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#### Abstract

The main purpose of this work is to study the limits of the Calogero-Sutherland system in the scalar and spin cases when the number of particles $N$ tends to infinity. In each case we study the bosonic and fermionic limit corresponding to the symmetric and antisymmetric wave functions of the system.

For the fermionic limit of the scalar system, we derive a limit expression for the Dunkl operator via free fermionic fields, see Theorem 2.1, which allows us to present the construction of commuting Hamiltonians in the Fock space, see Proposition 2.4. In the case of the value of the coupling constant $\beta=0$, we get an explicit formula for the generating function of Hamiltonians that differs from the previously known ones. The first one is given as a bosonic normal ordered answer, see Proposition 3.1. The second formula is given in terms of simple integral operator, but is not normal ordered, see Proposition 3.2.

The spin CS system has the Yangian symmetries. In fact the action of Yangian generators as well as Hamiltonians in scalar case do not form a projective system. So we study the projective properties of the Yangian action and formulate the results in Proposition 4.1 and Proposition 4.2.

For spin system we realize the bosonic and fermionic limit in a multicomponent Fock space. We introduce the maps to finite system and construct the pullback of finite Dunkl operators in terms of vertex operators in bosonic case and in terms of free fermion fields in fermionic case, see Proposition 5.1 and Proposition 6.1. The limit of Dunkl operator allows to construct the corresponding Yangian representation in the Fock space, see Theorem 5.1 and Theorem 6.1. In the bosonic case we investigate the classical limit, see Propositions 5.3 and 5.4.


## Historical review

The system of one-dimensional particles with inverse-square pairwise interactions has played a great role in mathematical and theoretical physics for the past 40 years. This model arises and has different applications in various fields of physics, such as condensed matter physics, spin chains, gauge theory, and string theory and constitutes the main example of integrable and solvable many-body system. In the literature, it is labeled by the names of F. Calogero, B. Sutherland and Y. Moser. The system of identical particles scattering on the line with inverse-square potential was as first introduced by F. Calogero in 1971 [7]. Its Hamiltonian is

$$
H=\sum_{i=1}^{N} \frac{p_{i}^{2}}{2}+\sum_{i<j} \frac{g}{\left(q_{i}-q_{j}\right)^{2}}
$$

where we use the standard notations of momentums and coordinates. Here the particle masses are scaled to unity, $g$ is the coupling constant. We consider a periodic version of the system (for example, with the period $2 \pi$ ), assuming that infinitely many images of particles interact, then the two-body potential becomes

$$
V(x)=\sum_{n=-\infty}^{\infty} \frac{g}{(x+2 \pi n)^{2}}=\frac{g}{2 \sin \frac{x}{2}}
$$

This was introduced by B. Sutherland in 1971 [44]. It is convenient to use the following parametrization of the coupling constant:

$$
g=\beta(\beta-1)
$$

We consider a system of $N$ identical particles on a circle of length $L$, which we will call the quantum Calogero-Sutherland system, with the following Hamiltonian

$$
\begin{equation*}
H=-\sum_{i=1}^{N}\left(\frac{\partial}{\partial q_{i}}\right)^{2}+2\left(\frac{\pi}{L}\right)^{2} \sum_{i<j}^{N} \frac{\beta(\beta-1)}{\sin ^{2}\left(\frac{\pi}{L}\left(q_{i}-q_{j}\right)\right)} \tag{1}
\end{equation*}
$$

which is the main point of our research. It is natural to consider periodic wave functions of the system

$$
\phi\left(q_{1}, \ldots, q_{i}+L, \ldots, q_{N}\right)=\phi\left(q_{1}, \ldots, q_{i}, \ldots, q_{N}\right)
$$

The function

$$
\phi_{0}(\mathbf{q})=\phi_{0}\left(q_{1}, \ldots, \ldots, q_{N}\right)=\prod_{i<j}\left|\sin \left(\frac{\pi}{L}\left(q_{i}-q_{j}\right)\right)\right|^{\beta}
$$

represents the vacuum state with eigenenergy [18]

$$
E_{0}=(\pi \beta / L)^{2} N\left(N^{2}-1\right) / 3
$$

Applying the transformation $\phi_{0}(\mathbf{q})^{-1} H^{C S} \Phi_{0}(\mathbf{q})$ and passing to the collective variables $x_{i}=e^{\frac{2 \pi i q_{i}}{L}}$, we arrive to the effective Hamiltonian

$$
\begin{equation*}
H=\sum_{i=1}^{N}\left(x_{i} \frac{\partial}{\partial x_{i}}\right)^{2}+\beta \sum_{i<j} \frac{x_{i}+x_{j}}{x_{i}-x_{j}}\left(x_{i} \frac{\partial}{\partial x_{i}}-x_{j} \frac{\partial}{\partial x_{j}}\right) . \tag{2}
\end{equation*}
$$

The Hamiltonian (2) is a differential-difference operator. It turns out that there is a family of commuting differential-difference operators that includes (2). This family can be constructed using the Heckman-Dunkl operators [11, 12]. We give the expressions of them in the form suggested in [37]:

$$
\begin{equation*}
D_{i}^{(N)}=x_{i} \frac{\partial}{\partial x_{i}}+\beta \sum_{j \neq i} \frac{x_{i}}{x_{i}-x_{j}}\left(1-K_{i j}\right), \tag{3}
\end{equation*}
$$

where $K_{i j}$ is a permutation operator. Symmetric polynomials in $D_{i}^{(N)}$ commute [12]. Denote by

$$
\begin{equation*}
H_{k}^{(N)}=\operatorname{Res}_{+}\left(\sum_{i}\left(D_{i}^{(N)}\right)^{k}\right) \tag{4}
\end{equation*}
$$

where $\mathrm{Res}_{+}$means a restriction on the space of symmetric polynomials. The operators $H_{k}^{(N)}$ can be chosen as the higher Hamiltonians of the Calogero-Sutherland model. In particular, $H=H_{2}^{(N)}$.

The eigenfunctions of commuting operators $H_{k}^{(N)}$ are symmetric polynomials in $N$ variables with the parameter $\alpha=\frac{1}{\beta}$, which are called Jack polynomials [16]. They are parametrized by the partitions and constitute a generalization of Schur polynomials and a special case of symmetric Macdonald polynomials with two parameters $q, t[24,25]$.

Putting $q, t \rightarrow 1$ and assuming that $q=t^{\alpha}$, we obtain Jack polynomials. It is known a family of difference operators for which Macdonald polynomials are eigenfunctions [24]. In the case of Jack polynomials these operators were introduced by J. Sekiguchi [42] and A. Debiard [9]. The Sekiguchi-Debiard operators are degeneration of Macdonald operators. In fact, they do not coincide with the operators given in (4), but can be expressed as a polynomial in (4).

The construction of Macdonald polynomials and corresponding commuting difference operators is also known for an arbitrary root system [8, 26, 27]. A generalization of Jack polynomials for arbitrary root systems was introduced by G. Heckman and E. Opdam and is called Jacobi polynomials associated with the root system [13, 14, 15, 35]. Jack polynomials is associated with the root system $A_{n}$. We consider only this case. We remark that the Calogero- Sutherland system is an integrable system corresponding to the root system $A_{N-1}$, following M. Olshanetsky and A. Perelomov [34].

Naturally, there is a question about the description of the model where the number of particles $N$ tends to infinity. In papers [2, 4, 5, 17, 37] from the 80's to early 90 's there were presented the explicit answers for the limit of the second Hamiltonian (2) in the bosonic Fock space. About 20 years later, the general construction of commuting Hamiltonians in the bosonic Fock space was presented by M. Nazarov and E. Sklyanin [32] and independently by A. Veselov and A. Sergeev [43]. Developing Macdonald's ideas, M. Nazarov and E. Sklyanin in [32] found the expressions for Sekiguchi-Debyard operators in the limit where $N$ tends to infinity. The main tool was the theory of symmetric functions. Symmetric functions can be considered as symmetric polynomials in infinite number of variables. The zero sector of the bosonic Fock space can be identified with the ring of symmetric functions, which is formally defined as the projective limit of rings of symmetric polynomials. Thus there was constructed a family of operators whose eigenfunctions are Jack symmetric functions.

In [31],[43] another construction of the limit for Calogero-Sutherland model in the bosonic Fock space was presented. The main idea was to use the family of Dunkl operators (3) as a quantum $L$-operator of the system. For Calogero systems the $L$-operator was already known [30] and was similar to the action of the family of Dunkl operators, written in matrix form in a suitable basis. Thus a precise construction of higher Hamiltonians in the bosonic Fock space was suggested and this allowed to show that the limiting system is integrable. The resulting system can be considered as a quantum analogue of the integrable hierarchy of the Benjamin-Ono equation [1, 38].

For special value of the coupling constant the symmetric Jack functions become Schur functions, and the Benjamin-Ono equation respectively degenerates into the dispersionless KdV equation (or the so-called Burger's equation). The exact construction of commuting Hamiltonians of the quantum dispersionless KdV equation can be obtained directly from the boson-fermion correspondence and was presented by A. Pogrebkov in [36]. Hamiltonians can be obtained recurrently [36] or in terms of the generating function [33, 41].

We consider the spin Calogero-Sutherland systems which are generalizations of these models, where extra degrees of freedom are involved, which are usually interpreted as spin variables. Integrability of the Calogero system has been studied in numerous papers, see for example [23]. The Calogero-Sutherland spin system is superintegrable due to N. Reshetikhin [39, 40]. In this paper, we will use a special case of the spin model corresponding to the root system $A_{N}$ and the representation of the higher weight of $\mathfrak{s l}_{N}$.

In this case, the numerator of the potential of Hamiltonian (1) will be $\beta\left(\beta-K_{i j}\right)$, where $K_{i j}$ is the coordinate exchange operator of $i$-th and $j$-th particles, and the dependence on spin is implicit.

The spin CS system has the Yangian symmetry, in other words the Hamiltonians of the Calogero-Sutherland system commute with the Yangian action, moreover they are expressed through the central elements of the Yangian elements. The presence of Yangian symmetry is directly related to the Dunkl operators. They satisfy the relations of the degenerate affine Hecke algebra, which in turn allows us to construct the representation of the Yangian $\mathrm{Y}\left(\mathfrak{g l}_{s}\right)$ according to the general construction [3, 10]. Thus, the higher Hamiltonians of the system can be chosen as the center of the Yangian, namely, as the coefficients of the quantum determinant.

In the symmetric case the limit expression $N$ for the second Hamiltonian in collective variables was obtained in [5]. The antisymmetric limit of the spin system was studied by D. Uglov in [45, 47]. D. Uglov studied the projective properties of the Yangian action for a finite system, namely, he presented a formula of renornalization of the transfer matrix of the Yangian in order form a projective system and the action was stabilized. Also D. Uglov decomposed the corresponding Fock space into irreducible components with respect to the Yangian action and found the spectrum of Hamiltonians.

## 1 Bosonic limit of Calogero-Sutherland system

In the first section we review recent results [31, 43] concerning the bosonic limit of Calogero-Sutherland system and rewrite them in a language of vertex operators. We use the notations differing from [31, 43] but more convinient for our purpose and clearifying the further exposition.

We begin with the description of the finite CS system restricted on the ring of symmetric polynomials in $N$ variables. The main idea is to regard the equivariant HeckmanDunkl operators as a quantum L-operator acting on the space of polynomial functions of one variable with coefficients being symmetric polynomials of the remaining $N-1$ variables. Clearly, the Dunkl operator $D_{i}^{(N)}$ itself preserves the symmetry involving all variables other than $x_{i}$ and therefore it acts on the space $\Lambda_{+}^{N, i}$ of functions symmetric in all variables except $x_{i}$.

The action of the higher Hamiltonian (4) can be obtained by the following procedure: we start with a symmetric polynomial $f\left(x_{1}, \ldots, x_{N}\right) \in \Lambda_{+}^{N}$ and construct a vector of its $N$ copies. The action the $k$-th power of Dunkl operator $\left(\tilde{D}_{i}^{(N)}\right)^{k}$ provides a family of $N$ equivariant functions: $f_{i}\left(x_{1}, \ldots ; x_{i} ; \ldots x_{N}\right)=\left(\tilde{D}_{i}^{(N)}\right)^{k} f\left(x_{1}, \ldots, x_{N}\right) \in \Lambda_{+}^{N, i}$ such that

- $f_{i}\left(x_{1}, \ldots ; x_{i} ; \ldots x_{N}\right)$ is a polynomial symmetric in all variables except $x_{i}$

$$
\begin{equation*}
K_{i j} f_{i}=f_{j} . \tag{5}
\end{equation*}
$$

For $g\left(x_{1}, \ldots ; x_{i} ; \ldots x_{N}\right) \in \Lambda_{+}^{N, i}$ we introduce an operator of its symmetrization

$$
\mathrm{E}_{N} g=\sum_{j=1}^{N} K_{i j} g
$$

Then we apply $\mathrm{E}_{N}$ to a function $f_{i}$ from an equivariant family (5):

$$
\mathrm{E}_{N} f_{i}=\sum_{j=1}^{N} f_{j}
$$

This procedure can be illustrated by the following matrix formula:

$$
\tilde{H}_{k}=(1,1, \ldots)\left(\begin{array}{ccc}
x_{1} \frac{\partial}{\partial x_{1}}+\beta \sum_{i=2}^{N} \frac{x_{1}}{x_{1}-x_{i}} & -\beta \frac{x_{1}}{x_{1}-x_{2}} & \ldots \\
-\beta \frac{x_{2}}{x_{2}-x_{1}} & x_{2} \frac{\partial}{\partial x_{2}}+\beta \sum_{i \neq 2} \frac{x_{2}}{x_{2}-x_{i}} & \vdots \\
\vdots & \ldots & \ddots
\end{array}\right)^{k}\left(\begin{array}{c}
f \\
f \\
f \\
f
\end{array}\right)
$$

which resembles the Lax matrix (see [30]) for CS system.
We reformulate the procedure in terms of the Newton polynomials $p_{k}^{(N)}=x_{1}^{k}+\cdots+x_{N}^{k}$ and express the Heckman-Dunkl operators via finite analogous $V_{+}(z), V_{+}^{\prime}(z)$ of the vertex operators $\Phi(z), \Phi^{-1}(z)$ and the negative part of derivative of the bosonic field $\varphi^{-}(z)$, given by the formulas:

$$
\Phi(z)=\exp \left(\sum_{n \geqslant 0} z^{n} \frac{\partial}{\partial p_{n}}\right), \quad \varphi^{-}(z)=\left(\sum_{n \geq 0} \frac{p_{n}}{z^{n}}\right) .
$$

To do that we present a symmetric polynomial in the following form

$$
f\left(p_{1}^{(N)}, p_{2}^{(N)}, p_{3}^{(N)}, \ldots\right)
$$

The operator $V_{+}\left(x_{i}\right)$ changes each occurrence of a Newton sum $p_{k}^{(N)}$ by $p_{k}^{(N-1)}+x_{i}^{k}$, so $V_{+}\left(x_{i}\right) f \in \Lambda_{+}^{N, i}$ is a Taylor decomposition of polynomial $f$ by variable $x_{i}$.

To symmetrize the function $F\left(x_{i},\left\{p_{n}^{(N-1)}\right\}\right) \in \Lambda_{+}^{N, i}$ we use the the formal intagral

$$
\mathrm{E}_{N} F\left(\left\{p_{n}\right\}\right)=\oint \frac{d \xi}{\xi} \varphi^{-}(\xi)\left(V_{+}^{\prime}(\xi) F\right)\left(\xi ;\left\{p_{n}\right\}\right)
$$

which counts the residue at infinity. The operator $V_{+}^{\prime}(\xi)$ changes each occurrence of a Newton sum $p_{k}^{(N-1)}$ by $p_{k}^{(N)}-\xi^{k}$. Then the integral $\oint \frac{d \xi}{\xi} \varphi^{-}(\xi)$ changes each item $\xi^{k}$ by $p_{k}^{(N)}$.

Further we realize the bosonic limit in the extended ring of symmetric functions $\hat{\Lambda}$. Let $\hat{\Lambda}=\Lambda\left[p_{0}\right]$ be a ring symmetric functions [24] extended by a free variable $p_{0}$. The space $\hat{\Lambda}$ is an irreducible representation of the Heisenberg algebra $\mathcal{H}$, generated by the elements $p_{n}$ and $\frac{\partial}{\partial p_{n}}$ and can be regarded as a polynomial version of the Fock space. It contains the vacuum vector $|0\rangle_{+}$, such that

$$
\frac{\partial}{\partial p_{n}}|0\rangle_{+}=0, \quad n=0,1, \ldots
$$

The dual vacuum vector $+\langle 0|$ satisfies the condition

$$
+\langle 0| p_{n}=0, \quad n=0,1, \ldots
$$

We define a projection $\tilde{\pi}_{N}: \hat{\Lambda} \rightarrow \Lambda_{+}^{N}$ for an element $|v\rangle_{+} \in \hat{\Lambda}$ as the following matrix element:

$$
\tilde{\pi}_{N}|v\rangle_{+}={ }_{+}\langle 0| \Phi\left(x_{N}\right) \ldots \Phi\left(x_{2}\right) \Phi\left(x_{1}\right)|v\rangle_{+} .
$$

This projection maps $p_{k}$ to the corresponding Newton polynomial in $N$ variables:

$$
\tilde{\pi}_{N}: p_{k} \rightarrow p_{k}^{(N)}=\sum_{i=1}^{N} x_{i}^{k}, \quad p_{0} \rightarrow N
$$

We define a linear map $\mathcal{S}: \hat{\Lambda} \otimes \mathbb{C}[z] \rightarrow \hat{\Lambda}$ as

$$
\mathcal{S} F\left(\left\{p_{n}\right\}\right)=\oint \frac{d \xi}{\xi} \varphi^{-}(\xi) \Phi^{-1}(\xi) F\left(\xi,\left\{p_{n}\right\}\right) .
$$

and prove that the map $\mathcal{S}$ is the pullback of the finite symmetrization $\mathrm{E}_{N}$ under the map $\tilde{\pi}_{N}:$

$$
\mathrm{E}_{N} \tilde{\pi}_{N-1} F\left(z,\left\{p_{n}\right\}\right)=\tilde{\pi}_{N} \mathcal{S}\left(F\left(z,\left\{p_{n}\right\}\right)\right.
$$

We present the main result of this section:
Theorem 1.1 The operator $\tilde{D}: \hat{\Lambda} \otimes \mathbb{C}[z] \rightarrow \hat{\Lambda} \otimes \mathbb{C}[z]$ given by

$$
\tilde{D}\left(F\left(z,\left\{p_{n}\right\}\right)\right)=z \frac{\partial}{\partial z} F\left(z,\left\{p_{n}\right\}\right)+\beta z \oint \frac{d \xi}{\xi^{2}} \frac{1}{1-\frac{z}{\xi}} \Phi^{*}(\xi) \Phi(z) F\left(\xi,\left\{p_{n}\right\}\right)
$$

is a limit of Dunkl operators $\tilde{D}_{i}^{(N)}$.
In other words, the operator $\tilde{D}$ is a pullback of $\tilde{D}_{i}^{(N)}$ under the map $\tilde{\pi}_{N}$. This result was formulated before in $[31,43]$ in other terms, here we present the formula in the language of vertex operators. This theorem implies the following

Proposition 1.2 The operators $\tilde{\mathscr{H}}_{k}=\mathcal{S} \tilde{D}^{k} \Phi(z): \hat{\Lambda} \rightarrow \hat{\Lambda}$,

$$
\tilde{\mathscr{H}}_{k}: \hat{\Lambda} \xrightarrow{\Phi(z)} \hat{\Lambda} \otimes \mathbb{C}[z] \xrightarrow{D^{k}} \hat{\Lambda} \otimes \mathbb{C}[z] \xrightarrow{\mathcal{S}} \hat{\Lambda}
$$

generate a commutative family of Hamiltonians of the limiting system [31, 43].
Also we show that in classical limit this system becomes the Benjamin-Ono hierarchy following [31].

## 2 Fermionic limit of CS system

The second section is devoted to the fermionic limit of Calogero-Sutherland model, we describe the results of paper [19]. In this case the particles are fermions and we deal with the antisymmetric wave functions.

As well as in bosonic case we begin with the description of the CS system restricted to the space of antisymmetric polynomials $\Lambda_{-}^{N}$ in terms of Heckman-Dunkl operators. We
then express Heckman-Dunkl operators via finite analogs $V_{-}(z) V_{+}(z)$ and $V_{-}^{\prime}(z) V_{+}^{\prime}(z)$ of vertex operators $\Psi(z)$ and $\Psi^{*}(z)$, where

$$
\begin{aligned}
\Psi(z) & =z^{p_{0}} \exp \left(-\sum_{n>0} \frac{p_{n}}{n z^{n}}\right) \exp \left(\sum_{n \geq 0} z^{n} \frac{\partial}{\partial p_{n}}\right) \\
\Psi^{*}(z) & =z^{-p_{0}} \exp \left(\sum_{n>0} \frac{p_{n}}{n z^{n}}\right) \exp \left(-\sum_{n \geq 0} z^{n} \frac{\partial}{\partial p_{n}}\right) .
\end{aligned}
$$

To do this we present any antisymmetric polynomial in $N$ variables as

$$
\prod_{i>j}\left(x_{i}-x_{j}\right) f\left(p_{1}^{(N)}, p_{2}^{(N)}, p_{3}^{(N)}, \ldots\right)
$$

where $p_{k}^{(N)}=x_{1}^{k}+\ldots+x_{N}^{k}$. The operator $V_{+}\left(x_{1}\right)$ changes each occurrence of $p_{k}^{(N)}$ by $p_{k}^{(N-1)}+x_{1}^{k}$, while the operator

$$
V_{-}\left(x_{1}\right)=x_{1}^{N} \exp \left(-\sum_{n>0} \frac{p_{n}^{(N-1)}}{n x_{1}^{n}}\right)
$$

is the multiplication by $\prod_{i=2}^{N}\left(x_{1}-x_{i}\right)$, so that the application of $V_{-}\left(x_{1}\right) V_{+}\left(x_{1}\right)$ to an antisymmetric polynomial $g\left(x_{1}, \ldots, x_{N}\right)$ is just its Taylor decomposition with respect to $x_{1}$. On the other hand, the operators $V_{-}^{\prime}(z) V_{+}^{\prime}(z)$ are used for the total antisymmetrization of the functions, antisymmetric with respect to all variables except one.

Then we realize a limit in the polynomial Fock space $\hat{\Lambda}$. To each vector $|v\rangle$ of $\hat{\Lambda}$ we attach a family $\left\{\bar{\pi}_{N}(v)\right\}$ of antisymmentric functions of $N$ variables, given by matrix elements

$$
\begin{equation*}
\bar{\pi}_{N}(v)=\langle 0| \Psi\left(x_{N}\right) \cdots \Psi\left(x_{1}\right)|v\rangle \tag{6}
\end{equation*}
$$

The goal is to construct operators in the space $\hat{\Lambda}$ which are compatible with finite CS Hamiltonians with respect to evaluation maps (6). This is done following E.Sklyanin ideology [31, 21]: we introduce an auxillary space $\left.U \subset \mathbb{C}\left[z, z^{-1}\right]\right] \otimes \hat{\Lambda}$ and its evaluations to the spaces of polynomials antisymmetric with respect to all variables except one. We present operators, acting in $U$ which are compatible with the above evaluation maps.

The key point of the construction is an operator of integral average $\left.\mathcal{A}: \mathbb{C}\left[z, z^{-1}\right]\right] \otimes$ $\hat{\Lambda} \rightarrow \hat{\Lambda}$, which is the limiting analogue of finite antisymmetrization. Let $F(z) \in$ $\left.\mathbb{C}\left[z, z^{-1}\right]\right] \otimes \hat{\Lambda}$, then we define $\mathcal{A} F \in \hat{\Lambda}$ by the following formula:

$$
\mathcal{A} F=\frac{1}{(2 \pi i)^{2}} \int_{z \circlearrowleft 0} d z \int_{u \circlearrowleft z} d u \frac{\Psi^{*}(u) F(z)}{u-z} .
$$

In Lemma 2.3 we show that $\mathcal{A}: U \rightarrow \hat{\Lambda}$ is a pullback of finite antisymmetrization.
Further we define an operator $\left.\left.D: \mathbb{C}\left[z, z^{-1}\right]\right] \otimes \hat{\Lambda} \rightarrow \mathbb{C}\left[z, z^{-1}\right]\right] \otimes \hat{\Lambda}$ by the following formula

$$
D F(z)=z \frac{\partial}{\partial z} F(z)+\beta \frac{1}{(2 \pi i)^{2}} \int_{w \circlearrowleft 0} d w \int_{u \circlearrowleft w} \frac{d u}{(u-w)} \frac{\Psi^{*}(u)}{\left(1-\frac{w}{z}\right)}(\Psi(w) F(z)-\Psi(z) F(w)) .
$$

and prove

Theorem 2.1 The operator $D$ acting on the auxillary space $U$ is a pullback of HeckmanDunkl operators $D_{i}^{(N)}$ under the map $\bar{\pi}_{N}$.

The Hamiltonians of finite system with $N$ particles in antisymmetric case can be expressed by meas of Dunkl operators analogously (4):

$$
\bar{H}_{k}^{(N)}=\operatorname{Res}_{-}\left(\sum_{i}\left(D_{i}^{(N)}\right)^{k}\right)
$$

where Res_ means the restriction on the space of antysymmetric polynomials $\Lambda_{-}^{N}$. We construct the limiting Hamiltonians $\mathscr{H}_{k}$ which are the pullbacks of finite Hamiltonians $\bar{H}_{k}^{(N)}:$

$$
\bar{H}_{k}^{(N)} \bar{\pi}=\bar{\pi} \mathscr{H}_{k} .
$$

We define the operators

$$
\begin{equation*}
\mathscr{H}_{k}=\mathcal{A} D^{k} \Psi(z): \hat{\Lambda} \rightarrow \hat{\Lambda} \tag{7}
\end{equation*}
$$

and formulate
Proposition 2.4 The operators $\mathscr{H}_{k}$ generate a commutative family of operators in the space $\hat{\Lambda}$.

The constructed Hamiltonians form a commutative family of operators in the space $\hat{\Lambda}$. Moreover, they commute inside the Heisenberg algebra and thus can be used as well in its other representations, for instance, in the bosonic Fock space. We can define the projection $\pi_{N}: \mathcal{F} \rightarrow \Lambda_{-}^{N}$ similar to (6)

$$
\pi_{N}(v)=\langle 0| \Psi\left(x_{N}\right) \cdots \Psi\left(x_{1}\right)|v\rangle .
$$

In fact it is nonzero only on the $N$-th sector $\mathcal{F}_{N}$ of the Fock space. Now the constructed Hamiltonians $\mathscr{H}_{k}$ are compatible with respect to the maps $\pi_{N}$, the commutativity $\pi_{N} \mathscr{H}_{k}=$ $\bar{H}_{k}^{(N)} \pi_{N}$ is nontrivial on the $N$-th sector $\mathcal{F}_{N}$. We reformulate the same construction in the fermionic Fock space represented as space of semi-infinite wedges, we define the projection analogous to $\pi_{N}$ which acts as a "cutting" of the wedge.

## 3 Generating functions of commuting Hamiltonians for some special values of coupling constant

In this section we consider the special case $\beta=0$ of antisymmetric limit. (we use the notations for Hamiltonians as in previous section, where we assume $\beta=0$ ). In this case the Hamiltonians (7) can be simply expressed as operators on the fermionic Fock space

$$
\mathscr{H}_{n}=\sum_{k} \vdots k^{n} \psi_{k}^{*} \psi_{k} \vdots
$$

The boson-fermion correspondence allows to express $\mathscr{H}_{n}$ in the bosonic Fock space, it was done by A. Pogrebkov [36] for the additive version and later by P. Rossi [41] on the circle.

Here we derive the two formulas for the densities for $\mathscr{H}_{n}$ that was not known before, we present the results given in [28]. In case $\beta=0$ the Dunkl operator is simply the differential operator $\left(z \frac{\partial}{\partial z}\right)$ and the Hamiltonians (7) are expressed from the densities $\mathscr{H}_{k}=\frac{1}{2 \pi i} \int_{z \circlearrowleft 0} \frac{d z}{z} \mathscr{W}_{k}(z)$ which is given by

$$
\mathscr{W}_{k}(z)=\frac{1}{2 \pi i} \int_{u \circlearrowleft z} d u \frac{\Psi^{*}(u)}{u-z}\left(z \frac{\partial}{\partial z}\right)^{k} \Psi(z) .
$$

These Hamiltonians are the pullbacks of simple differential operators $\bar{H}_{k}^{(N)}=\sum_{i=1}^{N}\left(x_{i} \frac{\partial}{\partial x_{i}}\right)^{k}$. We derive first formula by calculating the integral in variable $u$ in $\mathscr{W}_{k}(z)$ using the bosonic calculus. This gives the following answer

Proposition 3.1 The exponential generating function $\mathscr{W}(z, x)$ for densities $\mathscr{W}_{k}(z)$ is given by the formula

$$
\mathscr{W}(z, x)=\sum_{k=0}^{\infty} \frac{x^{k}}{k!} \mathscr{W}_{k}(z)=\frac{: \exp \left(x\left(z \frac{\partial}{\partial z}+\varphi(z)\right)\right):-1}{e^{x}-1}
$$

and satisfies the differential equation

$$
\frac{\partial \mathscr{W}(x, z)}{\partial x}=: \varphi(z) \mathscr{W}(z, x):+z \frac{\partial \mathscr{W}(x, z)}{\partial z}-\frac{e^{x} \mathscr{W}(z, x)-\varphi(z)}{e^{x}-1}
$$

Here the exponent of operator means the formal series acting on the identity:

$$
\exp \left(x\left(z \frac{\partial}{\partial z}+\varphi(z)\right)\right)=1+x \varphi(z)+\frac{x}{2!}\left(\varphi^{2}(z)+z \frac{\partial}{\partial z} \varphi(z)\right)+\ldots
$$

The second formula can be obtained by fermionic calculus and expressed in terms of integral operator. We introduce an integral operator $K: \mathcal{F} \otimes \mathbb{C}\left[\left[z, z^{-1}\right]\right] \rightarrow \mathcal{F} \otimes \mathbb{C}\left[\left[z, z^{-1}\right]\right]$, given by the formula

$$
K[f(z)]=\frac{1}{2 \pi i} \int_{w \circlearrowleft z} \frac{d w}{w-z} \varphi(w) f(z)
$$

here $f(z) \in \mathcal{F} \otimes \mathbb{C}\left[\left[z, z^{-1}\right]\right]$. Then we present the explicit formulas for the Hamiltonians by the following

Proposition 3.2 The exponential generating function for the Hamiltonians is given by

$$
\mathscr{H}(x)=\frac{1}{2 \pi i\left(e^{x}-1\right)} \int_{z \circlearrowleft 0} d z \frac{e^{x K}-1}{K}\left[\frac{\varphi(z)}{z}\right] .
$$

The Hamiltonians can be expressed by the formula:

$$
\begin{equation*}
\mathscr{H}_{n}=\frac{1}{2 \pi i} \int_{z \circlearrowleft 0} d z\left(\frac{1}{n+1} \sum_{l=0}^{n}\binom{n+1}{l+1} B_{n-l} K^{l}\left[\frac{\varphi(z)}{z}\right]\right) \tag{8}
\end{equation*}
$$

Here $B_{n}$ mean the Bernoulli numbers and the operator $\frac{e^{x K}-1}{K}$ means a formal power series in $K$ :

$$
\frac{e^{x K}-1}{K}=x+\frac{x^{2}}{2} K+\frac{x^{3}}{6} K^{2}+\frac{x^{4}}{24} K^{3}+\ldots
$$

We note that the answer for the Hamiltonians given in Proposition 3.2 is not normal ordered.

The Hamiltonians $\mathscr{H}_{n}$ commute, thus we can derive an hierarchy of time evolutions defined by these commutative flows as

$$
\varphi_{t_{n}}(z)=\left[\mathscr{H}_{n}, \varphi(z)\right] .
$$

We derive the explicit formulas and formulate the result by the following
Lemma 3.5 The hierarchy of time evolutions defined by commutative family (8) is given by

$$
\varphi_{t_{k}}(z)=\frac{1}{2} B(x): \int_{x \circlearrowleft 0} d x \frac{k!}{x^{k+1}} \sinh \left(x z \frac{\partial}{\partial z}\right) e^{x S\left(x z \frac{\partial}{\partial z}\right) \varphi(z)}: .
$$

The classical limit of this hierarchy is the dispersionless KdV hierarchy on the circle [36].

## 4 Dunkl operators and representation of the Yangian $\mathrm{Y}\left(\mathfrak{g l}_{s}\right)$

The phase space of the quantum spin Calogero-Sutherland (CS) system consists of functions with values in vector space $\left(\mathbb{C}^{s}\right)^{\otimes N}$ while the dependence on spin in the Hamiltonian

$$
H^{C S}=-\sum_{i=1}^{N}\left(\frac{\partial}{\partial q_{i}}\right)^{2}+\sum_{i, j=1}^{N} \frac{\beta\left(\beta-K_{i j}\right)}{\sin ^{2}\left(q_{i}-q_{j}\right)}
$$

is implicit [18]. Here $K_{i j}$ is the coordinate exchange operator of particles $i$ and $j$. After conjugating by the function $\prod_{i<j}\left|\sin \left(q_{i}-q_{j}\right)\right|^{\beta}$ which represents the degenerated vacuum state, and passing to the exponential variables $x_{i}=e^{2 \pi i q_{i}}$ and the parameter $\alpha=\beta^{-1}$ more common in mathematical literature, we arrive after simple rescaling to the effective Hamiltonian

$$
H=\alpha \sum_{i=1}^{N}\left(x_{i} \frac{\partial}{\partial x_{i}}\right)^{2}+\sum_{i<j} \frac{x_{i}+x_{j}}{x_{i}-x_{j}}\left(x_{i} \frac{\partial}{\partial x_{i}}-x_{j} \frac{\partial}{\partial x_{j}}\right)-2 \sum_{i<j} \frac{x_{i} x_{j}}{\left(x_{i}-x_{j}\right)^{2}}\left(1-K_{i j}\right)
$$

which we restrict to the spaces $\Lambda_{ \pm}^{s, N}$ of total invariants or respectively skewinvariants of the symmetric group $S_{N}$ in the space $V^{\otimes N}$,

$$
\Lambda_{ \pm}^{s, N}=\left(V^{\otimes N}\right)^{( \pm)}
$$

Here $V=\mathbb{C}[z] \otimes \mathbb{C}^{s}$. The (skew)invariants are taken with respect to the diagonal action of the symmetric groups, $\sigma_{i j} \mapsto K_{i j} P_{i j}$, where $K_{i j}$ is as above and $P_{i j}$ is the permutation of $i$-th and $j$-th tensor copy of the vector space $\mathbb{C}^{s}$.

Further we use the Heckman-Dunkl operators $\mathcal{D}_{i}^{(N)}: V \otimes \Lambda_{ \pm}^{s, N-1} \rightarrow V \otimes \Lambda_{ \pm}^{s, N-1}$ in the form suggested by Polychronakos [37]:

$$
\mathcal{D}_{i}^{(N)}=\alpha x_{i} \frac{\partial}{\partial x_{i}}+\sum_{j \neq i} \frac{x_{i}}{x_{i}-x_{j}}\left(1-K_{i j}\right)
$$

These operators satisfy the relations

$$
\begin{aligned}
K_{i j} \mathcal{D}_{i}^{(N)} & =\mathcal{D}_{j}^{(N)} K_{i j}, \\
{\left[\mathcal{D}_{i}^{(N)}, \mathcal{D}_{j}^{(N)}\right] } & =\left(\mathcal{D}_{j}^{(N)}-\mathcal{D}_{i}^{(N)}\right) K_{i j},
\end{aligned}
$$

which coincide with the relations of the degenerate affine Hecke algebra $H_{N}$. By Drinfeld duality [10], this representation of degenerate affine Hecke algebra transforms to the representation of the Yangian $\mathrm{Y}\left(\mathfrak{g l}_{s}\right)$ in $\Lambda_{ \pm}^{s, N}$, see $[3,22]$

$$
\begin{equation*}
t_{a b}(u)=\delta_{a b}+\sum_{i} \frac{E_{a b, i}}{u \pm \mathcal{D}_{i}^{(N)}} \tag{9}
\end{equation*}
$$

Here $E_{a b, i}$ describes the action of $\mathfrak{g l}_{s}$ on $i$-th tensor component,

$$
E_{a b, i}(\ldots \otimes \underbrace{\left(e^{c} \otimes x^{k}\right)}_{i} \otimes \ldots)=\delta_{b c}(\ldots \otimes \underbrace{\left(e^{a} \otimes x^{k}\right)}_{i} \otimes \ldots) .
$$

and $t_{a b}(u), a, b=1, \ldots s$,

$$
t_{a b}(u)=\delta^{a b}+\sum_{i=0}^{\infty} t_{a b, i} u^{-i-1}
$$

are generating functions of the generators $t_{a b, i}$ of the Yangian $\mathrm{Y}\left(\mathfrak{g l}_{s}\right)$. The defining relations of $\mathrm{Y}\left(\mathfrak{g l}_{s}\right)$ are [29]

$$
\left[t_{a b}(u), t_{c d}(v)\right]=\frac{t_{c b}(u) t_{a d}(v)-t_{c b}(v) t_{a d}(u)}{u-v} .
$$

Then the higher Hamiltonians of spin CS system can be chosen as coefficients of the quantum determinant

$$
q \operatorname{det} t(u)=\sum_{\sigma \in S_{m}}(-1)^{\operatorname{sgn}(\sigma)} t_{\sigma(1), 1}(u) t_{\sigma(2), 2}(u-1) \ldots t_{\sigma(m), m}(u-m+1)
$$

which generate the center of the $\mathrm{Y}\left(\mathfrak{g l}_{s}\right)[6,29]$.
Our main goal is to construct the limit of the above Yangian action when $N$ tends to infinity. In particular, we get the limits of the above commuting family of Hamiltonians. To construct the limit we need investigate the projective properties of the Yangian actions in phase spaces $\Lambda_{ \pm}^{s, N}$ of CS model. Such an analysis was done by D.Uglov in [46], but our description differs from that of [46].

The rings $\Lambda_{+}^{N}$ of scalar symmetric functions form the projective system with respect to the maps

$$
\omega_{N}^{+}: \Lambda_{+}^{N} \rightarrow \Lambda_{+}^{N-1}, \quad \omega_{N}^{+} f\left(x_{1}, \ldots, x_{N}\right)=f\left(x_{1}, \ldots, x_{N-1}, 0\right)
$$

Analogously, the spaces $\Lambda_{-}^{N}$ of scalar skewsymmetric functions form the projective system with respect to the maps

$$
\omega_{N}^{-}: \Lambda_{-}^{N} \rightarrow \Lambda_{-}^{N-1}, \quad \omega_{N}^{-} f\left(x_{1}, \ldots, x_{N}\right)=\left(x_{1} \ldots x_{N-1}\right)^{-1} f\left(x_{1}, \ldots, x_{N-1}, 0\right)
$$

Contrary to the ring of symmetric functions, the space $\hat{\Lambda}$ is not the projective limit of the spaces of (skew)symmetric functions due to the presence of zero mode $p_{0}$. On the other hand, CS Hamiltonians $H_{k}$ themselves in both symmetric and skewsymmetric cases do not form a projective family since they do not respect natural projections

$$
\omega_{N}^{+} H_{k}^{(N+1)} \neq H_{k}^{(N)} \omega_{N}^{+} .
$$

Let

$$
T(u)=\sum_{a, b=1}^{s} E_{a b} \otimes t_{a b}(u) \in \operatorname{End}\left(\mathbb{C}^{s}\right) \otimes \mathrm{Y}\left(\mathfrak{g l}_{s}\right)\left[u^{-1}\right]
$$

be the generating matrix of Yangian generators. Denote by $T_{N}(u)$ the transfer matrix corresponding to the representation (9), here index $N$ denotes the number of particles. In scalar case $(s=1)$ the transfer matrix $T_{N}(u)$ is the generating function of the Hamiltonians

$$
T_{N}(u)=1+\frac{1}{u} H_{0}^{(N)}+\frac{1}{u^{2}} H_{1}^{(N)}+\frac{1}{u^{3}} H_{2}^{(N)}+\ldots
$$

We formulate the projective property of $T_{N}(u)$ in scalar symmetric and skewsymmetric case

Proposition 4.1 (i) In scalar symmetric case we have the following identity of operators from $\tilde{\Lambda}_{+}^{N}\left[u^{-1}\right] \rightarrow \tilde{\Lambda}_{+}^{N-1}\left[u^{-1}\right]$ :

$$
\omega_{N}^{+} T_{N}(u)=\frac{u+1}{u} T_{N-1}(u+1) \omega_{N}^{+}
$$

(ii) In scalar skewsymmetric case the following identity of operators from $\tilde{\Lambda}_{-}^{N}\left[u^{-1}\right] \rightarrow$ $\tilde{\Lambda}_{-}^{N-1}\left[u^{-1}\right]$ holds:

$$
\omega_{N}^{-} T_{N}(u)=\frac{u+1}{u} T_{N-1}(u-\alpha-1) \omega_{N}^{-} .
$$

Iterating the relations from Proposition 4.1 we see, that the renormalized transfer matrices $\tilde{T}_{N}(u)$ and $\bar{T}_{N}(u)$ in symmetric and skewsymmetric case

$$
\tilde{T}_{N}(u)=\frac{u-N}{u} T_{N}(u-N) \quad \bar{T}_{N}(u)=T_{N}(u+\gamma N) \prod_{k=1}^{N} \frac{u+k \gamma}{u+k \gamma+1},
$$

are compatible with projection maps $\omega_{N}^{+}$and $\omega_{N}^{-}$, respectively

$$
\omega_{N}^{+} \tilde{T}_{N}(u)=\tilde{T}_{N-1}(u) \omega_{N}^{+} \quad \omega_{N}^{-} \bar{T}_{N}(u)=\bar{T}_{N-1}(u) \omega_{N}^{-}
$$

Here $\gamma=\alpha+1$. The coefficients of renormalized transfer matrices can be chosen as a projective system of Hamiltonians of CS system.

The statement of Proposition 4.1 can be generalized to skewsymmetric spin case. Regard an element $f$ of $\Lambda_{-}^{s, N}$ as $\left(\mathbb{C}^{s}\right)^{\otimes N}$ valued function $f=f\left(x_{1}, x_{2}, \ldots, x_{N}\right)$. We define a linear map $\omega_{N}: \Lambda_{-}^{s, N} \rightarrow \Lambda_{-}^{s, N-s}$ by the formula

$$
\omega_{N}^{-}(f)=\left(x_{1} \cdots x_{N-s}\right)^{-1}\left(1^{\otimes(N-s)} \otimes e_{1}^{\perp} \otimes e_{2}^{\perp} \cdots \otimes e_{s}^{\perp}\right) f\left(x_{1}, \ldots, x_{N-s}, 0, \ldots 0\right)
$$

and formulate the following
Proposition 4.2 The following identities of operators from $\mathbb{C}^{s} \otimes \tilde{\Lambda}_{-}^{s, N s}\left[u^{-1}\right] \rightarrow \mathbb{C}^{s} \otimes$ $\tilde{\Lambda}_{-}^{s,(N-1) s}\left[u^{-1}\right]$ holds:

$$
\omega_{N s}^{-} T_{N s}(u)=\frac{u+1}{u} T_{(N-1) s}(u-\alpha-s) \omega_{N s}^{-} .
$$

Set $\gamma=\alpha+s$ and

$$
\bar{T}_{N s}(u)=T_{N s}(u+\gamma N) \prod_{k=1}^{N} \frac{u+k \gamma}{u+k \gamma+1},
$$

treated as asymptotical series in $u^{-1}$. Then $\bar{T}_{N s}(u)$ satisfy compatibility conditions

$$
\omega_{N s}^{-} \bar{T}_{N s}(u)=\bar{T}_{(N-1) s} \omega_{N s}^{-}
$$

and form a projective system of transfer matrices.

## 5 Bosonic limit of spin Calogero-Sutherland system

In this section we observe the results of [21] using slightly different language.
Let $\mathcal{H}^{s}$ be the Heisenberg algebra with generators $a_{c, k}, c=1, \ldots, s, k=0,1, \ldots$ and $\left(q_{c}\right)^{ \pm 1}$, which satisfy the relations

$$
\left[a_{c, k}, a_{d, l}\right]=k \delta_{c d} \delta_{k,-l}, \quad q_{c} a_{d, k}=\left(a_{d, k}+\delta_{c d} \delta_{k 0}\right) q_{c} .
$$

Let $\hat{\Lambda}^{(s)}$ be a representation of the Heisenberg algebra $\mathcal{H}^{s}$ with the vacuum vector $|0\rangle_{+}$ such that

$$
a_{c, k}|0\rangle_{+}=0, c=1, \ldots, s, \quad k>0, \quad q_{c}|0\rangle_{+}=|0\rangle_{+}, c=1, \ldots, s .
$$

Denote by ${ }_{+}\langle 0|$ the vector of the dual space, which satisfies the relations

$$
+\langle 0| a_{c, k}=0, \quad c=1, \ldots, s, \quad k \leq 0 .
$$

For any $|v\rangle_{+} \in \hat{\Lambda}^{(s)}$ consider the matrix element $\tilde{\pi}_{N}\left(|v\rangle_{+}\right) \in V^{\otimes N}$

$$
\tilde{\pi}_{N}\left(|v\rangle_{+}\right)={ }_{+}\langle 0| \boldsymbol{\Phi}\left(z_{N}\right) \boldsymbol{\Phi}\left(z_{N-1}\right) \cdots \boldsymbol{\Phi}\left(z_{1}\right)|v\rangle_{+},
$$

where

$$
\begin{array}{ll}
\Phi_{c}(z)=\exp \left(\sum_{n>0} \frac{a_{c, n}}{n} z^{n}\right) q_{c}: & \hat{\Lambda}^{(s)} \rightarrow \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z], \quad c=1, \ldots, s, \\
\Phi(z)=\sum_{c} \Phi_{c}(z) \otimes e_{c}: & \hat{\Lambda}^{(s)} \rightarrow \hat{\Lambda}^{(s)} \otimes V
\end{array}
$$

are the vertex operators and by $\boldsymbol{\Phi}\left(z_{k}\right)$ we shortly denote $\boldsymbol{\Phi}\left(z_{k}\right) \otimes 1^{\otimes k-1}$. We show that $\tilde{\pi}_{N}\left(|v\rangle_{+}\right) \in \Lambda_{+}^{s, N}$ is symmetric invariant.

Our goal is to pull back the Yangian action (9) in $\Lambda_{+}^{s, N}$ through the map $\tilde{\pi}_{N}$. We use the similar procedure as in scalar case and decompose the application of each Yangian generator (9) to a vector $|w\rangle_{+} \in \Lambda_{+}^{s, N}$ into several steps. First we present the symmetric tensor $|w\rangle_{+} \in \Lambda_{+}^{s, N}$ as an element of $\left(\mathbb{C}\left[x_{i}\right] \otimes \mathbb{C}^{s}\right) \otimes \Lambda_{+}^{s, N-1}$ for each tensor component, producing an equivariant family of vectors, then we apply the power of Heckman operator $\mathcal{D}_{i}^{(N)}$ to the $i$-th vector of this equivariant family and get another equivariant family. The last step is the symmetrization $E_{N}(u)$ - the sum of all members of the equivariant family:

$$
E_{N}(u)=\sum_{j=1}^{N} \sigma_{1 j}(u)
$$

where $\sigma_{i j}=K_{i j} P_{i j}$ is the permutation of $i$-th and $j$-th tensor factors.
For each $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ define the element $\mathcal{S}(\boldsymbol{F}(z)) \in \hat{\Lambda}^{(s)}$ as the formal integral

$$
\mathcal{S}(\boldsymbol{F}(z))=\frac{1}{2 \pi i} \oint \frac{d z}{z} \boldsymbol{\Phi}^{*}(z) \boldsymbol{F}(z)
$$

which counts zero term of the Laurent series. Here

$$
\Phi^{*}(z)=\sum_{c} \varphi_{c}^{-}(z) \cdot \Phi_{c}^{-1}(z) \otimes e_{c}^{\perp}: \quad \hat{\Lambda}^{(s)} \otimes V \rightarrow \hat{\Lambda}^{(s)} \otimes \mathbb{C}[z]
$$

the series $\varphi_{c}^{-}(z)=\sum_{n \leq 0} a_{c, n} z^{n}$ and the operator $e_{c}^{\perp}: \mathbb{C}^{s} \rightarrow \mathbb{C}$ is given by the relation $e_{c}^{\perp}\left(e_{b}\right)=\delta_{b c}$. The key point of the construction is the following lemma which establishes the map $\mathcal{S}$ as the pullback of the finite symmetrization:

Lemma 5.2 For each $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ and any natural $N$ we have the equality of elements of $\Lambda_{+}^{s, N}$ :

$$
E_{N}\left(\tilde{\pi}_{N-1} \otimes 1\right)(\boldsymbol{F}(z))=\tilde{\pi}_{N} \mathcal{S}(\boldsymbol{F}(z))
$$

Let $\tilde{\mathcal{D}}: \hat{\Lambda}^{(s)} \otimes V \rightarrow \hat{\Lambda}^{(s)} \otimes V$ be the linear map, such that for any $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$

$$
\tilde{\mathcal{D}} \boldsymbol{F}^{(1)}(z)=\alpha z \frac{d}{d z} \boldsymbol{F}^{(1)}(z)+\frac{z}{2 \pi i} \oint \frac{d \xi}{\xi^{2}\left(1-\frac{z}{\xi}\right)} \boldsymbol{\Phi}^{*(2)}(\xi) \boldsymbol{\Phi}^{(2)}(z) \boldsymbol{F}^{(1)}(\xi)
$$

Here the upper index $(i), i=1,2$ indicates in which tensor copy of $\mathbb{C}^{s}$ the corresponding vector lives or an operator acts. We state that the operator $\tilde{\mathcal{D}}$ is the pullback of the equivariant family of Heckman operators $\mathcal{D}_{i}^{(N)}$.

Proposition 5.1 For any $\boldsymbol{F}(z) \in \hat{\Lambda}^{(s)} \otimes V$ we have

$$
\left(\tilde{\pi}_{N-1} \otimes 1\right) \tilde{\mathcal{D}}\left(\boldsymbol{F}\left(x_{1}\right)\right)=\mathcal{D}_{1}^{(N)}\left(\tilde{\pi}_{N-1} \otimes 1\right) \boldsymbol{F}\left(x_{1}\right)
$$

Let $E_{a b} \in \operatorname{End} \mathbb{C}^{s}$, be the matrix unit, $E_{a b}\left(e_{c}\right)=\delta_{b c} e_{a}$. Denote by $\mathcal{E}_{a b}$, the operator $1 \otimes 1 \otimes E_{a b}: \hat{\Lambda}^{(s)} \otimes V \rightarrow \hat{\Lambda}^{(s)} \otimes V:$

$$
\mathcal{E}_{a b} \boldsymbol{F}(z)=F_{b}(z) \otimes e_{a}
$$

Summarazing the statements above we get the following result [21]
Theorem 5.1 The operator $T_{a b, n}$ given by

$$
T_{a b, n}=\frac{(-1)^{n}}{2 \pi i} \oint \frac{d z}{z} \boldsymbol{\Phi}^{*}(z) \mathcal{E}_{a b} \tilde{\mathcal{D}}^{n} \boldsymbol{\Phi}(z)
$$

is the pullback of the Yangian generator $t_{a b, n}$, see (9):

$$
\tilde{\pi}_{N} T_{a b, n}=t_{a b, n} \tilde{\pi}_{N} \quad \text { for any } \quad N \in \mathbb{N} .
$$

Using this construction we derive the explicit expressions for the first Hamiltonians of CS system.

Further we investigate the classical limit of the system. We introduce the operator $\mathscr{H}^{\mathrm{cl}}$ which is the classical limit of the second Hamiltonian, the rule between the quantum commutator and Poisson bracket is $\beta^{-1}[,] \rightarrow\{$,$\} . In Proposition 5.3$ we present the equations of motion determined by $\mathscr{H}^{\mathrm{cl}}$ :

$$
\frac{d \phi_{a}(z)}{d t}=\left\{\phi_{a}(z), \mathscr{H}^{\mathrm{cl}}\right\} .
$$

Here and further $\phi_{a}(z)$ and $\mathcal{V}_{a}(z)$ are the classical analogues of field $\varphi_{a}(z)$ and vertex operator $\Phi_{a}(z)$ respectively.

The quantum system is integrable: it has an infinite number of integrals of motion that can be obtained from the $q$-determinant of the Yangian generator function $T_{a b}(u)$. It is natural to assume that the classical system is integrable as well. In particular, it should admit a Lax pair presentation. Consider the operators $L$ and $M$ acting on the analytic function $f(z)$ :

$$
\begin{aligned}
& L f=z \frac{\partial}{\partial z} f(z)+\sum_{a} \mathcal{V}_{a}(z)\left(\phi_{a}^{-}(z) \mathcal{V}_{a}^{-1}(z) f(z)\right)^{+} \\
& M f=\left(z \frac{\partial}{\partial z}\right)^{2} f(z)+2 \sum_{b}\left(\phi_{b}^{+}(z) \phi_{b}^{-}(z)\right)^{+} f(z)+2 \sum_{b} \mathcal{V}_{b}(z) z \frac{\partial}{\partial z}\left(\phi_{b}^{-}(z) \mathcal{V}_{b}^{-1}(z) f(z)\right)^{+} .
\end{aligned}
$$

Proposition 5.4 The operators $L$ and $M$ represent a Lax pair of the classical system:

$$
\frac{d L}{d t}=[M, L] .
$$

## 6 Fermionic limit for spin system

The fermionic limit of spin CS system was studied by D. Uglov. Here we suggest and develop another approach, which leads to the limiting integrable system closely related to [46], but realized by free fermionic fields, we mainly follow [20].

We start from the fermionic Fock space $\mathcal{F}^{s}$, which is the representation of algebra $\mathcal{H}_{-}^{s}$ of $s$ free fermion fields. We denote by $\Psi_{c}(z)$ and $\Psi_{c}^{*}(z)$ be the following generating functions of elements of $\mathcal{H}_{-}^{s}$ :

$$
\Psi_{c}(z)=\sum_{n \in \mathbb{Z}} \psi_{c n} z^{n}, \quad \Psi_{c}^{*}(z)=\sum_{n \in \mathbb{Z}} \psi_{c n}^{*} z^{n-1}
$$

For any $|v\rangle \in \mathcal{F}^{s}$ we define a matrix coefficients by the following formula

$$
\pi_{N}(|v\rangle)=\langle 0| \Psi\left(z_{N}\right) \boldsymbol{\Psi}\left(z_{2}\right) \cdots \boldsymbol{\Psi}\left(z_{1}\right)|v\rangle, \quad|v\rangle \in \mathcal{F}^{s}
$$

where $\Psi(z)=\sum_{c=1}^{s} \Psi_{c}(z) \otimes e_{c}$ and $e_{c} \in \mathbb{C}^{s}$ are again basic vectors of $\mathbb{C}^{s}$. The matrix element $\pi_{N}(|v\rangle)$ belongs to the space $\Lambda_{-}^{s, N}$, which is the phase space of finite spin CS system. Then we systematically construct the pullback with respect to the maps $\pi_{N}$ of all operation required for the construction of the Yangian action on the finite-dimensional spin CS system.

The crucial point of the construction is the operator which the pullback of total finite antisymetrization $\mathrm{A}_{N}: V \otimes \Lambda_{-}^{s, N-1} \rightarrow \Lambda_{-}^{s, N}$, given by

$$
\mathrm{A}_{N}(u)=u-\sum_{j=2}^{N} \sigma_{1 j}(u)
$$

For each $\boldsymbol{F}(z) \in \mathcal{H}_{-}^{s}(z) \otimes \mathbb{C}^{s}$ define the element $\mathcal{A}(\boldsymbol{F}(z)) \in \mathcal{H}_{-}^{s}$ as the integral

$$
\mathcal{A}(\boldsymbol{F}(z))=\frac{1}{(2 \pi i)^{2}} \int_{z \circlearrowleft 0} d z \int_{u \circlearrowleft z} d u \frac{\boldsymbol{\Psi}^{*}(u) \boldsymbol{F}(z)}{u-z} .
$$

As in scalar case we introduce a subspace $U \in \mathcal{H}_{-}^{s}(z) \otimes \mathbb{C}^{s}$ of elements which satisfy the special conditions, namely, preservation of the polynomial space and homogeneity. These conditions are preserved by pullbacks of Dunkl operators, which we define further. The following lemma establishes the map $\mathcal{A}$ as the pullback of the finite antisymmetrization:

Lemma 6.2 For each $\boldsymbol{F}(z) \in U$, any $|v\rangle \in \mathcal{F}^{s}$ and any natural $N$ we have the equality of elements of $\Lambda_{-}^{s, N}$ :

$$
\mathrm{A}_{N} \pi_{N-1,1}(\boldsymbol{F}(z) \otimes|v\rangle)=\pi_{N} \mathcal{A}(\boldsymbol{F}(z))|v\rangle
$$

Define an operator $\mathcal{D}: \mathcal{H}_{-}^{s}(z) \otimes \mathbb{C}^{s} \rightarrow \mathcal{H}_{-}^{s}(z) \otimes \mathbb{C}^{s}$ by the relation

$$
\begin{aligned}
& \mathcal{D} \boldsymbol{F}(z)=\alpha z \frac{d}{d z} \boldsymbol{F}(z)+ \\
& \frac{z}{(2 \pi i)^{2}} \int_{\substack{w \circlearrowleft 0}} d w \int_{u \circlearrowleft w \mid z} d u \boldsymbol{\Psi}^{*(2)}(u) \frac{\boldsymbol{\Psi}^{(2)}(w) \boldsymbol{F}^{(1)}(z)-\boldsymbol{\Psi}^{(2)}(z) \boldsymbol{F}^{(1)}(w)}{(u-w)(z-w)} .
\end{aligned}
$$

By means of Lemma 6.2 we now can identify the operator $\mathcal{D}$ as a pullback of the equivariant family of Heckman operators $\mathcal{D}_{i}^{(N)}$ acting in the space of partially antisymmetric tensors

Proposition 6.1 For any $\boldsymbol{F}(z) \in U,|v\rangle \in \mathcal{F}^{s}$ and $N \in \mathbb{N}$ we have the equality

$$
\pi_{N-1,1}\left(\mathcal{D} \boldsymbol{F}\left(x_{1}\right) \otimes|v\rangle\right)=\mathcal{D}_{1}^{(N)} \pi_{N-1,1}\left(F\left(x_{1}\right) \otimes|v\rangle\right)
$$

As in bosonic spin case we introduce operators :

$$
\mathrm{T}_{a b, n}=\mathcal{A} \mathcal{E}_{a b} \mathcal{D}^{n} \boldsymbol{\Psi}(z)=\frac{1}{(2 \pi i)^{2}} \int_{z \circlearrowleft 0} d z \int_{u \circlearrowleft z} d u \frac{\boldsymbol{\Psi}^{*}(u) \mathcal{E}_{a b} \mathcal{D}^{n} \boldsymbol{\Psi}(z)}{u-z}
$$

Summarizing the statements above we establish the operator $\mathrm{T}_{a b, n}$ as the pullback of the Yangian generator $t_{a b, n}$ in $\Lambda_{-}^{s, N}$.

Proposition 6.2 For any $|v\rangle \in \mathcal{F}^{s}$ and $N \in \mathbb{N}$ we have the equality

$$
\pi_{N}\left(\mathrm{~T}_{a b, n}|v\rangle\right)=t_{a b, n} \pi_{N}|v\rangle .
$$

We note the importance of the polynomial property of the total zero mode in the constructed Yangian action on the Fock space $\mathcal{F}^{s}$, which we prove by using projective properties of the Yangian action in the phase spaces of CS models, it allows to formulate the following

Theorem 6.1 The operators $\mathrm{T}_{a b, n}$ satisfy Yangian relations.

In particular, the coefficients of the quantum determinant $q \operatorname{det} \mathrm{~T}(u)$ form a commutative family which can be regarded as the limits of the higher Hamiltonians of CS system.

## Applications of the results

Here we mention some applications of the results concerning the limits of integrable systems of Calogero-Sutherland type:

- representation theory
- theory of symmetric function
- knot theory, combinatorics of Hurwitz numbers


## The main results of the thesis are presented in two papers:

1. Khoroshkin S. M., Matushko M., Sklyanin E., On spin Calogero-Moser system at infinity, Journal of Physics A: Mathematical and Theoretical. 2017. Vol. 50. P. 1-26
2. S.M. Khoroshkin, M. G. Matushko, Fermionic limit of the Calogero-Sutherland system, Journal of Mathematical Physics. 2019. Vol. 60. No. 7. P. 071706-1-071706-22.

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