism has been worked out for two families of algebras: the *RTT-algebras* [FRT], and the *reflection equation algebras* [Ch1, KS]. These algebras play an exceptional role in the theory of quantum groups and in applications. They are quite different in their structure and in the representation theory. However, investigations of these algebras reveal many constructions, such as q-versions of trace and determinant, q-generalizations of the Newton's relations and of the Hamilton-Cayley theorem, which are common for these algebras and for the classical matrix algebras (see  cite EOW, NT, PS, Zh). Therefore, it was a natural next step to introduce a unifying framework for description of both families — the framework of *quantum matrix* (QM-) algebras [H, IOP1]. The structure theory of these algebras will be the main subject of our study in this and the next sections.

Quantum matrix algebras are associative unital algebras which are generated by a set of matrix components $\{M_{ij}\}_{i,j=1}^{\rm N}$, satisfying quadratic relations of a special kind. These relations are defined using the so-called R-matrix representations of the braid group. Roughly speaking they determine the permutation rules for generators $M_{ij}$. It is the type of the R-matrix representation which controls the form of the structure results for the QM-algebras, namely, their Hamilton-Cayley identities and the structure of their characteristic subalgebras.

We begin with a brief overview of the Hecke type R-matrix representations of the braid group. Such representations are used in the construction of families of QM-algebras of linear types: $GL({\rm N})$, $SL({\rm N})$, $GL({\rm N}|{\rm M})$. The first two families will be the main objects of study in this section.

Consider a finite dimensional $\mathbb{C}$-linear space $V$, $\dim V = {\rm N}$. Fixing a linear basis $\{v_j\}_{j=1}^{\rm N}$ in $V$ we identify operators $X \in {\rm End}(V^{\otimes n})$ with their matrices $X^{k_1 k_2 \dots k_n}_{j_1 j_2 \dots j_n}$.

For any $X \in {\rm End}(V^{\otimes m})$ and for all $i \geq 1$ symbol $X_i \in {\rm End}(V^{\otimes n})$, $n \geq i+m-1$, denotes an operator whose matrix in a chosen basis has a form

$$(X_i)^{k_1\dots k_n}_{j_1\dots j_n} \; = \; I^{k_1\dots k_{i-1}}_{j_1\dots j_{i-1}} \, X^{k_i\dots k_{i+m-1}}_{j_i\dots j_{i+m-1}} \, I^{k_{i+m}\dots k_n}_{j_{i+m}\dots j_n} \, . \tag{1.1}$$

Here and below symbol $I$ denotes the identity operator.

Any operator $R \in {\rm Aut}(V^{\otimes 2})$, satisfying matrix relation

$$R_1 \, R_2 \, R_1 \; = \; R_2 \, R_1 \, R_2, \tag{1.2}$$

is called an *R-matrix.*

Permutation operator $P \in {\rm Aut}(V^{\otimes 2})$: $P(u \otimes v) = v \otimes u \quad \forall \, u, v \in V$ is the R-matrix. $R^{-1}$ is the R-matrix iff $R$ is.

Any R-matrix $R$ generates representations $\rho_R$ of the series of braid groups $\mathcal{B}_n$, $n = 2, 3, \dots$

$$\rho_R : \, \mathcal{B}_n \to {\rm Aut}(V^{\otimes n}) \; , \quad g_i \mapsto R_i, \quad 1 \leq i \leq n-1. \tag{1.3}$$

Here $g_i$ are Artin's generators of the braid group.

R-matrix is called *skew invertible* if there exists an operator $\Psi_R \in \mathrm{End}(V^{\otimes 2})$ such that

$$\mathrm{Tr}_{(2)} R_{12} \Psi_{R23} = \mathrm{Tr}_{(2)} \Psi_{R12} R_{23} = P_{13}\,. \tag{1.4}$$

Here subscripts of the symbol $X_{ij}$ show explicitly indices of the spaces $V$ in which operator $X$ acts nonidentically (e.g., $P_{13} = P_{i_1 i_3}^{j_1 j_3}\, I_{i_2}^{j_2}$). Here and in what follows symbol $\mathrm{Tr}_{(i)}$ denotes taking the trace in space $V$ with index $i$.

By any skew invertible R-matrix $R$ define an operator $D_R \in \mathrm{End}(V)$

$$D_{R1} = \mathrm{Tr}_{(2)} \Psi_{R12}. \tag{1.5}$$

$R$ is called *strict skew invertible*, if $D_R$ is invertible. If the R-matrix $R$ is strict skew invertible, then $R^{-1}$ is strict skew invertible too [Gu], and

$$(D_{R^{-1}})_2^{-1} == \mathrm{Tr}_{(1)} \Psi_{R12}.$$

With any skew invertible R-matrix $R$ one associates a $\mathbb{C}$-linear map $\mathrm{Tr}_R$ : $\mathrm{End}(V) \otimes W \rightarrow W$ from the space of $\mathrm{N}\times\mathrm{N}$ matrices, whose components belong to a $\mathbb{C}$-linear space $W$, into the space $W$

$$\mathrm{Tr}_R(M) = \textstyle\sum_{j,k=1}^{\mathrm{N}} D_R{}_j^k M_k^j \qquad \forall\, M \in \mathrm{End}(V) \otimes W. \tag{1.6}$$

This map is called the *R-trace*. The permutation $P$ is strict skew invertible and the map $\mathrm{Tr}_P$ coincides with the usual matrix trace. Following property is characteristic for the R-trace

$$\mathrm{Tr}_{R(2)} R_1 = I_1. \tag{1.7}$$

The R-matrix $R$ is called *Hecke type* if its minimal polynomial is quadratic. Rescaling the Hecke type R-matrix $R$ we turn its minimal polynomial to a form

$$(R - qI)(R + q^{-1} I) = 0. \tag{1.8}$$

Here $q \in \mathbb{C}\backslash\{0\}$ is a nonzero complex parameter.

The R-matrix representations (1.3) associated to the Hecke type R-matrices are the representations of the Iwahori-Hecke algebras $\mathcal{H}_n(q)$ whose Artin's generators $g_i$ satisfy the so-called Hecke relations $(g_i - q1)(g_i + q^{-1}1) = 0$. In these algebras (but not in the group algebras of the braid groups) one can introduce a set of the *baxterised* generators

$$g_i(x) \;=\; 1 + \tfrac{x-1}{q-q^{-1}}\, g_i, \quad x \in \mathbb{C}\backslash\{0\}, \;\; i = 1, \ldots, n-1, \tag{1.9}$$

satisfying relations

$$
\begin{aligned}
g_i(x)\, g_{i+1}(xy)\, g_i(y) &= g_{i+1}(y)\, g_i(xy)\, g_{i+1}(x), && (1.10)\\
g_i(x) g_i(x^{-1}) &= \tfrac{(qx-q^{-1})(qx^{-1}-q^{-1})}{(q-q^{-1})^2}\, 1. && (1.11)
\end{aligned}
$$

Relations (1.10) and (1.11) are called, respectively, the *Yang-Baxter equations* and the *unitarity conditions.* Baxterized elements play an important role in the representation theory of the Iwahori-Hecke algebras. In particular, they can be used for construction of the two series of primitive central idempotents $a^{(n)}, s^{(n)} \in \mathcal{H}_n(q)$ [J, Gy]:

$$
\begin{aligned}
& a^{(1)} = s^{(1)} = 1,\\
& a^{(i+1)} = \tfrac{q^i}{[i+1]_q}\, a^{(i)}\, g_i(q^{-2i})\, a^{(i)}, \qquad s^{(i+1)} = \tfrac{q^{-i}}{[i+1]_q}\, s^{(i)}\, g_i(q^{2i})\, s^{(i)} \quad \forall\, i \geq 1, \qquad (1.12)\\
& g_j a^{(n)} = a^{(n)} g_j = -q^{-1} a^{(n)}, \qquad g_j s^{(n)} = s^{(n)} g_j = q s^{(n)} \quad \forall j : 1 \leq j \leq n-1.
\end{aligned}
$$

Here $[i]_q = (q^i - q^{-i})/(q - q^{-1})$ is a standard notation for the Euler's $q$-number. To avoid singularities in the definitions of $a^{(n)}$, $s^{(n)}$ we impose additional restrictions on the parameter $q$

$$
[i]_q \neq 0 \quad \forall\, i :\; 2 \leq i \leq n. \qquad (1.13)
$$

**Definition 1.1.** *Let $R$ be strict skew invertible R-matrix of the Hecke type. If additionally in the associated representation $\rho_R$ relations*

$$
\operatorname{rank} \rho_R(a^{(i)}) > 0 \;\; \forall\, i = 1, \dots, n, \qquad \rho_R(a^{(n+1)}) = 0, \qquad (1.14)
$$

*are fulfilled, then $R$ is called $GL(n)$ type.*

In [Gu] it was proved that the $GL(n)$ type R-matrices verify also condition

$$
\operatorname{rank} \rho_R(a^{(n)}) = 1.
$$

This allows one to define the notion of a q-determinant in the quantum matrix algebras associated with the $GL(n)$ type R-matrices (see below).

A most well known (although not the unique) series of the $GL(n)$ type R-matrices is the canonical Drinfeld-Jimbo series

$$
R_{(DJ)} \;=\; \sum_{j,k=1}^{\mathrm{N}} q^{\delta_{jk}} E_{jk} \otimes E_{kj} \;+\; (q - q^{-1}) \sum_{j<k} E_{jj} \otimes E_{kk}. \qquad (1.15)
$$

Here $(E_{jk})_l^m \;:=\; \delta_{jm}\, \delta_{kl},\;\; j, k, m, l \;=\; 1, \dots, \mathrm{N},$ is the basis of matrix units. In this case $n \;=\; \dim V \;=\; \mathrm{N}$, although in general this is not alsways true (for the counterexamples, see [Gu]). Among other series of $GL(n)$ type R-matrices we mention the multparametric generalizations of R-matrices (1.15) [R1] and Cremmer-Gervais R-matrices [CG, IO].

An ordered pair $\{R, F\}$ of two R-matrices $R$ and $F$ is called *a compatible R-matrix pair* if conditions

$$R_1\, F_2\, F_1 \;=\; F_2\, F_1\, R_2\,, \qquad R_2\, F_1\, F_2 \;=\; F_1\, F_2\, R_1\,, \tag{1.16}$$

are satisfied. Conditions (1.16) are called *twist relations*. Obviously, the R-matrix pairs $\{R, P\}$ and $\{R, R\}$ are compatible.

To a pair of compatible strict skew invertible R-matrices $\{R, F\}$ we shall put into correspondence a *quantum matrix alebra* $\mathcal{M}(R, F)$. To define it we use a free associative algebra $W = \mathbb{C}\langle 1, M_a^b\rangle$, which is generated by the unit and by components of $\mathrm{N}\times\mathrm{N}$ matrix $M = \|M_a^b\|_{a,b=1}^{\mathrm{N}}$. We also prepare $\bar{i}$*-copies* of the matrix $M$: $M_{\bar{i}}$, $i \geq 1$, which are defined recurrently with the help of matrix $F$

$$M_{\overline{1}} := M_1, \quad M_{\bar{i}} := F_{i-1} M_{\overline{i-1}} F_{i-1}^{-1} \quad \forall\, i > 1. \tag{1.17}$$

In a particular case $F = P$ $\bar{i}$-copies of the matrix $M \in \mathrm{End}(V)\otimes$ coincide with its usual copies (1.1): $M_{\bar{i}} = M_i$.

**Definition 1.2.** *[H, IOP1]. Let $\{R, F\}$ be a compatible pair of strict skew invertible R-matrices. A quotient algebra of the free associative algebra $W = \mathbb{C}\langle 1, M_a^b\rangle$, $a, b = 1, \dots, \mathrm{N}$, by a two sided ideal generated by relations*

$$R_1 M_{\overline{1}} M_{\overline{2}} = M_{\overline{1}} M_{\overline{2}} R_1, \tag{1.18}$$

*is called the quantum matrix (QM-) algebra. For its notation we use symbol $\mathcal{M}(R, F)$.*

*If $R$ is the Hecke type (resp., the $GL(n)$ type) R-matrix, the algebra $\mathcal{M}(R, F)$ is called the Hecke type (resp., the $GL(n)$ type) QM-algebra.*

For any adjacent pair of $\bar{i}$-copies of the generating matrix $M$, following relations are fulfilled in the QM-algebra $\mathcal{M}(R, F)$ [IOP1]

$$R_i M_{\bar{i}} M_{\overline{i+1}} = M_{\bar{i}} M_{\overline{i+1}} R_i, \qquad \forall\, i \geq 1, \tag{1.19}$$

For different values of the index $i$ these relations are pairwise identical.

Perhaps the most important subfamilies of the family of quantum matrix algebras are algebras corresponding to the compatible pairs $\{R, P\}$ и $\{R, R\}$.

To the pair $\{R, P\}$ the definition 1.2 puts into correspondence the so-called *RTT-algebra* [D1, FRT]. The matrix of generators of this algebra is traditionally denoted as $T$. Relations (1.18) in this case look like

$$R_1 T_1 T_2 \,=\, T_1 T_2 R_1. \tag{1.20}$$

To the pair $\{R, R\}$ we associate the *reflection equation* algebra [Ch1, KS]. We use symbol $L$ for its matrix of generators. Relations (1.18) in this case can be transformed to a form

$$R_1 L_1 R_1 L_1 \,=\, L_1 R_1 L_1 R_1. \tag{1.21}$$

The RTT-algebras and the reflection equation algebras are used extensively in investigations of the integrable models of mathematical physics. Both families have additional algebraic structures. In the case $R = R_{(DJ)}$ these algebras can be given a nice geometric interpretation. Namely, the RTT-algebra can be treated as the *algebra of quantized functions on the group* $GL(n)$. In turn, the reflection equation algebra represents *quantized algebra of right-invariant differential operators on* $GL(n)$. These interpretations will be important for us in Sections 3 and 5, where we discuss differential geometric constructions on quantum groups.

In the next section we present general structure results for generic $GL(n)$ type QM-algebras.

## 2 Structure of characteristic subalgebra and q-analogue of the Cayley-Hamilton theorem

For the QM-algebra $\mathcal{M}(R, F)$, consider a linear span of the unit and all elements of the form

$$ch(x^{(k)}) := \mathrm{Tr}_{R(1\dots k)} \Big(\rho_R(x^{(k)})\, M_{\overline{1}} M_{\overline{2}} \dots M_{\overline{k}}\Big) \quad \forall\, k \geq 1. \tag{2.1}$$

Here $x^{(k)}$ is an arbitrary element of the group algebra $\mathbb{C}[\mathcal{B}_k]$. We denote this space $\mathcal{C}(R, F)$.

**Proposition 2.1.** *$\mathcal{C}(R, F)$ is the abelian subalgebra of the QM-algebra $\mathcal{M}(R, F)$. It is called the characteristic subalgebra of $\mathcal{M}(R, F)$.*

*In case of the reflection equation algebra the characteristic subalgebra belongs to its center: $\mathcal{C}(R, R) \subset \mathbb{Z}[\mathcal{M}(R, R)]$.*

**Comments.**

- Commutativity of the characteristic subalgebra of the RTT-algebra was first observed in [Mai], where the commutativity was checked for the series of elements of $\mathcal{C}(R,P)$ called power sums (see below). A complete proof of the commutativity of $\mathcal{C}(R,F)$ is given in [IOP1].

- in [D2, R2], centrality of the characteristic subalgebra for the reflection equation algebras was carried out by the use of the quasitriangular Hopf algebra structure. A proof using the QM-algebra formalism is given in [IP2].

When studying multiplication in characteristic subalgebras, the following sets of elements are considered.

*Power sums $p_i$, $i \geq 0$:*

$$
\begin{aligned}
p_0 &:= \mathrm{Tr}_{R}(I)\,1 \ \scriptstyle{(=q^{-n}[n]_q 1 \ \text{ in the } GL(n) \text{ case})}, \qquad p_1(M) := \mathrm{Tr}_{R}(M), \\
p_k(M) &:= \mathrm{Tr}_{R(1\dots k)}\left(R_{k-1}\dots R_2 R_1\, M_{\overline{1}} M_{\overline{2}}\dots M_{\overline{k}}\right), \quad k = 2,3,\dots .
\end{aligned} \tag{2.2}
$$

In considering QM-algebras of the Hecke type, *elementary symmetric polynomials are also used $e_i$, $i \geq 0$:*

$$
\begin{aligned}
e_0 &:= 1, \\
e_k(M) &:= \mathrm{Tr}_{R(1\dots k)}\Big((\rho_R(a^{(k)})\, M_{\overline{1}} M_{\overline{2}}\dots M_{\overline{k}}\Big), \quad k = 1,2,\dots .
\end{aligned} \tag{2.3}
$$

For correctness of the definition of $e_k$ we impose additional conditions on the parameter $q:\ [i]_q \neq 0\ \ \forall i = 2,\dots,k$.

Note that for the $GL(n)$ type QM-algebras the series of elementary symmetric olinomials truncates: $e_{n+i} = 0\ \forall i > 0$.

In what follows, we will show the argument in the notation of $p_i$ and $e_i$ in cases where its absence can make a confusion only (see, e.g., proposition 3.2).

**Proposition 2.2.** *The characteristic subalgebra of the Hecke type QM-algebra is generated by the set of power sums $\{p_i\}_{i\geq 0}$. Under consditions $[i]_q \neq 0\ \ \forall i \geq 2$, it is also genrated by the set of elementary symmetric polynomials $\{e_i\}_{i\geq 0}$. The two sets of generators are related by the following $q$-analogues of the Newton relations*

$$
\sum_{i=0}^{k-1} (-q)^i\, e_i\, p_{k-i} \;=\; (-1)^{k-1}\,[k]_q\, e_k \qquad \forall\, k \geq 0. \tag{2.4}
$$

The matrix presentation $\{M_a^b\}_{a,b=1,\dots,\mathrm{N}}$ for the set of generators of the QM-algebra is not only a way to write concisely the algebraic relations (1.18). This matrix structure is essential for the formulation of a q-analogue of the Cayley-Hamilton theorem in the case of QM-algebras. To state this theorem, we have to generalise a notion of the matrix powers for the QM-algebra generating matrix.

A matrix

$$(M^{\overline{k}})_1 \;=\; \mathrm{Tr}_{R(2\dots k)}\,(R_{k-1}\dots R_2R_1\,M_{\overline{1}}M_{\overline{2}}\dots M_{\overline{k}})\,, \quad k=2,3,\dots \tag{2.5}$$

is called $\overline{k}$-th matrix power of the generating matrix $M$ of the QM-algebra $\mathcal{M}(R,F)$. Here we use the overlined character $\overline{k}$ in the definition (2.5) to avoid possible confusion with the notation of the classical matrix power $M^k$. We also put $M^{\overline{1}} = M$ and $M^{\overline{0}} = I$.

For the matrix $L$ of generators of the reflction equation algebra $\mathcal{M}(R,R)$ (1.21) notions of the overlined and the usual matrix powers turn to be identical: $L^{\overline{k}} \equiv L^k$.

The overlined matrix power, like the usual one, can be realised by an iterative application of the binary associative operation $\star$ to pairs of matrices $M$ and $N$:

$$M\star N \;=\; M\cdot\phi(N), \tag{2.6}$$

where $\phi$ is a linear map

$$\phi(N)_1 \;=\; \mathrm{Tr}_{R(2)}\,(N_{\overline{2}}\,R_{12})\,/ \tag{2.7}$$

Here symbol "$\cdot$"stays for the usual matrix multiplication. The map $\phi$ is defined by the compatible pair of the strict skew infertible $R$-matrices $\{R,F\}$, and is invertible [OP].

An equivalent form of the definition (2.5) is

$$M^{\overline{k}} \;=\; \underbrace{M\star M\star\dots\star M}_{k\ \text{раз}}. \tag{2.8}$$

The associativity of the operation $\star$ leads to the additivity of the overlined matrix power:

$$M^{\overline{k}}\star M^{\overline{j}} \;=\; M^{\overline{k+j}}.$$

**Theorem 2.3.** *(q-analogue of the Cayley-Hamilton theorem).*
*For the matrix $M$ generating the $GL(n)$ type QM-algebra following characteristic identity is fulfilled*

$$\sum_{i=0}^{n}(-q)^i\,M^{\overline{n-i}}\,e_i \;=\; 0\,. \tag{2.9}$$

**Comments.**

- Fir the family of reflection equation algebras the Newton relations were derived in [PS, GPS1]. Their generalisation to generic QM-algebras of the Hecke type was obtained in [IOP1]. Note that the definition of the QM-algebra used in [IOP1] is different from 1.2. Equivalence of these definitions is proved in [IOP2].

- The Cayley-Hamilton identities for the reflection equation algebras constructed by the Drinfeld-Jimbo R-matrix (1.15), were derived in [NT].Their generalization for arbitrary $GL(n)$ type reflection equation algebras was presented in [GPS1], and for arbitrary $GL(n)$ type QM-algebras in [IOP1]. Analogues of the Cayley-Hamilton theorem for particular examples of the RTT-algebras were discussed earlier in [EOW].

- In the works of I. Gelfand, V. Retach, et al. (See [G-T] and references therein), a different generalization of the Hamilton–Cayley theorem to the case of non-commutative associative algebras was considered. In this approach, the concept of *quasi-determinant* plays central role. This concept allows construction of analogues of all formulas and statements of the classical matrix theory for very wide family of algebras freely generated by the matrix components. In this approach, the R-matrices are not used and specific quadratic relations are not imposed on the matrix of generators (1.18). The classical definition of the matrix multiplication is used for construction of powers of the generating matrix. The scalar coefficients of the characteristic identity are substituted by diagonal matrices, and the components of these matrices are not polynomials, but rational functions of the generators of the algebra. Such a price has to be paid for a generality of the approach. In the particular case of the reflection equation algebras, corresponding to the Drinfeld-Jimbo R-matrix ( ref DJ), the characteristic identities of Gelfand, Retach, et al. coincide with (2.9). However, for the QM-algebras of the general form, the relation of characteristic identities in the two approaches is not obvious.

From the theorem 2.3 it follows that a necessary and sufficient condition for the invertibility of the matrix of generators of the $GL(n)$ type QM-algebra both with respect to the classical and the $\star$ matrix products, is invertibility of the element $e_n$. For the $GL(n)$ type algebras $\mathcal{M}(R, F)$, the notion of *quantum determinant* of the generators matrix $M$ coincides, up to numeric factor, with $e_n$ (see below).

Consider possibility of factorization of the $GL(n)$ type QM-algebra by a relation of the type $e_n \sim 1$. This factorization would lead to a controlled and an meaningful result in case if $e_n$ belongs to the center of the QM-algebra. This is always the case for the reflection equation algebras. For the case of RTT-algebras, the centrality condition of $e_n$ is described in the following proposition.

**Proposition 2.4.** [Gu] *In the $GL(n)$ type RTT-algebra (1.20) the element $e_n$ and the components of the generating matrix $T$ satisfy relations*

$$T\, e_n \;=\; e_n\, (O_{\scriptscriptstyle R})^{-1}\, T\, O_{\scriptscriptstyle R}. \tag{2.10}$$

*Here the mutually inverse matrices $O_{\scriptscriptstyle R}$ and $(O_{\scriptscriptstyle R})^{-1}$ are given by formulas*

$$\begin{aligned} O_{\scriptscriptstyle R 1} \;&=\; [n]_q\, \mathrm{Tr}_{R(2,\dots,n+1)} \Big( P_1 P_2 \dots P_n\, \rho_R(a^{(n)}) \Big)\;, \qquad (2.11)\\ (O^{-1}_{\scriptscriptstyle R})_1 \;&=\; [n]_q\, \mathrm{Tr}_{R(2,\dots,n+1)} \Big( \rho_R(a^{(n)})\, P_n \dots P_2 P_1 \Big)\;. \end{aligned}$$

Guided by this result, we can separatee a subfamily of RTT-algebras of the type $SL(n)$.

**Definition 2.5.** *Consider the $GL(n)$ type RTT-algebra $\mathcal{M}(R,P)$. An element*

$$\mathrm{det}_{\scriptscriptstyle R} T \;:=\; \mathrm{Tr}_{\,(1\dots n)} \Big( \rho_R(a^{(n)})\, T_1 T_2 \dots T_n \Big) \;=\; q^{n^2}\, e_n(T) \tag{2.12}$$

*is called the quantum determinant of the matrix of the algebra generators $T$.*[1]

*Assume that the R-matrix $R$ is such that its corresponding matrix $O_{\scriptscriptstyle R}$ (2.11) is scalar. In this case, the quantum determinant belongs to the center of the RTT-algebra, and the quotient algebra of $\mathcal{M}(R,P)$ by the relation* $\mathrm{det}_{\scriptscriptstyle R} T = 1$ *is called the $SL(n)$ type RTT-algebra.*

An important property of the $SL(n)$ type RTT-algebra is the invertibility of its matrix of generators $T$ or, in other words, the existence of the antipode mapping $T \mapsto T^{-1}:\;\; T\,T^{-1} = T^{-1}\,T = I$, which makes it the Hopf algebra (see [FRT]). Explicit expression for $T^{-1}$ is given by formula

$$(T^{-1})_1 \;=\; q^{n(n-1)}\, [n]_q\, \mathrm{Tr}_{R(2,\dots,n)} \Big( T_2 \dots T_n\, \rho_R(a^{(n)}) \Big)\,. \tag{2.13}$$

Well-known examples of the $SL(n)$ type RTT-algebras are the quantized algebras of functions on Lie groups $SL(n)$ [FRT], which are constructed by the Drinfeld-Jimbo R-matrices (1.15). Note however, that for the multiparameter $GL(n)$ type

[1] The numeric coefficient in the definition of the quantum determinant is chosen in such a way, that $\mathrm{det}_{\scriptscriptstyle R} T$ is group-like with respect to the comultiplication $\Delta(T^j_k) = \sum_{m=1}^n T^j_m \otimes T^m_k$.

R-matrices [R1], the $SL(n)$ type defining conditions (2.5), as a rule, are not satisfied.

In the next section, we investigate, using the results obtained here, the Heisenberg double algebras (see the definition below) over $SL(n)$ type RTT-algebras.

## 3 Quantization of differential operators over linear groups

A notion of a Heisenberg double over quantum group has been formulated and attracted substantial researcher's interest in the early 90-s [S, AF1, SWZ2]. From the algebraic point of view it is a smash product algebra (see [Mon]) of the quantum group (or, the quantized universal enveloping algebra) and its dual Hopf algebra. In the differential geometric interpretation it may be viewed as an algebra of quantized differential operators over group or, equivalently, as an algebra of quantized functions over cotangent bundle of the group. Since the group's cotangent bundle serve a typical phase space for integrable classical dynamics, it is natural to attach the same role to the Heisenberg double over quantum group for quantum physical models.

In this section, we study the structure of the Heisenberg double over RTT-algebra of the type $SL(n)$. We define its spectral extension and introduce a new set of generators in the spectrally extended double. Calculation of permutation relations for these generators gives a new way to construct dynamical R-matrices of the $GL(n)$ type. We construct two series of dynamical R-matrices: one of them is well known, the other one seems to be new. In the next section, we will use the new generators for calculation of the evolution operator of the q-deformed quantum isotropic top [AF1, AF2].

**Definition 3.1.** *Consider two QM-algebras associated to the strict skew invertible R-matrix $R$: the RTT-algebra $\mathcal{M}(R,P)$, generted by the components of matrix $T$ satisfying relations (1.20), and the reflection equation algebra $\mathcal{M}(R,R)$, generated by the components of matrix $L$ subject to relations (1.21). A Heisenberg double of these algebras is the algebra generated by the components of two matrices $T$ and $L$, satisfying eqs. (1.20), (1.21), and the permutation relaitons (3.3) between the components of $T$ and $L$. For readers convenience we put all the relations together:*

$$R_1T_1T_2 \;=\; T_1T_2R_1, \qquad (3.1)$$

$$R_1L_1R_1L_1 \;=\; L_1R_1L_1R_1, \qquad (3.2)$$

$$R_1L_1R_1T_1 \;=\; \gamma^2\, T_1L_2, \quad \textit{где} \;\; \gamma\in\{\mathbb{C}\backslash 0\}. \qquad (3.3)$$

*This Heisenberg double algebra is further denoted as $\mathcal{HD}_\gamma(R)$.*

*The algebra $\mathcal{HD}_\gamma(R)$ is called the Hecke type ($GL(n)$ type) Heiseberg double in case if it is constructed with the use of the Hecke type ($GL(n)$ type) R-matrix $R$.*

For the $SL(n)$ type reduction of this algebra we shall study permutation relations of two elements $e_n(T)$ and $e_n(L)$.

**Proposition 3.2.** *[IP2]. In the $GL(n)$ type Heisenberg double $\mathcal{HD}_\gamma(R)$ following relations are satisfied*

$$\begin{aligned} L\, e_n(T) &= q^{-2}\gamma^{2n}\, e_n(T)\, (O_{\scriptscriptstyle R})^{-1}\, L\, O_{\scriptscriptstyle R}, &(3.4)\\ T\, e_n(L) &= q^{2}\gamma^{-2n}\, e_n(L)\, T. &(3.5)\end{aligned}$$

**Definition 3.3.** *Consider the $GL(n)$ type Heisenberg double $\mathcal{HD}_\gamma(R)$ associated with the R-matrix $R$. Assume that the corresponding matrix $O_{\scriptscriptstyle R}$ (2.11) is scalar and the parameters $\gamma$ and $q$ are related by the equality $\gamma^n = q$. In such case the quotient algebra of $\mathcal{HD}_\gamma(R)$ by relations* [2]

$$\det{}_{\scriptscriptstyle R} T = q^{n^2}\, e_n(T) \;=\; 1, \qquad e_n(L) \;=\; q^{-1}\, 1 \tag{3.6}$$

*is called the $SL(n)$ type Heisenberg double. We denote this algebra $\mathcal{HD}_{\scriptscriptstyle SL(n)}(R)$.*

Let us comment on the classical limit $q \to 1$ of the algebra $\mathcal{HD}_{\scriptscriptstyle SL(n)}(R)$, associated with the Drinfeld-Jimbo R-matrix (1.15).

Let us make the linear change of the generators

$$T \to t:\; t_k^j = T_k^j, \qquad L \to \ell:\; \ell_k^j \;=\; \left(\delta_k^j 1 - q^{\frac{1-n^2}{n}} L_k^j\right)/(q-q^{-1}). \tag{3.7}$$

Taking into account the limiting formulas

$$R_{\scriptscriptstyle (DJ)} \stackrel{q\to 1}{\longrightarrow} P, \quad R^2_{\scriptscriptstyle (DJ)} \stackrel{q\to 1}{\longrightarrow} I + (q-q^{-1})P, \quad \gamma^2 = q^{2/n} \stackrel{q\to 1}{\longrightarrow} 1 + (q-q^{-1})\tfrac{1}{n}. \tag{3.8}$$

the Heisenberg double relations (3.1)–(3.3) and the $SL(n)$ reduction condtions (3.6) in the $q \to 1$ limit assume a form

$$\begin{aligned} [t_1, t_2] &= 0,\\ [\ell_1, \ell_2] &= P_{12}(\ell_2 - \ell_1),\\ [\ell_1, t_2] &= (P_{12} - \tfrac{1}{n}\, I_{12})\, t_2,\\ \mathrm{Tr}\, \ell = 0, &\qquad \det t = 1. \end{aligned} \tag{3.9}$$

[2] A numeric factor $q^{-1}$ appearing in the relation for $e_n(L)$ will be explained below.

This is nothing but the defining relations of the Lie algebra $sl(n)$ of the right-invariant vector fields $\ell^j_k$, acting on the coordinate functions $t^j_k$ of the Lie group $SL(n)$. Such a classic limit justifies the interpretation of the Heisenberg double $\mathcal{HD}_{SL(n)}(R)$ as a quantization (i.e., a q-deformation) of the algebra of differential operators over $SL(n)$.

To investigate the Heisenberg double algebra in more details we now define a central extension of the $GL(n)$ type reflection equation algebra $\mathcal{M}(R,R)$ introducing a new set of generators $\{\mu_i\}_{i=1,\dots,n}$ playing a role of the eigenvalues of the matrix $L$.

Consider an algebra of rational fenctions in $n$ variables $\mathbb{C}(\mu_1,\dots,\mu_n)$. Assuming the algebraic independence of generators $e_i(L)$ of the characteristic subalgebra $\mathcal{C}(R,R) \subset \mathcal{M}(R,R)$ consider a monomorphism $\mathcal{C}(R,R) \hookrightarrow \mathbb{C}(\mu_1,\dots,\mu_n)$ which sends generators $e_i(L)$ to the elementary symmetric polynomials

$$e_i(L) \mapsto e_i(\mu_1,\dots,\mu_n) := \sum_{1\le j_1<\dots<j_i\le n} \mu_{j_1}\mu_{j_2}\dots\mu_{j_i} \qquad \forall\, i = 0,1,\dots,n\,, \quad (3.10)$$

This map defines the $\mathcal{C}(R,R)$-module structure on $\mathbb{C}(\mu_1,\dots,\mu_n)$.

**Definition 3.4.** *An algebra*

$$\overline{\mathcal{M}(R,R)} \,:=\, \mathcal{M}(R,R) \mathop{\otimes}_{\mathcal{C}(R,R)} \mathbb{C}(\mu_1,\dots,\mu_n)$$

*is called the spectral extension of the $GL(n)$ type reflection equaion algebra $\mathcal{M}(R,R)$. In other words, the algebra $\overline{\mathcal{M}(R,R)}$ is an extension of the algebra $\mathcal{M}(R,R)$ by the set of rational functions of $n$ mutually commutative variables $\mu_1,\mu_2,\dots,\mu_n$, satisfying conditions*

$$\begin{array}{rcl} e_\alpha(L) &=& e_\alpha(\mu_1,\dots,\mu_n), \qquad L^i_j\,\mu_\alpha \,=\, \mu_\alpha\, L^i_j \\ && \forall\, i,j = 1,\dots,\mathrm{N}, \quad \forall\,\alpha = 1,\dots,n. \end{array} \quad (3.11)$$

*Further on, we call $\mu_\alpha$ the spectral variables.*

*A quotient algebra of $\overline{\mathcal{M}(R,R)}$ by the relation*

$$e_n(L) \,=\, \prod_{\alpha=1}^{n} \mu_\alpha \,=\, q^{-1}1 \quad (3.12)$$

*is called the $SL(n)$ type spectral extension of the reflection equation algebra.*

As we have already mentioned, the number N of rows (columns) of the matrix $L^i_j$ does not have to match the number $n$ of its spectral variables. However, for QM-algebras associated with the Drinfeld-Jimbo R-matrix (1.15), this is indeed she case.

In the spectrally extended algebra $\overline{\mathcal{M}(R,R)}$ the characteristic identity (2.9) of the Cayley-Hamilton theorem assumes a factorized form:

$$\prod_{\alpha=1}^{n}(L - q\mu_\alpha I) \,=\, 0\,. \tag{3.13}$$

We stress that, in case ot the reflection equation algebra, the $\star$-product of matrices (2.6) coincides with the usual matrix multiplication. Therefore, in the formula (3.13) the matrix multiplication of factors is assumed.

Using eq. (3.13), one can construct a set of $n$ mutually orthogonal matrix projectors

$$P^\alpha \,=\, \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{(L - q\mu_\beta I)}{q(\mu_\alpha - \mu_\beta)} \;:\quad P^\alpha P^\beta \,=\, \delta_{\alpha\beta}\, P^\alpha\,, \quad \sum_{\alpha=1}^{n} P^\alpha \,=\, I. \tag{3.14}$$

The columns (rows) of the matrix $P^\alpha$ are the right (left) eigenvectors of the matrix of generators $L$ with the eigenvalues $q\mu_\alpha$:

$$L\,P^\alpha \,=\, P^\alpha L \,=\, q\mu_\alpha P^\alpha\,. \tag{3.15}$$

Our next goal is a spectral extension of the Heisenberg double algebra $\mathcal{HD}_{SL(n)}(R)$.

**Definition-theorem 3.5.** *[IP2]. A spectral extension of the $SL(n)$ type Heisenberg double algebra $\mathcal{HD}_{SL(n)}(R)$ (see Def.3.3) is its extension by the set of rational functions in $n$ mutually commutative variables $\mu_1, \ldots, \mu_n$, satisfying conditions (3.11), (3.12), and relations*

$$(P^\beta T)\,\mu_\alpha \,=\, q^{2(\delta_{\alpha\beta}-\frac{1}{n})}\mu_\alpha\,(P^\beta T) \qquad \forall\, \alpha, \beta = 1, \ldots, n, \tag{3.16}$$

*where $P^\alpha$ are the matrix projectors (3.14).*

For correctness of the definition 3.5 one has to prove consistency of the relations (3.16) and (3.3) with the expression (3.11) for the generators of the characteristic subalgebra $e_\alpha(L)$ in terms of the elementary symmetric polynomials in the spectral variables $\mu_\alpha$. This proof is carried out in [IP2].

In [AF1] A.Alekseev and L.Faddeev have observed at the classical level and postulated at the quantum level an appearance of dynamical R-matrices in the permutation relations for certain elements of the Heisenberg double algebra. These elements turn out to be suitable variables for investigation of a dynamics of the q-deformed Euler top.

In the rest of this section we describe spectral extension of the Heisenberg double algebra in terms of new set of generators. In this description the dynamical R-matrices appear in a natural and coherent way.

Conside a set of matrices $W^\alpha$ with their components belonging to $\mathcal{HD}_{SL(n)}(R)$:

$$W^\alpha \;=\; P^\alpha T, \qquad \alpha = 1, \ldots, n. \tag{3.17}$$

Taking into account invertibility of the matrix $T$, one can express all generators of the Heisenberg double algebra in terms of the spectral variables $\mu_\alpha$ and the matrix components of $W^\alpha$:

$$T \;=\; \textstyle\sum_{\alpha=1}^n W^\alpha, \qquad L \;=\; \sum_{\alpha=1}^n q\mu_\alpha W^\alpha T^{-1}.$$

**Theorem 3.6.** *[IP2] Let us introduce two $n^2 \times n^2$ matrices $R^S(q;\mu)_{\alpha\beta}^{\gamma\delta}$ u $R^A(q;\mu)_{\alpha\beta}^{\gamma\delta}$, $\alpha, \beta, \gamma, \delta = 1, \ldots, n$.*
*The nonzero components of $R^S(q;\mu)$ are following*

$$R^{S\,\alpha\alpha}_{\;\;\alpha\alpha} \;=\; q, \;\; R^{S\,\alpha\beta}_{\;\;\alpha\beta} \;=\; -\frac{(q-q^{-1})\mu_\beta}{\mu_\alpha - \mu_\beta}, \;\; R^{S\,\alpha\beta}_{\;\;\beta\alpha} \;=\; \frac{q^{-1}\mu_\alpha - q\mu_\beta}{\mu_\alpha - \mu_\beta} \;\; \forall\, \alpha \neq \beta, \tag{3.18}$$

*whereas the nonzero components of $R^A(q;\mu)$ are*

$$\begin{aligned} & R^{A\,\alpha\alpha}_{\;\;\alpha\alpha} \;=\; -q^{-1}, \;\; R^{A\,\alpha\beta}_{\;\;\alpha\beta} \;=\; -\frac{(q-q^{-1})\mu_\beta}{\mu_\alpha - \mu_\beta}, \;\; R^{A\,\alpha\beta}_{\;\;\beta\alpha} \;=\; \frac{q^{-1}\mu_\beta - q\mu_\alpha}{\mu_\alpha - \mu_\beta} \;\; \forall\, \alpha \neq \beta, \\ & R^{A\,\beta\alpha}_{\;\;\alpha\alpha} \;=\; -R^{A\,\alpha\beta}_{\;\;\alpha\alpha} \;=\; \frac{(q^4-1)\,\mu_\alpha\,\varphi_{\alpha\beta}}{q(\mu_\alpha - \mu_\beta)} \;\; \forall\, \alpha \neq \beta, \\ & \textit{where} \quad \varphi_{\alpha\beta} := \prod_{\sigma \neq \alpha,\beta} \frac{\mu_\sigma - q^2\mu_\alpha}{\mu_\sigma - \mu_\beta} \end{aligned} \tag{3.19}$$

*Both these two matrices satisfy the dynamical Yang-Baxter equation*

$$R(\mu)_{12} R(\nabla_1(\mu))_{23} R(\mu)_{12} \;=\; R(\nabla_1(\mu))_{23} R(\mu)_{12} R(\nabla_1(\mu))_{23}. \tag{3.20}$$

*where $\nabla^1$ is the diagonal matrix, composed of operators of finite shifts of the spectral variables*

$$\begin{aligned} \nabla_1 = \mathrm{diag}\{\nabla_\alpha\}_{\alpha=1}^n : \quad & \nabla_\alpha(\mu_\beta) = q^{2X_{\alpha\beta}} \mu_\beta, \\ \textit{where} \quad & X_{\alpha\beta} \;=\; \delta_{\alpha\beta} - \tfrac{1}{n} \quad \forall\, \alpha, \beta = 1, \ldots, n. \end{aligned} \tag{3.21}$$

*Spectral extension of the Heisenberg double algebra $\mathcal{HD}_{SL(n)}(R)$ can be defined in terms of the components of the matrices $W^\alpha$ satisfying relations*

$$
\begin{aligned}
&W^\alpha\,\mu_\beta - q^{2(\delta_{\alpha\beta}-\frac{1}{n})}\mu_\beta\,W^\alpha \;=\; 0 \;\;\forall\,\alpha,\beta = 1,\ldots,n,\\
&S_1^{(2)}[W_1^\alpha W_2^\beta R_1 \;-\; \sum_{\alpha',\beta'=1}^{n} R^S(q;\mu)_{\alpha'\beta'}^{\alpha\;\beta}\,W_1^{\alpha'}W_2^{\beta'}] \;=\; 0,\\
&A_1^{(2)}[W_1^\alpha W_2^\beta R_1 \;-\; \sum_{\alpha',\beta'=1}^{n} R^A(q;\mu)_{\alpha'\beta'}^{\alpha\;\beta}\,W_1^{\alpha'}W_2^{\beta'}] \;=\; 0.
\end{aligned}
\tag{3.22}
$$

*Here we use following shorthand notation*

$$
A^{(2)} = \rho_R(a^{(2)}) = \frac{qI-R}{q+q^{-1}}\,, \qquad S^{(2)} = \rho_R(1-a^{(2)}) = \frac{q^{-1}I+R}{q+q^{-1}}. \tag{3.23}
$$

**Comments.**

- The dynamical Yang-Baxter equation was first encountered in the work of J.-L. Gervais and A. Neveu [GN]. Several families of the dynamical R-matrices, including the series $R^S(q;\mu)$, were constructed using various methods in [F, AF1, Is, EV1, EV2]. A systematic approach to the dynamical Yang-Baxter equations and dynamical R-matrices was proposed by J. Felder [Fe] and further developed in the works of P. Etingof, A. Varchenko and others (see the review [ES]). The series of R-matrices $R^A(q;\mu)$, as far as we know, does not fit into the framework of this classification scheme and has not been encountered in the literature before.

- Notice that in the proof of the theorem 3.6 dynamical R-matrices $R^{A/S}(q;\mu)$ are calculated by solving systems of (at most three) linear equations. At the same time, they themselves are solutions of the system of nonlinear finite difference equations (3.20).

- Note also that the dynamical R-matrices $R^{A/S}(q;\mu)$ in relations (3.22) are defined completely by the type $SL(n)$ of the non-dynamical R-matrix $R$ and do not depend on its particular form.

## 4 Evolution of the q-deformed isotropic top

In this section we investigate evolution of a quantum model of q-deformed Euler top. This model was introduced into consideration by A. Alexeev and L. Faddeev

[AF1, AF2]. Observables of the model are operators acting in the Heisenberg double algebra. We will define natural dynamical variables of the model and construct its evolution operator.

Consider a sequence of mappings $\theta$ acting on the generating matrices of the Heisenberg double

$$\begin{gathered}\{T,\ L\} \xrightarrow{\theta^k} \{T(k),\ L(k)\} \quad \forall\, k \geq 0\ : \\ T(0) = T, \quad T(k+1) = LT(k) = L^{k+1}T, \quad L(k) = L. \end{gathered} \tag{4.1}$$

It is easy to see that the map $\theta$ is consistent with the relations (3.1)–(3.3) of the Heisenberg double definition 3.1. Less trivial is checking consistency of this map with the $SL(n)$ reduction conditions (3.6). A crucial role for this consistency plays the normalization of the matrix $L$: $e_n(L) = q^{-1}\,1$.

**Proposition 4.1.** *The map $\theta$ (4.1) gives rise to an automorphism of the Heisenberg double $\mathcal{HD}_{SL(n)}(R)$.*

In [AF1, AF2] series of the automorphisms (4.1) have been treated as a discrte time evolution of the q-deformed quantum isotropic top. It is crucial for this interpretation that the map $\theta$ is to be realised as an inner algebra automorphism

$$T(k+1) \;=\; L\,T(k) \;=\; \Theta\, T(k)\, \Theta^{-1}. \tag{4.2}$$

Here operator $\Theta$, being an element of the Heisenberg double, plays a role of the model's (discrete time) evolution operator. Unfortunately, in the algebra $\mathcal{HD}_{SL(n)}(R)$ such an operator was not found. It turns out that one can construct $\Theta$ in the spectral extension of the algebra $\mathcal{HD}_{SL(n)}(R)$. In this case, it is also possible to define the evolution of the q-deformed isotropic top in the continuous time.

Taking into account invariance of the matrix $L$ under automorphisms (4.1) and an algebraic dependence of the spectral variables (3.12), we will use following ansatz fot the operator $\Theta$:

$$\Theta = \Theta(\mu_1, \ldots, \mu_{n-1}). \tag{4.3}$$

Sricktly speaking, we will search $\Theta$ as a formal power series in the variables $\mu_\alpha$, $\alpha = 1, \ldots, n-1$.

**Proposition 4.2.** *In the Heisenberg double algebra $\mathcal{HD}_{SL(n)}(R)$ the evolution operator $\Theta$ (4.2), (4.3) satisfy finite difference relations*

$$q\mu_\alpha\, \Theta(\nabla_\alpha(\mu_\beta)) \;=\; \Theta(\mu_\beta) \quad \forall\, \alpha = 1, \ldots, n-1. \tag{4.4}$$

*Here $\nabla_\alpha$ are the finite shift operators (3.21).*

Motivated by a physical interpretation we pass to a logarithmic parameterization of the spectral variables and the parameter $q$

$$q = \exp(2\pi \mathrm{i}\,\tau), \qquad q^{1/n}\mu_\alpha = \exp(2\pi \mathrm{i}\,x_\alpha) : \quad \sum_{\alpha=1}^{n} x_\alpha = 0. \tag{4.5}$$

**Theorem 4.3.** *[IP2] In the class of formal series in the variables $x_\alpha$, $\alpha = 1, \ldots, n-1$, two functions $\Theta^{(a)}$, $a = 1, 2$,*

$$\Theta^{(1)}(x_\alpha) \;=\; \sum_{\vec{k}\in\mathbb{Z}^{n-1}} \exp\left\{\pi \mathrm{i}\,(\vec{k},\,\Omega\,\vec{k}) + 2\pi \mathrm{i}\,(\vec{k}\,,\,\vec{x})\right\}, \quad \Omega_{\alpha\beta} = 2\tau(\delta_{\alpha\beta} - \tfrac{1}{n}), \tag{4.6}$$

$$\begin{aligned}\Theta^{(2)}(x_\alpha) \;&=\; \exp\left\{-\tfrac{\pi \mathrm{i}}{2\tau}\textstyle\sum_{\beta=1}^{n} x_\beta^2\right\} \\ &=\; \exp\{-\pi \mathrm{i}\,(\vec{x},\,\Omega^{-1}\vec{x})\}, \quad (\Omega^{-1})_{\alpha\beta} = \tfrac{1}{2\tau}(\delta_{\alpha\beta} + 1).\end{aligned} \tag{4.7}$$

*both solve the equations (4.4).*

*Here $\Theta^{(1)} = \theta(\vec{x}, \Omega)$ is the Riemann theta function, $\tau$ is its modular parameter, $\Omega$ — $(n-1)\times(n-1)$ is its matrix of periods (see [Mum, Ig]).*

*The series $\Theta^{(1)}$ converges in case $|q| < 1$. If $q$ is the rational root of unity, the series $\Theta^{(1)}$ can be truncated to a final sum.*

*The series $\Theta^{(2)}$ is well defined for any values of the parameters $\tau \neq 0$ and $x_\alpha$. In contrast to the case of $\Theta^{(1)}$, the logarithmic parameterization $\mu_\alpha \mapsto x_\alpha$ (4.5) is essential for the solution $\Theta^{(2)}$.*

*Relation between the solutions $\Theta^{(1)}$ and $\Theta^{(2)}$ is given by a particular case of the modular functional equation (see [Mum], chapter 2, section 5).*

$$\Theta^{(2)}(\vec{x}) \;=\; \left(\tfrac{(-2\,\mathrm{i}\,\tau)^{n-1}}{n}\right)^{1/2} \frac{\theta(\vec{x},\,\Omega)}{\theta(\Omega^{-1}\vec{x},\,-\Omega^{-1})}\,. \tag{4.8}$$

*In case $|q| = 1$, the solution $\Theta^{(2)}$ is interpreted as the unitary evolution operator of the quantum $q$-deformed Euler top for a unit period of time. The Hamiltonian of this model is $H = -\pi(\vec{x},\,\Omega^{-1}\vec{x})$.*

## 5 Quantization of the differential calculus over linear groups

Studies of the differential calculus over quantum groups began almost simultaneously with the discovery of the quantum groups. General frameworks for these investigations were given by the bicovariance postulates of S. Woronowicz [W] and the R-matrix ideology [FRT]. Quickly enough, the external algebras of the differential forms on the series of linear groups $GL(n)$ were quantized and the exterior derivatives over these algebras were constructed (see papers [Ju, Mal1, Mal2, Su1, Su2, Tz, SWZ1, SWZ2, Zu] and reviews [Is, KSch]). However, all attemps to quantize de Rahm complexes over series $SL(n)$ following Woronowich's approach ended to no avail. It turned out that the $SL$-reduction procedure is incompatible with the classical graded Leibniz rule postulated for the quantized exterior derivative.

In this section, we present a scheme for constructing differential calculus over quantum groups of the series $SL(n)$ which uses q-deformed version of the Leibniz rule. This scheme has been suggested in papers [FP1, FP2]. Note that for the quantum groups of the series $O(n)$, $Sp(2n)$, the satisfactory construction of the differential calculus is still not known (see [AIP]).

We consider a differential calculus algebra on a linear quantum matrix group generated by the components of four matrices of generators:

$\|T^i_j\|_{i,j=1}^{\mathrm{N}}$ – coordinate functions over quantum matrix groups,

$\|\Omega^i_j\|_{i,j=1}^{\mathrm{N}}$ – right-invariant 1-forms,

$\|L^i_j\|_{i,j=1}^{\mathrm{N}}$ – right-invariant Lie derivatives,

$\|K^i_j\|_{i,j=1}^{\mathrm{N}}$ – left-invariant Lie derivatives.

The matrices of generators $T$, $L$ and the algebras they generate were already discussed in the previous sections. In the absence of differential forms the Lie derivatives $L$ can be identified with the vector fields.

Following theorem defines quantized differential calculi algebras of the types $GL(n)$ and $SL(n)$.

**Definition-theorem 5.1.** *To any $GL(n)$ type R-matrix $R$ (see Def. 1.1) we put into correspondence an algebra $\mathcal{DC}_{GL(n)}(R)$ generated by the components of four matrices $T$, $\Omega^g$, $L$, $K$, subject to relations*

$$\begin{aligned} R_1T_1T_2 &= T_1T_2R_1, && (5.1)\\ R_1L_1R_1L_1 &= L_1R_1L_1R_1, && (5.2)\\ R_1L_1R_1T_1 &= q^{2/n}\, T_1L_2, && (5.3)\\ R_1K_2R_1K_2 &= K_2R_1K_2R_1, && (5.4)\\ T_2R_1K_2R_1 &= q^{2/n}\, K_1T_2, && (5.5) \end{aligned}$$

$$\begin{aligned} R_1\Omega^g_1R_1\Omega^g_1 &= -\,\Omega^g_1R_1\Omega^g_1R_1^{-1}, & (5.6)\\ R_1\Omega^g_1R_1^{-1}T_1 &= T_1\Omega^g_2, & (5.7)\\ R_1L_1R_1\Omega^g_1 &= \Omega^g_1R_1L_1R_1, & (5.8) \end{aligned}$$

$$K_1\Omega^g_2 \,=\, \Omega^g_2K_1, \qquad K_1L_2 \,=\, L_2K_1. \tag{5.9}$$

*Assuming that the $GL(n)$ type $R$-matrix $GL(n)$ is such that its corresponding matrix $O_{\scriptscriptstyle R}$ (2.11) is scalar, we define an algebra $\mathcal{DC}_{SL(n)}(R)$. It is generated by the components of four matricrs $T$, $\Omega$, $L$, $K$ subject to relations which are identical to relations for $T$, $\Omega^g$, $L$, $K$ with the only exception: the relation (5.6) is substituted by*

$$R_1\Omega_1R_1\Omega_1 \,+\, \Omega_1R_1\Omega_1R_1^{-1} \;=\; \kappa_q\left(\Omega_1^2 \,+\, R_1\Omega_1^2R_1\right), \tag{5.10}$$

$\kappa_q = \frac{q^n(q-q^{-1})}{n_q+q^n(q-q^{-1})}$, *under condition that* $n_q + q^n(q-q^{-1}) \neq 0.$

*The generators of $\mathcal{DC}_{SL(n)}(R)$ are also subject to the $SL(n)$-reduction conditions*

$$\det{}_{\scriptscriptstyle R} T = q^{n^2}\, e_n(T) \,=\, 1, \quad e_n(L) \,=\, e_n(K) \,=\, q^{-1}\,1, \quad \mathrm{Tr}_{\scriptscriptstyle R}(\Omega) \,=\, 0. \tag{5.11}$$

*Here expression for the element $e_n(K)$ is given by the same formulas as for $e_n(L)$ (см. (2.3)), where the $R$-matrix $R = R_{12}$ is to be substituted by $R_{21} = PRP$.*

*A mapping of generators*

$$\{T, L, K\} \mapsto \{T, L, K\}. \quad \Omega \,\mapsto\, \Omega^g - \tfrac{q^n}{n_q}\mathrm{Tr}_{\scriptscriptstyle R}(\Omega^g)\, I \tag{5.12}$$

*generates the homomorphic map $\mathcal{DC}_{SL(n)}(R) \hookrightarrow \mathcal{DC}_{GL(n)}(R)$.*

**Comments.**

- Quantization of the $GL(n)$ type exterior algebra of differential forms $\Omega^g$, that is, the set of relations (5.6)-(5.8), was presented for the first time in [SWZ1, Zu]. Formulas for the $SL(n)$-reduction of the exterior algebra, i.e., the relations (5.10)–(5.12), were suggested and checked for their consistency with relations (5.1)-(5.9) in [IP2, FP1].

- In a standard way one can supply the differential calculi algebras $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$ with the bicovariant bimodule and the Hopf algebra structures (5.1) (see, e.g., [KSch]). Ii is the presence of these additional structures that justify interpretation of $\mathcal{DC}_{GL(n)}(R)$ and $\mathcal{DC}_{SL(n)}(R)$ as differential calculi algebras over quantum groups.

The next theorem describes construction of an exterior derivative map over the differential calculi algebras.

**Theorem 5.2.** *[FP1]. To calculate the action of the exterior derivative on an arbitrary element of algebras $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$ we first transform the element using the permutation relations (5.1)–(5.10) to a linear combination of monomials, where components of the generating matrices appear in a following order:*

$$K - \Omega - T - L.$$

*The exterior derivative d on each ordered monomial is given by a following deformed graded Leibnitz rule;*

$$\begin{aligned} d(K\ldots) &= Kd(\ldots), \quad d(\ldots L) \;=\; d(\ldots)L, && (5.13)\\ d(\Omega\ldots) &= (\Omega^2 \,-\, \Omega\, d)(\ldots), && (5.14)\\ d(T_1T_2\ldots T_k) &= \Big\{I + \kappa_q(S^{(k)}_{R}(I) - I)\Big\}^{-1} S^{(k)}_{R}(\Omega)\, T_1T_2\ldots T_k, && (5.15) \end{aligned}$$

*where for any* $\mathrm{N}\times\mathrm{N}$ *matrix X*

$$S^{(k)}_{R}(X_1) \;=\; X_1 \;+\; \sum_{i=1}^{k-1} R_i\ldots R_2R_1X_1R_1R_2\ldots R_i. \qquad (5.16)$$

*The graded linear map d defined in this way obeys nilpotency property* $d^2 = 0$, *and is a homomorphism of the differential calculi algebras* $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$.

**Comments.**

- In the classical limit (3.7), (3.8) algebras $\mathcal{DC}_{GL(n)}(R)$ and $\mathcal{DC}_{SL(n)}(R)$ turn into the exterior algebras of differential forms supplied with the action of Lie derivatives, respectively, over the Lie groups $GL(n)$ and $SL(n)$. Components of the matrix $\Omega^j_k$ become the basic right-invariant 1-forms: $\sum_{p=1}^{\mathrm{N}}(t^{-1})^j_p\, dt^p_k$. Relations (5.14)-(5.16) turn into the usual rules for the action of the exterior derivative on functions and on differential forms.

- For correctness of the action of $d$ on functions (5.15) we demand that matrices $\{I + \kappa_q(S^{(k)}_{R}(I) - I)\}\;\forall\, k \geq 2$ be invertible. These conditions impose further restrictions on the possible values of the quantization parameter $q$. Using the techniques developed in [FP2] one can find these restrictions explicitly

$$q:\; \sum_{k=1}^{n} q^{m_k} \neq 0 \quad \forall\, m_k \in \mathbb{Z}. \qquad (5.17)$$

All values of $q$ forbidden by conditions (5.17) belong to the annulus $1/2 < |q| < 2$. The limiting points of the set of forbidden values are lying on the unit circle $|q| = 1$. For the allowed values of the parameter $q$ cohomologies of the differential complex (5.14)-(5.16) coincide with de Rahm cohomologies of Lie groups $GL(n)$ and $SL(n)$ [FP2].

# 6 Spectral extension and anti-involution for the differential calculi algebras

In this section we construct unitary type anti-involution for the differential calculi algebras of the previous section: $\mathcal{DC}_{GL(n)}(R)$ and $\mathcal{DC}_{SL(n)}(R)$. To this end we define two additional structures on these algebras. The first additional structure is the Gaussian decomposition for the matrices of Lie derivatives $L$ and $K$. The second structure is the spectral extension of the differential calculi algebras which is analogous to the spectal extension discussed in section 3.

Let us first deal with the spectral extension, which we carry out using sets of spectral variables of three matrices: two generating matrices of Lie derivatives $L$ and $K$, and the matrix

$$F^j_k \,=\, (L\,T\,K\,T^{-1})^j_k, \tag{6.1}$$

whose components also satisfy the reflection equation. The $SL(n)$-reduction conditions (5.11) impose following relation on the matrix $F$ [P2]

$$e_n(F) \,=\, q^{-n^2}1. \tag{6.2}$$

By analogy with the spectral extension of the reflection equation algebra of right-invariant Lie derivatives $L$ (see Def. 3.4) we introduce spectral extension of the two other reflction equation algebras generated by the matrices $K$ and $F$

$$\begin{array}{rcl} e_\alpha(K) &=& e_\alpha(\nu_1,\dots,\nu_n), \qquad K^i_j\,\nu_\alpha \,=\, \nu_\alpha\,K^i_j, \\ e_\alpha(F) &=& e_\alpha(\rho_1,\dots,\rho_n), \qquad F^i_j\,\rho_\alpha \,=\, \rho_\alpha\,F^i_j \\ && \forall\, i,j = 1,\dots,\mathrm{N}, \quad \forall\,\alpha = 1,\dots,n. \end{array} \tag{6.3}$$

The $SL(n)$-reduction conditions for the sets of spectral variables $\{\nu_\alpha\}$ and $\{\rho_\alpha\}$ read

$$e_n(K) \,=\, \prod_{\alpha=1}^n \nu_\alpha \,=\, q^{-1}1, \qquad e_n(F) \,=\, \prod_{\alpha=1}^n \rho_\alpha \,=\, q^{-n^2}1. \tag{6.4}$$

Using the characteristic identities from the Cayley-Hamilton theorem 2.3 we construct two sets of orthogonal matrix projectors

$$Q^{\alpha} \;=\; \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{(K - q\nu_\beta I)}{q(\nu_\alpha - \nu_\beta)} \;:\quad Q^{\alpha}Q^{\beta} \;=\; \delta_{\alpha\beta}\, Q^{\alpha}\,, \quad \sum_{\alpha=1}^{n} Q^{\alpha} \;=\; I, \tag{6.5}$$

$$S^{\alpha} \;=\; \prod_{\substack{\beta=1\\ \beta\neq\alpha}}^{n} \frac{(F - q\rho_\beta I)}{q(\rho_\alpha - \rho_\beta)} \;:\quad S^{\alpha}S^{\beta} \;=\; \delta_{\alpha\beta}\, S^{\alpha}\,, \quad \sum_{\alpha=1}^{n} S^{\alpha} \;=\; I. \tag{6.6}$$

**Definition-theorem 6.1.** *[P2].*
*For the differential calculi algebras $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$ consider their extension by the set of rational functions of $3n$ mutually commutative generators $\mu_1,\dots,\mu_n$, $\nu_1,\dots,\nu_n$, $\rho_1,\dots,\rho_n$, satisfying relations (3.11), (3.16), (6.3), and also relations*

$$\mu_\alpha\, K \;=\; K\, \mu_\alpha, \qquad \nu_\alpha\, Y \;=\; Y\, \nu_\alpha \quad \forall\, Y = L, K, \Omega, \tag{6.7}$$

$$\rho_\alpha\, Z \;=\; Z\, \rho_\alpha \quad \forall\, Z = T, L, K, \tag{6.8}$$

$$\nu_\alpha\,(TQ^{\beta}) \;=\; q^{2(\delta_{\alpha\beta}-\frac{1}{n})}\,(TQ^{\beta})\,\nu_\alpha, \tag{6.9}$$

$$q^{2\delta_{\alpha\sigma}}(P^{\beta}\,\Omega\,P^{\sigma})\,\mu_\alpha \;=\; q^{2\delta_{\alpha\beta}}\mu_\alpha\,(P^{\beta}\,\Omega\,P^{\sigma}), \tag{6.10}$$

$$q^{2\delta_{\alpha\sigma}}(S^{\beta}\,\Omega\,S^{\sigma})\,\rho_\alpha \;=\; q^{2\delta_{\alpha\beta}}\rho_\alpha\,(S^{\beta}\,\Omega\,S^{\sigma}) \quad \forall\, \alpha,\beta,\sigma = 1,\dots,n. \tag{6.11}$$

*For the extension of $\mathcal{DC}_{SL(n)}(R)$ one also adds the $SL(n)$-reduction conditions (3.12), (6.4).*[3]

*Such extensions are called spectral extensions of the differential calculi algebras.*

In the differential calculi algebras over the linear quantum groups there exist three series of maps which are similar to the Heisenberg double automorphism (4.1). The next theorem describes these maps and gives their presentations as inner algebra automorphisms in the spectral extensions of the algebras.

**Theorem 6.2.** *[P2]. The maps $\theta_L$, $\theta_K$ and $\theta_F$ defined on the generators as*

$$\theta_L:\; T \mapsto LT, \quad \Omega \mapsto L\Omega L^{-1}, \quad L \mapsto L, \quad K \mapsto K; \tag{6.12}$$

$$\theta_K:\; T \mapsto TK, \quad Y \mapsto Y \quad \forall\, Y = L, K, \Omega; \tag{6.13}$$

$$\theta_F:\; T \mapsto T, \qquad \Omega \mapsto F\Omega F^{-1}, \quad L \mapsto L, \quad K \mapsto K. \tag{6.14}$$

*give rise to automorphisms of algebras $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$. These automorphisms are mutually commutative.*

[3] In [P2] the set of formulas (6.7)-(6.11) defining the spectral extension is supplemented by relations $[\rho_\alpha, \mathrm{Tr}_R X] = 0$, where $X$ is arbitrary matrix polynomial in $\Omega$ and $F$. Using the technical trick from Remark 3.4 [P2] one can show that the latter relations follow from eqs. (6.11).

*Passing to the logarithmic parameterization of $q$ and the spectral variables $\{\mu_\alpha, \nu_\alpha, \rho_\alpha\}$*

$$
\begin{gathered}
q = e^{2\pi\mathrm{i}\,\tau}, \quad q^{1/n}\mu_\alpha = e^{2\pi\mathrm{i}\,x_\alpha}, \quad q^{1/n}\nu_\alpha = e^{2\pi i y_\alpha}, \quad q^n\rho_\alpha = e^{2\pi i z_\alpha}: \\
\textstyle\sum_{\alpha=1}^n x_\alpha \;=\; \sum_{\alpha=1}^n y_\alpha \;=\; \sum_{\alpha=1}^n z_\alpha \;=\; 0,
\end{gathered} \tag{6.15}
$$

*we consider the spectral extensions of the differential calculi algebras by the formal power series in variables $x_\alpha, y_\alpha, z_\alpha$. For such spectral extensions of the differential calculi algebras $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$ the element*

$$
\Theta_{(t_1,t_2,t_3)} \;=\; \exp\{-\tfrac{i\pi}{2\tau}\textstyle\sum_{\alpha=1}^n(t_1x_\alpha^2 - t_2y_\alpha^2 + t_3z_\alpha^2)\} \tag{6.16}
$$

*gives 3-parametric family of their inner algebra automorphisms*

$$
\theta_{(t_1,t_2,t_3)}: \quad u \mapsto \Theta_{(t_1,t_2,t_3)}\, u\, (\Theta_{(t_1,t_2,t_3)})^{-1} \;\; \forall\, u \in \mathcal{DC}_{GL(n)}(R), \mathcal{DC}_{SL(n)}(R). \tag{6.17}
$$

*For particular values of parameters $t_i$ this automorphism coincides with $\theta_L$, $\theta_K$ and $\theta_F$:*

$$
\theta_L \;=\; \Theta_{(1,0,0)}, \quad \theta_K \;=\; \Theta_{(0,1,0)}, \quad \theta_F \;=\; \Theta_{(0,0,1)}. \tag{6.18}
$$

We now turn to description of the Gaussian decomposition of the matrices of quantized Lie derivatives.

In so doing we restrict ourselves to considering algebras $\mathcal{DC}_{GL(n)}(R_{(DJ)})$, $\mathcal{DC}_{SL(n)}(R_{(DJ)})$ corresponding to the Drinfeld-Jimbo R-matrix (1.15). This is because the Gaussian decomposition of the generating matrices is currently only developed for the reflection equation algebras associated with the Drinfeld-Jimbo R-matrices. Further on, in all formulas of this section, $R$ stands for $R_{(DJ)}$.

Following [FRT], we introduce two mutually commutative pairs of the RTT-algebras, which are generated by the components of upper/lower-triangular matrices[4]

$$
L^{(\pm)} = ||L^{(\pm)}{}_i^j||_{i,j=1}^{\;n}, \qquad K^{(\pm)} = ||K^{(\pm)}{}_i^j||_{i,j=1}^{\;n},
$$

subject to relations

$$
\begin{gathered}
R_1L_2^{(\pm)}L_1^{(\pm)} \;=\; L_2^{(\pm)}L_1^{(\pm)}R_1, \qquad R_1L_2^{(+)}L_1^{(-)} \;=\; L_2^{(-)}L_1^{(+)}R_1, \\
R_1K_2^{(\pm)}K_1^{(\pm)} \;=\; K_2^{(\pm)}K_1^{(\pm)}R_1, \qquad R_1K_2^{(+)}K_1^{(-)} \;=\; K_2^{(-)}K_1^{(+)}R_1, \\
(L^{(-)})_i^i(L^{(+)})_i^i \;=\; (K^{(-)})_i^i(K^{(+)})_i^i \;=\; 1 \quad \forall\, i = 1, \dots, n.
\end{gathered} \tag{6.19}
$$

[4]Recall that for theDrinfeld-Jimbo R-matrices $n = \dim V = \mathrm{N}$.

For the $SL(n)$ case these relations are to be supplemented by the reduction conditions

$$\prod_{i=1}^{n}(L^{(\pm)})_i^i \,=\, 1, \qquad \prod_{i=1}^{n}(K^{(\pm)})_i^i \,=\, 1. \tag{6.20}$$

As is known, the reflection equation algebra can be realized in terms of a pair of upper/lower triangular RTT algebras (see, e.g., [KSch], pp.345-347). Such a realization is called *Gaussian decomposition*. For the matrices of left/right-invariant Lie derivatives it reads

$$L \,=\, q^{n-1/n}(L^{(-)})^{-1}L^{(+)}\,, \qquad K \,=\, q^{n-1/n}K^{(+)}(K^{(-)})^{-1}\,. \tag{6.21}$$

The Gaussian decomposition can extendend in a consistent way to the spectral extensions of the differential calculi algebras $\mathcal{DC}_{GL(n)}(R)$, $\mathcal{DC}_{SL(n)}(R)$, that is, one can define permutation relations of the components of triangular matrices $L^{(\pm)}$ and $K^{(\pm)}$ with the components of all generating matrices $T$, $L$, $K$, $\Omega$ and with the spectral variables $\mu_\alpha$, $\nu_\alpha$, $\rho_\alpha$ (see, e.g., [P2], section 4).

In the algebras (6.19) there is known unitary type anti-involution (see., e.g., [AF2])

$$(L^{(\pm)})^\dagger = (L^{(\mp)})^{-1}, \qquad (K^{(\pm)})^\dagger = (K^{(\mp)})^{-1}. \tag{6.22}$$

Here symbol "$\dagger$" denotes the operation of Hermite conjugation of matrices, that is a composition of the matrix transposition and the anti-involution. This operation is compatible with the algebra relations (6.19), if the R-matrix $R$ fulfills the condition

$$R^\dagger = PR^{-1}P. \tag{6.23}$$

For the Drinfeld-Jimbo R-matrices this relation is satisfied if the parameter $q$ belongs to the unit circle in the complex plane: $|q| = 1$.

The main result of this section is a generalization of the operation $\dagger$ for the differential calculi algebras.

**Theorem 6.3.** *Let us fix the quantization parameter $q$ as: $q = e^{2\pi i\tau}$, $\tau \in \mathbb{R}$. For the spectral extension of the differential calculus algebra $\mathcal{DC}_{GL(n)}(R)$, corresponding to the Drinfeld-Jimbo R-matrix (1.15) the Hermite conjugation $\dagger$ (6.22) on the generating matrices can be defined in a following way*

$$\mu_\alpha^\dagger \;=\; \mu_\alpha^{-1}, \qquad \nu_\alpha^\dagger \,=\, \nu_\alpha^{-1}, \qquad \rho_\alpha^\dagger \,=\, \rho_\alpha^{-1}, \tag{6.24}$$

$$L^\dagger \;=\; q^{1/n-n}L^{(-)}(L^{(+)})^{-1}, \tag{6.25}$$

$$K^\dagger \;=\; q^{1/n-n}(K^{(+)})^{-1}K^{(-)}, \tag{6.26}$$

$$
\begin{aligned}
T^\dagger &= q^{n-1/n}(K^{(-)})^{-1}\,\Theta_{(\frac12,\frac12,\frac12)}\big(T^{-1}\big)\,(L^{(-)})^{-1}, && (6.27)\\
(\Omega^g)^\dagger &= -\,L^{(-)}\,\Theta_{(\frac12,\frac12,\frac12)}\big(F^{-1}\Omega^g\big)\,(L^{(-)})^{-1}, && (6.28)\\
F^\dagger &= L^{(-)}\,\Theta_{(\frac12,\frac12,\frac12)}\big(F^{-1}\big)\,(L^{(-)})^{-1}. && (6.29)
\end{aligned}
$$

*Here the automorphism $\Theta_{(\frac12,\frac12,\frac12)}$ is given by (6.16), (6.17).*

*The anti-linear algebra anti-homomorphism $\dagger$, given on generators by eqs. (6.24)-(6.29), is anti-involutive: $\dagger^2 = id$.*

*Restriction of the map $\dagger$ to the spectral extension of the differential calculus algebra $\mathcal{DC}_{SL(n)}(R)$ is the anti-involutive anti-homomorphiasm of a pair of two subalgebras. Their matrices of right-invariant 1-forms are, respectively, $\Omega$ (5.12) and $\Omega^\dagger$:*

$$
\Omega^\dagger \,=\, (\Omega^g)^\dagger - \tfrac{q^n}{n_q}\mathrm{Tr}_{R_{op}}(\Omega^g)^\dagger I, \qquad \mathrm{Tr}_{R_{op}}(\Omega^g)^\dagger \,=\, -\,\mathrm{Tr}_{R}(F^{-1}\Omega^g), \tag{6.30}
$$

*where $R_{op} = PRP$.*

# 7 Factorized solutions of the quantum Knizhnik-Zamolodchikov equations

In this section, we will discuss application of the R-matrix techniques for construction of solutions of the quantum Knizhnik-Zamolodchikov equations.

The quantum (also called the q-deformed, or finite difference) Knizhnik-Zamolodchikov equations (qKZ) first appeared in literature as equations for form factors and correlation functions of the integrable quantum field theory and statistical physics models [Sm, JM]. In mathematics, these equations arise in the study of representations of affine quantum groups [FR] and affine (double) Hecke algebras [Ch2]. Originally, in studying solutions of the qKZ equations the researchers interests were mainly focused on representations of affine quantum groups and Hecke algebras in the tensor product spaces (see the review [EFK] and references therein).

A new area of applications of the qKZ equations arised in connection with the discovery by A. Razumov and Yu. Stroganov of remarkable manifestations of the combinatorics of alternating matrices and plane partitions in the properties of ground states of the XXZ spin chains [RS1]. although in their non-Hermitian regimes. Subsequently, the Razumov-Stroganov observations have been given a physical interpretation in applicaations to the stochastic O(1) loop model [BGN, RS2] and to the stochastic process called "Raise and Peel"[dG-R, P-N]. Connection of the Razumov-Stroganov observations to solutions of the qKZ equations have been established by V. Pasquier, F. Di Francesco, and P. Zinn-Justin [Pa, DFZ1]. By analyzing solutions of the qKZ equations a substantial progress in understanding the Rasumov-Stroganov phenomena was achieved. Some of their hypotheses were finally proved, a number of new hypotheses were stated (see, for example, [KP, DF2, DFZ2]).

In this section, motivated by the Razumov-Stroganov hypothesis we consider solutions of the qKZ equations related to the irreducible representations of Temperley-Lieb algebras[5] on the Dyck paths and to their generalizations. We obtain factorized expressions for these solutions. We are treating the case of qKZ equations on an open segment with boundary conditions of two types, denoted below as type A and type B. For the boundary conditions of type A, similar to ours expressions were obtained earlier by A. Kirillov and A. Lascoux in their studies of Kazhdan-Lustig elements for Grassmannians [KL]. We note that the use of the factorized formulas allows making new observations of combinatorics in the stationary states of the Raise and Peel model and finding new connections of the qKZ solutions with enumerations of the plane partitions and the alternating matrices, and also with solutions of the discrete Hirota equations (see [P1, APR, dGPZ]).

In formulating the qKZ equations, we use basic information about Temperley-Lieb algebras and their representations.

**Definition 7.1.** *Temperley-Lieb algebra $\mathcal{T}^A_L(q)$ of type A is a quotient algebra of the Hecke algebra $\mathcal{H}_L(q)$ by relation $a^{(3)} = 0$ (see (1.12)). For our purposes it is suitable to describe this algebra in terms of new generators $e_i$, related to Artin's generators $g_i$ as follows*

$$e_i = g_i - q1 \quad \forall\, i = 1, \dots, L-1.$$

*In terms of the new set of generators the algebra $\mathcal{T}^A_L(q)$ is given by relations*

$$\begin{aligned} e_i^2 \,&=\, \tau e_i, \;\; \tau = -[2]_q, \qquad e_i e_{i\pm 1} e_i \,=\, e_i, \\ e_i e_j \,&=\, e_j e_i \qquad \forall\, i,j = 1,\dots,L-1 :\; |i-j| > 1, \end{aligned} \tag{7.1}$$

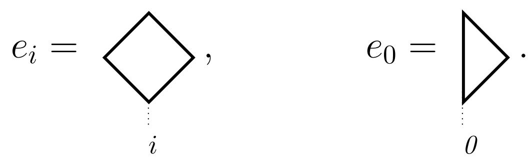
*Temperley-Lieb algebra $\mathcal{T}^B_L(q,\omega)$ of type B is an extension of the algebra $\mathcal{T}^A_L(q)$ by a new generator $e_0$, satisfying relations*

$$e_0^2 \,=\, -\tfrac{[\omega]_q}{[\omega+1]_q} e_0, \quad e_1 e_0 e_1 \,=\, e_1, \quad e_0 e_i \,=\, e_i e_0 \quad \forall\, i = 2,\dots,L-1. \tag{7.2}$$

[5]Temperley-Lieb algebras are the quotient algebras of the Hecke algebras. Their R-matrix representations are Schur–Weyl dual to the representations of the quantum group $U_q(sl_2)$.

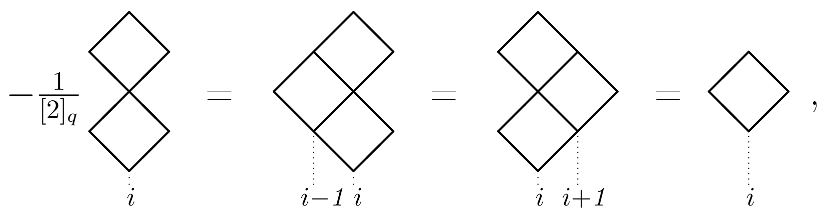
*Here $\omega \in \mathbb{Q}$ is a new algebra parameter satisfying condition $[\omega+1]_q \neq 0$.*

We will use a graphical presentation of the Temperley-Lieb algebras. In this presentation the generators $e_i$ and $e_0$ are drawn, respectively, as squares and triangles (tiles) which move freely along vertical lines. The horizontal coordinates of these tiles are determined by the indices of the corresponding generators:



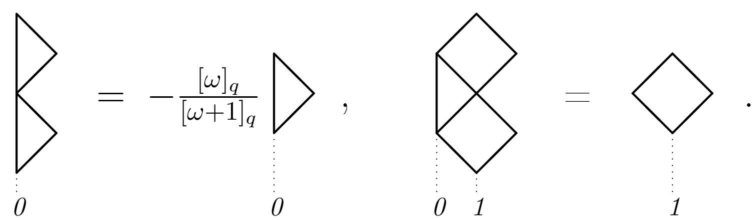

(7.3)

Nontrivial relations among generators of the Temperley-Lieb algebras (7.1) and (7.2) in the graphical presentation look, correspondingly:




(7.4)

и




(7.5)

Consider left ideals in the Temperley-Lieb algebras generated by the following idempotents

$$\mathcal{T}_L^A(q) \ : \qquad I_L^A = \ e_1 e_3 \cdot \ldots \cdot e_{2\lfloor \frac{L}{2} \rfloor - 1}, \tag{7.6}$$

$$\mathcal{T}_L^A(q) \ : \qquad I_L^B = \begin{cases} e_1 e_3 \cdot \ldots \cdot e_{L-1}, & \text{if } L \text{ is odd,} \\ e_0 e_2 \cdot \ldots \cdot e_L, & \text{if } L \text{ is even.} \end{cases} \tag{7.7}$$

Idempotents $I_L^A$ and $I_L^B$ are primitive. Hense, in their corresponding ideals irreducible representations of the Temperley-Lieb algebras $\mathcal{T}_L^A(q)$ and $\mathcal{T}_L^B(q,\omega)$ are realized. Elements of linear bases of these representations can be presented graphically as, respectively, the Dyck paths and the ballot paths.

**Definition 7.2.** *Dyck path $\alpha$ of length $L$ is a sequence of $L+1$ nonnegative integers $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_L)$ satisfying conditions $\alpha_0 = 0$, $\alpha_{i+1} - \alpha_i = \pm 1$ and $\alpha_L = 0/1$ for $L$ even/odd.*

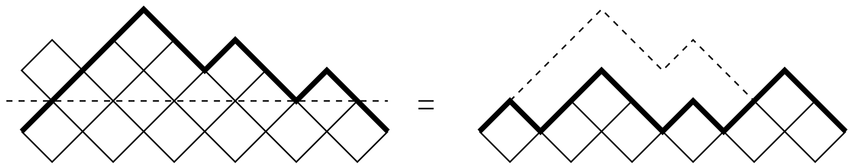
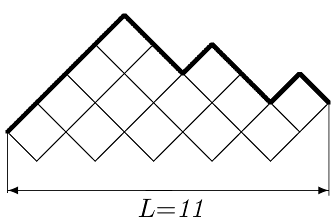


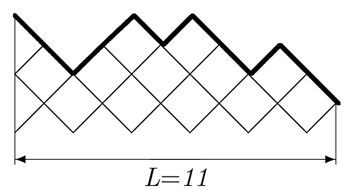
Figure 1. The action of generator $e_1$ on element $|\alpha\rangle = e_4(e_3e_5e_7)(e_2e_4e_6e_8e_{10})I^{A}{}_{12}$: $e_1|\alpha\rangle = e_4e_{10}I^{A}{}_{12}$.

*Ballot path $\alpha$ of length $L$ is a sequence of $L+1$ nonnegative integers $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_L)$ such, that $\alpha_L = 0$ and $\alpha_{i+1} - \alpha_i = \pm 1$.*

*We denote the sets of Dyck paths and ballot paths of length L as, respectively, $\mathcal{D}_L^A$ and $\mathcal{D}_L^B$.*

To any Dyck path $\alpha$ (ballot path $\beta$) of length $L$ we put into correspondence an element of the linear basis $|\alpha\rangle$ $(|\beta\rangle)$ in the ideal generated by the idempotent $I_L^A$ $(I_L^B)$ according to a folowing rule:

$|\alpha\rangle =$ 


$|\beta\rangle =$ 


$$\alpha = (0,1,2,3,4,3,2,3,2,1,2,1), \qquad \beta = (3,2,1,2,3,2,3,2,1,2,1,0),$$
$$|\alpha\rangle = e_4(e_3e_5e_7)(e_2e_4e_6e_8e_{10})I_{11}^A, \qquad |\beta\rangle = (e_0e_4e_6)(e_1e_3e_5e_7e_9)I_{11}^B.$$

This correspondence is the bijection.

The action of the generators $e_i$, $i \geq 0$, on the elements $w_\alpha$ is graphically represented as the process of deposition of tiles on the profile specified by Dyck, or ballot path. It can lead to a profile growth (tile falls on the point of local minimum), reflection from the profile's surface (tile hits the profile at its local maximum), or to avalanche-like shedding of the profile (tile falls onto the slope, see Fig.1). Such a graphical presentation allows us to interpret the action of Temperley-Lieb algebras in their representations on Dyck paths (ballot paths) as a stochastic process of growth of a one-dimensional surface on a segment with various types (A, or B) of boundary conditions (see, e.g., [P1]).

Consider two spaces of vectors $|\Psi\rangle^A$ and $|\Psi\rangle^B$, being linear combinations of the basic elements $|\alpha\rangle$ from the space of Dyck paths (type A), or the space of ballot paths (type B) with coefficients $\psi_\alpha$ taking values in the space of formal series in variables $q^{\pm x_i}$, $i = 1, 2, \ldots, L$:

$$|\Psi(x_1, \ldots, x_L)\rangle^{A/B} = \sum_{\alpha \in \mathcal{D}_L^{A/B}} \psi_\alpha(x_1, \ldots, x_L)|\alpha\rangle.$$

*Quantum Knizhnik-Zamolodchikov equations* on a segment of size $L$ with type A/B boundary conditions is a system of finite difference equations for the coefficients of vector $|\Psi\rangle^{A/B}$:

$$\begin{aligned} R_i(x_i - x_{i+1})|\Psi\rangle^{A/B} &= \pi_i|\Psi\rangle^{A/B} \qquad \forall\, i = 1, \ldots, L-1, && (7.8)\\ K_0^{A/B}(-x_1)|\Psi\rangle^{A/B} &= \pi_0|\Psi\rangle^{A/B}, && (7.9)\\ |\Psi\rangle^{A/B} &= \pi_L|\Psi\rangle^{A/B}. && (7.10) \end{aligned}$$

The right hand sides of these equations contain operators $\pi_i$, $0 \le i \le L$, acting by on the arguments of the coefficient functions $\psi_\alpha$ by permuations:

$$\begin{aligned} &\pi_i\psi_\alpha(\ldots, x_i, x_{i+1}, \ldots) = \psi_\alpha(\ldots, x_{i+1}, x_i, \ldots),\\ &\pi_0\psi_\alpha(x_1, \ldots) = \psi_\alpha(-x_1, \ldots), \qquad \pi_N\psi_\alpha(\ldots, x_L) = \psi_\alpha(\ldots, -\lambda - x_L). \end{aligned}$$

Here $\lambda \in \mathbb{C}$ is a parameter related to the characteristics of the qKZ equations called their *level* (see [EFK]).

The left-hand sides of the qKZ equations contain operators $R_i(u)$ and $K_0(u)$. These operators are the baxterized versions of the Temperley-Lieb algebra generators $e_i$ and $e_0$, respectively. Operators $R_i(u)$ are similar to the elements of $g_i(x)$ (1.9), which were encountered earlier in considerations of the QM-algebras. They satisfy additive versions of the Yang-Baxter equations (1.10) and the unitarity conditions (1.11). Operator $K_0(u)$ also satisfies the additive version of the reflection equation with spectral parameter, and the unitarity condition (see [dGP]). All the mentioned relations for $R_i(u)$ and $K_0(u)$ are nothing but the compatibility conditions of the qKZ equations.

Explicit expressions for $R_i(u)$ and $K_0(u)$ read

$$\begin{aligned} R_i(u) &= \frac{[1-u]_q - [u]_q\, e_i}{[1+u]_q},\\ K_0^A(u) &= 1, \qquad K_0^B(u) = \frac{k(u,\delta) + [2u]_q[\omega+1]_q\, e_0}{k(-u,\delta)}, && (7.11)\\ k(u,\delta) &= [\tfrac{\omega+\delta}{2} + u]_q[\tfrac{\omega-\delta}{2} + u]_q. \end{aligned}$$

Here $\delta \in \mathbb{Q}$ is yet another parameter, influencing behavior of the solution $|\Psi\rangle^B$ at the left boundary of the segmeent.

Relations (7.8)-(7.10) are representing the qKZ equations in a form suitable for physical applications. In constructing factorized formulas of their solutions, an alternative algebraic presentation of the qKZ equations is useful.

**Proposition 7.3.** *Relations (7.8) and type B relations (7.9) can be equivalently written as*

$$\sum_{\alpha\in\mathcal{D}_L^{A/B}} \psi_\alpha(x_1,\ldots,x_L)\,(e_i|\alpha\rangle) = \sum_{\alpha\in\mathcal{D}_L^{A/B}} (T_i - q1)\psi_\alpha(x_1,\ldots,x_L)|\alpha\rangle, \tag{7.12}$$

*u*

$$\sum_{\alpha\in\mathcal{D}_L^{B}} \psi_\alpha(x_1,\ldots,x_L)\,(e_0|\alpha\rangle) = \sum_{\alpha\in\mathcal{D}_L^{B}} \left(\frac{T_0+q^{-\omega}1}{q^{\omega+1}-q^{-\omega-1}}\right)\psi_\alpha(x_1,\ldots,x_L)|\alpha\rangle. \tag{7.13}$$

*Here operators $T_i$, $T_0$ are given by*

$$T_i \;=\; q - \frac{1}{[x_i-x_{i+1}]_q}(\pi_i-1)[x_{i+1}-x_i+1]_q \tag{7.14}$$

$$T_0 \;=\; -q^{-\omega} + (\pi_0+1)\,\frac{(q-q^{-1})\,k(-x_1,\delta)}{[2x_1]_q} \tag{7.15}$$

*These operators realize representations of type A and type B Hecke algebras in the space of formal series in variables $q^{\pm x_i}$, $i=1,2,\ldots,L$.*

**Comment.** The algebraic presentation to the qKZ equations was given[Pa]. Operators $T_0$ and $T_i$ were introduced into consideration inthe work of M. Noumi [N]. Operators $T_i$, $i>0$ are also known as Demazurre-Lesztig operators [Ch2].

First of all, let us consider conditions imposed by the qKZ equations on the coefficients of the basic elements $|\Omega^A\rangle$ and $|\Omega^B\rangle$ corresponding to a unique Dyck path and a unique ballot path that do not have local minima inside the segment:

$$\Omega^{\mathrm{A}} = (0,1,\ldots,\lfloor\tfrac{L-1}{2}\rfloor,\lfloor\tfrac{L+1}{2}\rfloor,\lfloor\tfrac{L-1}{2}\rfloor,\ldots,L(\mathrm{mod}\ 2)), \qquad \Omega^{\mathrm{B}} = (L,L-1,\ldots,1,0).$$

Considerations below will show that elements $|\Omega^A\rangle$ and $|\Omega^B\rangle$ are naturally treated as the highest vectors in representations of the Temperley-Lieb algebras on Dyck paths and ballot paths, respectively.

Let us introduce more notation

$$\Delta^{\pm}_{\mu}(x_k,\ldots,x_m) := \prod_{k\le i<j\le m} [\mu + x_i \pm x_j]_q.$$

Here $\mu\in\mathbb{Q}$ is a parameter.

**Proposition 7.4.** *Coefficients $\psi^{\mathrm{A}}_{\Omega}$ and $\psi^{\mathrm{B}}_{\Omega}$ standing for vectors $|\Omega^A\rangle$ and $|\Omega^B\rangle$ in solutions of the qKZ equations (7.8)-(7.10) $|\Psi\rangle^A$ u $|\Psi\rangle^B$ can be normalized to a following form*

$$
\begin{aligned}
{}^{\mathrm{A}}_{\Omega}(x_1,\dots,x_L) &= \Delta^-_1(x_1,\dots,x_\ell)\,\Delta^+_{-1}(x_1,\dots,x_\ell)\,\Delta^-_1(x_{n+1},\dots,x_L) \\
&\quad \Delta^+_{\lambda+1}(x_{\ell+1},\dots,x_L)\,\xi^{\mathrm{A}}(x_1,\dots,x_\ell|x_{\ell+1}+\frac{\lambda}{2},\dots,x_L+\frac{\lambda}{2}), \qquad (7.16)\\
{}^{\mathrm{B}}_{\Omega}(x_1,\dots,x_L) &= \Delta^-_1(x_1,\dots,x_L)\,\Delta^+_{\lambda+1}(x_1,\dots,x_L) \\
&\quad \xi^{\mathrm{B}}(x_1+\frac{\lambda}{2},\dots,x_L+\frac{\lambda}{2}). \qquad (7.17)
\end{aligned}
$$

*Here $\ell=\lfloor (L+1)/2\rfloor$; $\xi^{\mathrm{A}}(x_1,\dots x_\ell|x_{\ell+1},\dots,x_L)$ is an even symmetric function in each of the two sets of variables $x_i$, $1\le i\le \ell$ and $x_j$, $\ell+1\le j\le L$ separately; $\xi^{\mathrm{B}}(x_1,\dots,x_L)$ is an even symmetric function in all arguments $x_i$, $1\le i\le L$.*

For any Dyck path (ballot path) $\alpha$ we consider the procedure of its completion to the highest path $\Omega^{\mathrm{A}}$( $\Omega^{\mathrm{B}}$) by adding the tiles (squares and triangles). To each added tile we assign a natural number $u_{i,j}$ — the tile's *content*, where $i$ and $j$ are the horizontal and vertical coordinates of the tile. The rule for computing the content $u_{i,j}$ of the tile is following:

- $u_{i,j}=1$ if the added tile falls on the local minimum of the Dyck path (ballot path) $\alpha$;
- otherwise we put $u_{i,j}=\max\{u_{i+1,j-1},u_{i-1,j-1}\}+1$, where $u_{i\pm1,j-1}$ are the contents of the previously added tiles, the current tile falls atop.

Examples of graphical presentation of this procedure are given in Figures 2 and 3 below.

Using this procedure to each Dyck path (ballot path) $\alpha$ we put into correspondence following operator $H_\alpha$:

$$
H_\alpha = \prod_{i,j}^{\nearrow u} T_i(u_{ij}). \qquad (7.18)
$$

Here $T_i(u)$ are baxterized versions of operators $T_i$ (7.19), (7.20)[6]

$$
T_i(u) = \frac{q^{-u}}{[u]_q} + T_i = \frac{[x_i-x_{i+1}+u]_q}{[u]_q[x_i-x_{i+1}]_q} - \frac{[x_i-x_{i+1}+1]_q}{[x_i-x_{i+1}]_q}\,\pi_i, \qquad (7.19)
$$

[6] Operators $T_i(u)$, $i>0$, are nothing but the rescaled operators $g_i(q^{2u})$ (1.9).

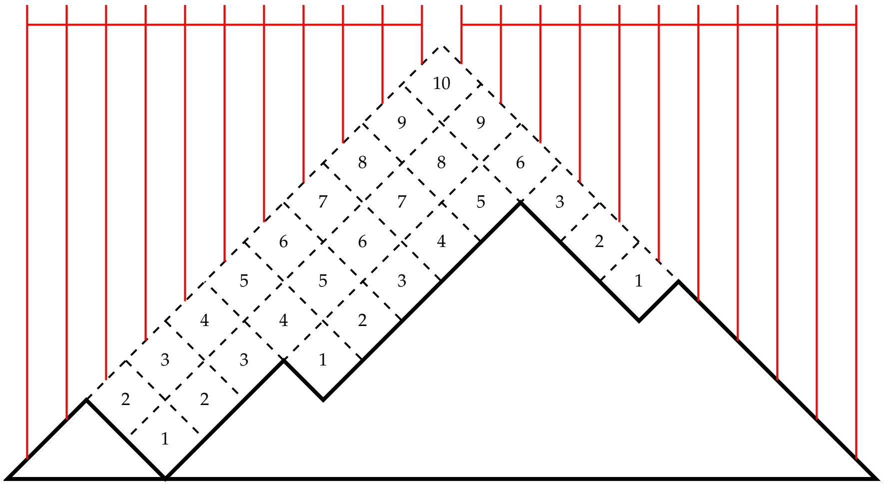


Figure 2: For the Dyck path $(0,1,2,1,0,1,2,3,2,3,4,5,6,7,6,5,4,5,4,3,2,1,0)$ added tiles are drawn in dashed lines. The tile's contents are placed inside them.

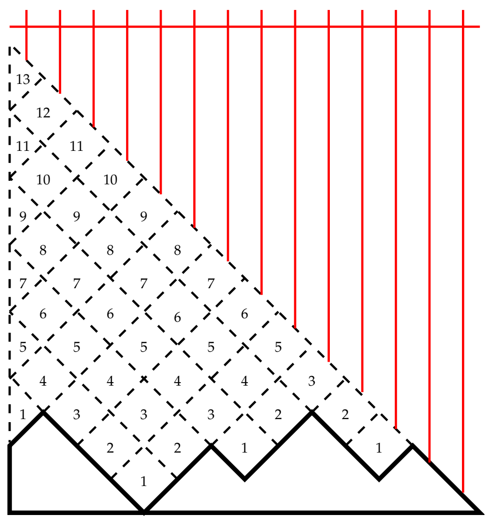


Figure 3: For the ballot path $(2,3,2,1,0,1,2,1,2,3,2,1,2,1,0)$ added tiles are drawn in dashed lines. The tile's contents are placed inside them.

$$T_0(u) \;=\; \frac{1}{(q^{\omega+1}-q^{-\omega-1})}\left(T_0 + q^{\omega} - \frac{(q-q^{-1})[\lfloor u/2\rfloor]_q[\omega+\lfloor (u+1)/2\rfloor]_q}{[u]_q}\right). \quad (7.20)$$

The product in (7.18) is taken over all added tiles and the factors of the product are ordered in such a way that their arguments $u_{i,j}$ do not decrease when reading from left to right (note that factors with the same argument commute).

Now we are ready to formulate this section's main result.

**Theorem 7.5.** *[KL, dGP] The coefficient function $\psi_\alpha$ in the solution of the qKZ equations (7.8)-(7.10) is given by formula*

$$\psi_\alpha = H_\alpha \; \Omega^{\mathrm{A/B}}, \quad (7.21)$$

*where* $\psi_\Omega^{\mathrm{A/B}}$ *are coefficient functions of the highest vectors* $|\Omega^{A/B}\rangle$ *(7.16), (7.17), and the factorized operators* $H_\alpha$ *for the Dyck paths (type A boundary conditions) and for the ballot paths (type B boundary conditions) are defined by the rule (7.18) (for illustrations see figures 2 and 3).*

*In the problem with type A boundary conditions, denote* $\beta(k)$, $k = 1, \ldots, \lfloor \frac{L}{2} \rfloor$, *the path of length* $L$, *which has only one minimum occuring at the horizontal position* $2k-1$ *with vertical coordinate* $-1$ *(see illustration on Fig.4). Similarly, in the problem with type B boundary conditions, denote* $\beta(k)$, $k = 1, \ldots, \lfloor \frac{L+1}{2} \rfloor$, *the path of length* $L$ *with the only minimum, occuring at the horizontal position* $2k - \epsilon_L - 1$, $\epsilon_L = L (\mathrm{mod}\ 2)$ *with vertical coordinate* $-1$ *(see illustration on Fig.5). For the problems of both types denote* $H_{\beta(k)}$ *a factorized operator which one puts into correspondence to the path* $\beta(k)$ *following the rule (7.18).*

*Coefficients functions* $\psi_\Omega^{\mathrm{A/B}}$ *of the highest vectors* $|\Omega^{\mathrm{A/B}}\rangle$ *in the solutions of the qKZ equations satisfy relations*

$$H_{\beta(k)} \quad {}_{\Omega}^{\mathrm{A/B}} = 0 \quad \forall\, k. \tag{7.22}$$

*Equations (7.22) complemented by formulas (7.16), (7.17), (7.21) are equivalent to the whole system of the qKZ equations (7.8)-(7.10).*

**Comment.** For the special values of parameters $\lambda$ and $\omega$ simple solutions of the qKZ equations are known:

- for $\lambda = -3$ type A problem has a solution $\xi^{\mathrm{A}} = 1$ (см. (7.16)) [DF1];
- for $\lambda = -3/2$ and $\omega = -1/2$type B problem has a solution $\xi^{\mathrm{B}} = 1$ (см. (7.17)) [ZJ].

Namely these solutions were used for investigations of the properties of statioonary states of the stochastic process "Raise and Peel"(see [DF1, ZJ, dGP, dGPZ] and references therein).

# 8 Main results of the dissertation

- Definition of a family of *quantum matrix* (QM-) algebras of $GL(n)$ type and a proof of the structure results for these algebras. The definition 1.2 is given in the article [4] from the list below. Q-generalizations of Newton's relations between two sets of generators of the characteristic subalgebras of the Hecke type QM-algebras – the so-called *power sums* and *elementary symmetric*

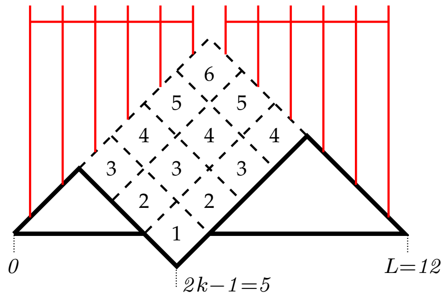



Figure 4: The path $\beta(3)$ of the length $L = 12$ is drawn in bold. It is not the Dyck path. The corresponding factorised operator $H_{\beta(3)}$ is an ordered product of the Baxterisedelements $T_i(u_{ij})$ representedby dashed tileson the picture.It is constructedby the rule (7.18).

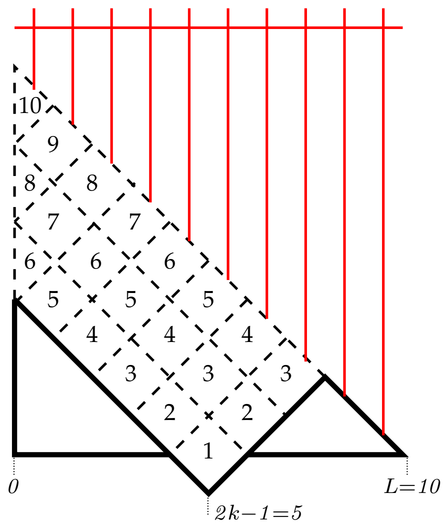



Figure 5: The path $\beta(3)$ of the length $L = 10$ is drawn in bold. It is not the ballot path. The corresponding factorised operator $H_{\beta(3)}$ is an ordered product of the baxterisedelements $T_i(u_{ij})$ representedby dashed tileson the picture.It is constructedby the rule (7.18).

*polynomials* – were obtained in varying degrees of generality in [1,3,4] (see Proposition 2.2). A q-generalization of the Cayley-Hamilton theorem was proved in [3,4] (see Theorem 2.3).

- Construction of a spectral extension for type $SL(n)$ Heisenberg double algebras (see definition-theorem 3.5). Derivation of a new presentation for the spectrally extended Heisenberg double algebra in terms of generating matrices $W^\alpha$ and spectral variables $\mu_\alpha$ and construction of the two families of dynamical R-matrices of the type $SL(n)$ (see theorem 3.6). These results were obtained in [7].

- Construction of an evolution operator in the model of q-deformed quantum Euler top (Alexeev-Faddeev model), carried out in [7] (see theorem 4.3).

- Itroduction of series of differential calculi algebras for the quantum groups of the types $GL(n)$ and $SL(n)$ (see definition-theorem 5.1) and construction of an exterior derivative for these algebras (see theorem 5.2). These results were obtained in [2,10].

- Construction of spectral extensions for the quantized differential calculi algebras of the types $GL(n)$ and $SL(n)$ (see definition-theorem 6.1) and construction a unitary type anti-involution map for the quantized differential calculus over general linear quantum groups (see theorems 6.2, 6.3), carried out in [10].

- Derivation of factorized expressions for solutions of the quantum Knizhnik-Zamolodchikov equations related to representations of Temperley-Lieb algebras in the space of Dyck paths and in the space of ballot paths (see theorem 7.5). These results were obtained in [9].

## 9 List of publications


1. P. Pyatov, P. Saponov,
   *'Characteristic relations for quantum matrices'*.
   Journ. Phys. A: Math.Gen. **28** (1995) 4415–4421.

2. P. Pyatov, L. Faddeev,
   *'The differential calculus on quantum linear groups'*.
   In *'Contemporary Mathematical Physics'*. Eds. R.L.Dobrushin, A.Minlos, M.A.Shubin and A.M.Vershik, Amer. Math. Soc. Transl. - Ser.2, **175** (1996) 35–47.

3. D. I. Gurevich, P.N. Pyatov and P. A. Saponov,
   *'Hecke Symmetries and Characteristic Relations on Reflection Equation Algebras'*.
   Letters in Mathematical Physics **41** (1997) 255—264.

4. A. Isaev, O. Ogievetsky, P. Pyatov,
   *'On quantum matrix algebras satisfying the Cayley-Hamilton-Newton identities'*.
   J. Phys. A: Math. Gen. **32** (1999) L115–L121.

5. P. Pyatov,
   *'Raise and Peel Models of fluctuating interfaces and combinatorics of Pascal's hexagon'*.
   J. Stat. Mech.: Theor. Exp. **09** (2004) P003.

6. F.C.Alcaraz, P.Pyatov, V.Rittenberg,
*'Density profiles in the raise and peel model with and without a wall. Physics and combinatorics'.*
J. Stat. Mech.: Theor. Exp., (2008) P01006.

7. A. Isaev, P. Pyatov,
*'Spectral extension of the quantum group cotangent bundle'.*
Commun. Math. Phys. **288** (2009) no.3, 1137–1179.

8. J. de Gier, P. Pyatov, P. Zinn-Justin,
*'Punctured plane partitions and the q-deformed Knizhnik–Zamolodchikov and Hirota equations'.*
J.Comb. Theory Ser. A **116** (2009) 772–794.

9. J. de Gier, P. Pyatov,
*'Factorised solutions of Temperley-Lieb qKZ equations on a segment'.*
Adv. Theor. Math. Phys. **14** (2010) 795–877.

10. P.Pyatov,
*'On the construction of unitary quantum group differential calculus'.*
J. Phys. A: Math. Theor. **49** (2016) 415202 (25pp).

## 10 Further developments and applications

In conclusion, we briefly comment on recent advances in the lines of the dissertation researches and discuss prospects for further investigations.

Investigations of a structure of the characteristic subalgebra and a proof of q-analogues of the Cayley-Hamilton theorem for the quantum matrix algebras of the type $GL(m|n)$ were carried out in [GPS2, GPS3]. This type QM-algebras together with the algebras of the $GL(n)$ type complete the list of all known series of the QM-algebras related to the Hecke R-matrices. These results generalize and unify earlier researches of Cantor and Trishin on characteristic identities for the matrix superalgebras [KT1, KT2] and investigations by Jarvis, Green, Gould and coauthors on characteristic identities in the universal enveloping Lie superalgebras [JG, GJ, Gou2].

Studies of the characteristic identities for the QM-algebras of Birman-Murakami-Wenzl (BMW) type associated with quantum orthogonal and symplectic groups were undertaken in [OP]. In the classical limit, these results reproduce the characteristic identities for the generators of orthogonal and symplectic Lie algebras (see

[Gou1, Mol]). In case of the Drinfeld-Jimbo's quantized universal enveloping algebras, the characteristic identities were considered in [MRS]; their images in the highest weight representations were studied in [Mudr]. An analogue of the Cayley-Hamilton theorem for the series of ortho-symplectic QM-algebras is not yet known. Preliminary work in this direction, that is a study of the characteristic subalgebras of the BMW type QM-algebras was carried out in [OP].

In [GS1, DM1, DM2, GS2], the Cayley-Hamilton identities in a factorized form were used for explicit quantization of the semisimple orbits of $GL(n)$'s coadjoint action and for the quantization of the line bundles over them.

Another perspective field of applications for the q-versions of the Newton relations and the Cayley-Hamilton theorem is a derivation of explicit PBW-ordered expressions for generators of the center of the reflection equation algebras (see [JW, Fl]).

In the theory of Heisenberg double algebras the spectral extension procedure allows simple construction of (possibly new) families of dynamical R-matrices. It also provides a techniques for studying a series of the Heisenberg double automorphisms with possible applications to a construction of quantum dynamical models. This procedure is currently implemented for the Heisenberg double algebras of the types $GL(n)$ and $SL(n)$ only. It would be promising to derive this procedure for the Heisenberg double algebras of the types $GL(m|n)$, $O(n)$, $Sp(2n)$, and to construct $SL(m|n)$ and $SO(n)$ type reductions of these algebras.

Yet another perspective field of applications of the spectral extension technique would be its use in the implementation of the quantum version of the Poisson reduction and in studying of quantum integrable models obtained in this way (see [AKO]).

As for the study of factorized formulas for solutions of the qKZ equations, their original applications were in the studies of combinatorial properties of the stationary states of the stochastic Raise and Peel model (see [P1, M-B, dGP, dGPZ, APR]). In our opinion, it would be also promising to find out a connection of polynomial solutions of the qKZ equations related to the stochastic model of ASEP (Asymmetric Simple Exclusion Process) with the factorized formulas. These solutions for the model with open boundary conditions on a segment, as well as their relationship with the asymmetric Coornwinder polynomials, were discussed in [C-W, FV].

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