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On Bethe vectors of $\mathfrak{gl}(2|1)$-invariant integrable models

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Introduction

Quantum integrable models are a special class of physical models. These models describe non trivial systems of interacting particles and at the same time they can be studied accuracy using mathematical tools. They offer us a unique training ground for a deep study of non trivial physical phenomena explicitly.

A wide class of quantum integrable models is associated with higher rank algebras. Integrable models with symmetries of high rank appear in condensed matter physics, in particular in the $\mathfrak{gl}_{m|n}$ invariant XXX Heisenberg spin chain, in multi-component Bose/Fermi gas [25], and in the study of models of cold atoms (the Hubbard model [21], the t-J model [22, 23, 24]). Also spin chains of higher rank are interesting in the context of computing correlation functions in $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [10, 11].

The role of the scalar product of Bethe vectors is extremely important in the study of correlation functions of local operators of the underlying quantum models [15, 17, 18]. One can reduce the problem of calculation of the form factors and the correlation functions of local operators to the calculation of the scalar products of the Bethe vectors [28, 29].

The study of integrable systems with high rank symmetry is still a challenging task. Until recently, such models have either not been studied at all, or have been studied under various simplifying hypotheses. The results presented in the thesis are the first in this direction.

My thesis presents the results of four articles in which I am one of the co-authors. The articles are devoted to the study of Bethe vectors and their scalar products in quantum integrable models with $\mathfrak{gl}_{2|1}$-algebra symmetry. This research is the development of mathematical apparatus of the study of correlation functions of these systems. In fact, this thesis is completely devoted to the description of Bethe vectors and to study of their properties.

This section contains an overview of the results of the thesis, where we present generalization of Algebraic Bethe Anzats to the case $\mathfrak{gl}_{2|1}$-invariant integrable models and scalar products of Bethe vectors in this case.

1 Quantum R-matrix structure

The most fundamental structure of algebraic Bethe ansatz is R-matrix. Depending on point of view one can perceive it as scattering matrix of some $2 \to 2$ scattering process [1, 2, 3] or as a set of structure functions of bilinear algebra which depends on spectral parameter [5, 4, 16]. This algebra is called RTT-algebra. Elements of the algebra can be encoded in 3×3 matrix $T(u)$

which is called monodromy matrix. The R-matrix of $\mathfrak{gl}(2|1)$-based models acts in the tensor product $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1}$, where $\mathbb{C}^{2|1}$ is 3-dimensional Z_2-graded vector space with the grading $[1] = [2] = 0, [3] = 1$:

$$R_{12}(u, v)\; T_1(u) T_2(v) = T_2(v) T_1(u)\; R_{12}(u, v). \tag{1}$$

Here subscripts mean tensor multiplier in which the operator acts. Here $T_1(u) = T(u) \otimes 1$ and $T_2(u) = 1 \otimes T(u)$, their elements act in some space $\mathcal{H}$ called physical space. Arguments u, v of the monodromy matrix are called spectral parameters. The spectral parameter is a complex number. R-matrix acts in both spaces. In our work we use R-matrix associated with ($\mathfrak{gl}_{2|1}$) algebra

$$R_{12}(u) = u\, \mathbf{1} + c\, P_{12}, \tag{2}$$

where $\mathbf{1}$ is the unity operator, c is a constant and P_{12} is a graded permutation operator [7]. In the $\mathfrak{gl}(2|1)$-case permutation operator P_{12} has form

$$P_{12} = \sum_{i,j=1}^{3} (-1)^{[j]} e_{ij} \otimes e_{ji}, \tag{3}$$

where e_{ij} are the elementary units $(e_{ij})_{ab} = \delta_{ia}\delta_{jab}$ with supersymmetric grading

$$(\mathbf{1} \otimes e_{ij})(e_{kl} \otimes \mathbf{1}) = (-1)^{([i]+[j])([k]+[l])} e_{kl} \otimes e_{ij}. \tag{4}$$

The R-matrix (2) satisfies the graded Yang–Baxter equation

$$R_{12}(u-v) R_{13}(u-w) R_{23}(v-w) = R_{23}(v-w) R_{13}(u-w) R_{12}(u-v) \tag{5}$$

written in the tensor product of graded spaces $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1}$.

Let us multiply (1) by the inverse matrix to $R_{12}(u, v)$ and take the trace over space $\mathbb{C}^{2|1} \otimes \mathbb{C}^{2|1}$. Using property of the trace one can obtain commutativity relation

$$[t(u), t(v)] = 0, \tag{6}$$

for the transfer matrix

$$t(u) = \sum_{i} (-1)^{[i]} T_{ii}(u). \tag{7}$$

Due to equation (6) coefficients in a series expansion at some point u_0 of the transfer matrix $t(u) = \sum_k (u-u_0)^k H_k$ commute

$$[H_n, H_m] = 0. \tag{8}$$

These coefficients are called Hamiltonians. One can say that the transfer matrix is a generating function of the commuting Hamiltonians of some integrable system.

Thus, the presence of R-matrix structure implies the presence of a large number of conservation laws in the system and indicates the integrability of this system.

To use algebraic Bethe ansatz approach, besides quantum R-matrix structure one needs an existence of special vector $|0\rangle \in \mathcal{H}$ called vacuum. This vector must have several properties

$$\begin{aligned} &T_{ji}(u)|0\rangle = 0, \qquad \text{with } \ i<j \\ &T_{ii}(u)|0\rangle = \lambda_i(u)|0\rangle, \end{aligned} \tag{9}$$

where $\lambda_i(u)$ are some functions depending on the concrete quantum integrable model. Action of $T_{ij}(u)$ with $i<j$ onto vacuum $|0\rangle$ is nontrivial. In the quantum integrable models multiple action of upper-triangular elements of monodromy matrix onto $|0\rangle$ generates a basis in the physical space $\mathcal{H}$.

Generalized model. In the framework of the our work we assume that $\lambda_i(u)$ are free functional parameters and we do not specify any of their concrete dependencies [17, 27, 18]. It means that one can find concrete quantum integrable model for any specific choice of $\lambda_i(u)$.

2 Spin chain as basic example

In the past, the structure of the R-matrix was discovered in a large number of quantum systems [22, 23, 24, 21, 25]. Usually it is very non-trivial problem to find R-matrix structure. One of the most simple examples is a spin chain. One can construct quantum integrable system inductively using general properties of R-matrix.

To construct a spin chain we use rational R-matrix (2). In this case the monodromy matrix of the spin chain is

$$T_0(u) = R_{01}(u-\xi_1)R_{02}(u-\xi_2)\dots R_{0n}(u-\xi_n). \tag{10}$$

Here R_{0i}-matrix acts non-trivially in the space $V_0 \otimes V_i$, and as unity in the rest spaces V_j (with $j \neq i$). The monodromy matrix acts in the space $V_0 \otimes V_1 \otimes V_2 \otimes \dots \otimes V_n$. This space is divided into two parts: physical space $\mathcal{H} = V_1 \otimes V_2 \otimes \dots \otimes V_n$ and auxiliary space V_0. We consider the monodromy matrix as matrix acting in the 3-dimensional auxiliary space with noncommutative elements acting in the physical space $\mathcal{H}$. The parameters ξ_i are called inhomogeneities. The monodromy matrix satisfies the RTT-relation (1).

The model described by monodromy matrix (10) is called the inhomogeneous XXX spin chain. It is the most typical example of quantum integrable model with quantum R-matrix structure. It exists for any R-matrix.

One can set all parameters $\xi_i = 0$. Then, the model becomes homogeneous spin chain. To describe the quantum integrable system obtained from this monodromy matrix let us consider one very special Hamiltonian in the expansion of transfer matrix of homogeneous spin chain.

$$H = (t(0))^{-1} t'(0). \tag{11}$$

From (11) one can obtain that

$$H = c \sum_i P_{i,i+1}. \tag{12}$$

This Hamiltonian is the sum of operators, each of them acting in two adjacent spaces. This property is called ultra locality.

In the case of spin chain vacuum vector is $|0\rangle = \mathbf{e}_1^{(1)} \otimes \ldots \otimes \mathbf{e}_1^{(n)}$, where $\mathbf{e}_1^{(i)}$ is a vector $(1,0,0)^T$ from the space $\mathbb{C}^{2|1}$. According to (9) the lower triangular elements of the monodromy matrix annihilate the vacuum. The vacuum is eigenvector for the diagonal elements with eigenvalues

$$\begin{aligned} \lambda_1(u) &= \prod_{k=1}^{n} (u - \xi_k + c)\,, \\ \lambda_i(u) &= \prod_{k=1}^{n} (u - \xi_k)\,, \quad i = 2, 3. \end{aligned} \tag{13}$$

The monodromy matrix of the inhomogeneous XXX spin chain (10) satisfies all the necessary properties for the application of the algebraic Bethe ansatz approach.

3 Algebraic Bethe ansatz for $\mathfrak{gl}_2$

Let us consider how algebraic Bethe ansatz works in the most simple case of $n = 2$ [16]. In this case the monodromy matrix is 2×2 matrix

$$T(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{14}$$

To apply algebraic Bethe ansatz we need a vector $|0\rangle \in \mathcal{H}$ called vacuum. Vacuum should have the following properties:

$$\begin{aligned} A(u)|0\rangle &= a(u)|0\rangle, \\ D(u)|0\rangle &= d(u)|0\rangle, \\ C(u)|0\rangle &= 0, \end{aligned} \tag{15}$$

where $a(u)$ and $d(u)$ are the eigenvalues of corresponding operators on vacuum.

To simplify all the next expressions let us introduce shorthand notation [20]. Symbol "bar" in $\bar{u}$ means that it is a set of variables $\bar{u} = \{u_1, u_2, \dots, u_n\}$. Subscript near $\bar{u}$ means that one element of the set is excluded $\bar{u}_i = \bar{u} \backslash \{u_i\}$. If some function depends on set instead of a variable then one should understand that this expression is a product of this function over all elements in this set. One can use also this notation for function depending on two sets of variables. For example

$$a(\bar{u}) = \prod_{u_i \in \bar{u}} a(u_i), \quad f(\bar{u}, \bar{v}_i) = \prod_{u_k \in \bar{u}} \prod_{v_j \in \bar{v}, j \neq i} f(u_k, v_j). \tag{16}$$

Using RTT-relation (1) with R-matrix (2) one can show that

$$[T_{ij}(u), T_{ij}(v)] = 0. \tag{17}$$

So, we can also extend the shorthand notation to the product of operators

$$T_{ij}(\bar{u}) = T_{ij}(u_1) T_{ij}(u_2) \dots T_{ij}(u_n). \tag{18}$$

In the case of $\mathfrak{gl}_2$ there is only one monodromy matrix elements acting nontrivial onto vacuum $|0\rangle$. It is upper triangular element $B(u)$. One can introduce a Bethe vector associated with set $\bar{u} = \{u_1, u_2, \dots, u_n\}$

$$\mathbb{B}(\bar{u}) = B\left(\bar{u}\right)|0\rangle = B(u_1)B(u_2)\dots B(u_n)|0\rangle. \tag{19}$$

Due to (17) Bethe vector is symmetric in elements of set $\bar{u}$. We suppose that Bethe vector can become eigenvector of transfer matrix $t(u) = A(u) + D(u)$. To find it out we need the commutation relations of diagonal elements with $B(u)$. These commutation relations follow from the RTT relation (1):

$$\begin{aligned} A(u)B(v) &= f(v,u)B(v)A(u) + g(u,v)B(u)A(v), \\ D(u)B(v) &= f(u,v)B(v)D(u) + g(v,u)B(u)D(v). \end{aligned} \tag{20}$$

where

$$f(v,u) = \frac{v-u+c}{v-u}, \quad g(v,u) = \frac{c}{v-u}. \tag{21}$$

Action of the transfer matrix $t(u) = A(u) + D(u)$ on the Bethe vector (19) gives us equation

$$t(z)\mathbb{B}(\bar{u}) = \tau(z|\bar{u})\ \mathbb{B}(\bar{u}) + \sum_{i=1}^{n}\ g(z,u_i)\Lambda_i\ \mathbb{B}(\bar{u}_i \cup \{z\}), \tag{22}$$

where

$$\tau(z|\bar{u}) = a(z)f(\bar{u}, z) + d(z)f(z, \bar{u}), \tag{23}$$

and

$$\Lambda_i = a(u_i)f(\bar{u}_i, u_i) - d(u_i)f(u_i, \bar{u}_i). \tag{24}$$

If we set all $\Lambda_i = 0$ then Bethe vector $\mathbb{B}(\bar{u})$ becomes eigenvector with eigenvalue $\tau(z|\bar{u})$ (23). The conditions $\Lambda_i = 0$ are called the system of Bethe equations.

Unfortunately, generalization of this scheme to algebras of higher rank is not so simple.

4 $\mathfrak{gl}_{2|1}$-invariant Bethe vector

In the $\mathfrak{gl}_{2|1}$-case formula for the Bethe vector is more sophisticated. It has form [8]:

$$\mathbb{B}(\bar{u}, \bar{v}) = \sum \frac{g(\bar{v}_{\mathrm{I}}, \bar{u}_{\mathrm{I}})}{\lambda_2(\bar{v}_{\mathrm{II}})\lambda_2(\bar{u})} \frac{f(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}})g(\bar{v}_{\mathrm{II}}, \bar{v}_{\mathrm{I}})h(\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{I}})}{f(\bar{v}, \bar{u})} \mathbb{T}_{13}(\bar{u}_{\mathrm{I}})T_{12}(\bar{u}_{\mathrm{II}})\mathbb{T}_{23}(\bar{v}_{\mathrm{II}})|0\rangle, \tag{25}$$

where $\mathbb{T}_{i3}(\bar{v})$ for $i = 1, 2$ is symmetric combination of matrix elements

$$\mathbb{T}_{i3}(\bar{v}) = \frac{T_{i3}(v_1) \dots T_{i3}(v_n)}{\prod_{l>m} h(v_l, v_m)}, \tag{26}$$

with

$$h(x, y) = \frac{x - y + c}{c}. \tag{27}$$

Here sets of the Bethe parameters $\bar{u}$ and $\bar{v}$ are divided into two subsets $\bar{u} \Rightarrow \{\bar{u}_{\mathrm{I}}, \bar{u}_{\mathrm{II}}\}$ and $\bar{v} \Rightarrow \{\bar{v}_{\mathrm{I}}, \bar{v}_{\mathrm{II}}\}$, such that $\#\bar{u}_{\mathrm{I}} = \#\bar{v}_{\mathrm{I}}$. The sum is taken over all possible partitions of this type.

Let us notice that this Bethe vector depends on two sets of variables $\bar{u}, \bar{v}$. All the upper triangular elements of the monodromy matrix are involved in a construction of Bethe vector. Now it is not a monomial in the elements of the monodromy matrix and the coefficients are extremely nontrivial. The number of terms grows exponentially with the sizes of sets $\bar{u}, \bar{v}$.

Our works contain a generalization of the Bethe vector and its properties (like co-product property and recurrence equation for Bethe vectors) that help to apply Algebraic Bethe ansatz scheme to models with super-$\mathfrak{gl}_{2|1}$ symmetries.

We prove that the action of all monodromy matrix entries $T_{ij}(z)$ onto Bethe vector can be expressed as linear combination of Bethe vectors with various arguments. A separated our paper is dedicated to these action formulas.

Particularly, we prove that Bethe vector becomes eigenvector for transfer matrix (7)

$$t(z)\,\mathbb{B}(\bar{t})=\tau(z|\bar{t})\,\mathbb{B}(\bar{t}), \tag{28}$$

if Bethe parameters satisfy the system of equations (this system is called Bethe equations)

$$\begin{aligned} \frac{\lambda_1(u_i)}{\lambda_2(u_i)}&=\frac{f(u_i,\bar{u}_i)}{f(\bar{u}_i,u_i)}f(\bar{v},u_i),\\ \frac{\lambda_3(v_j)}{\lambda_2(v_j)}&=f(v_j,\bar{u}). \end{aligned} \tag{29}$$

with eigenvalue

$$\tau(z|\bar{t})=\lambda_1(z)f(\bar{u},z)+\lambda_2(z)f(\bar{v},z)f(z,\bar{u})-\lambda_3(z)f(\bar{v},z). \tag{30}$$

This result prove that out scheme of Algebraic Bethe anzats is applicable to $\mathfrak{gl}_{2|1}$ case.

5 Scalar product of Bethe vectors

We need a dual Bethe vector to define a scalar product of Bethe vectors. The dual Bethe vector belongs to dual physical space $\mathcal{H}^*$. We suppose that the dual physical space $\mathcal{H}^*$ contains a dual vacuum $\langle 0|$ (such that $\langle 0|0\rangle=1$) with properties

$$\begin{aligned} &\langle 0|T_{ij}(u)=0, \qquad \text{with } \ i<j\\ &\langle 0|T_{ii}(u)=\lambda_i(u)\langle 0|, \end{aligned} \tag{31}$$

where functions λ_i are the same as in (9). Then the dual Bethe vector $\mathbb{C}(\bar{u},\bar{v})$ can be obtained from Bethe vector $\mathbb{B}(\bar{u},\bar{v})$ using antimorphism Ψ defined by

$$\begin{aligned} \Psi(AB)&=(-1)^{[A][B]}\Psi(B)\Psi(A),\\ \Psi(T_{ij}(u))&=(-1)^{[i][j]+[i]}T_{ji}(u),\\ \Psi(|0\rangle)&=\langle 0|. \end{aligned} \tag{32}$$

The dual Bethe vector is

$$\mathbb{C}(\bar{u},\bar{v})=\Psi(\mathbb{B}(\bar{u},\bar{v})). \tag{33}$$

Now we can define the scalar product of the Bethe vectors

$$S(\bar{u}^C,\bar{v}^C|\bar{u}^B,\bar{v}^B)=\mathbb{C}(\bar{u}^C,\bar{v}^C)\mathbb{B}(\bar{u}^B,\bar{v}^B). \tag{34}$$

Applying antimorphism Ψ one can prove that scalar product is symmetric $S(\bar{u}^C,\bar{v}^C|\bar{u}^B,\bar{v}^B) = S(\bar{u}^B,\bar{v}^B|\bar{u}^C,\bar{v}^C)$.

In our work we prove that the scalar product of Bethe vectors has the following form:

$$S(\bar{u}^C,\bar{v}^C|\bar{u}^B,\bar{v}^B) = \sum r_1(\bar{u}^C_{\rm II})r_1(\bar{u}^B_{\rm I})r_3(\bar{v}^C_{\rm II})r_3(\bar{v}^B_{\rm I})f(\bar{u}^C_{\rm I},\bar{u}^C_{\rm II})f(\bar{u}^B_{\rm II},\bar{u}^B_{\rm I})g(\bar{v}^C_{\rm I},\bar{v}^C_{\rm II})$$
$$g(\bar{v}^B_{\rm II},\bar{v}^B_{\rm I})\frac{f(\bar{v}^C_{\rm I},\bar{u}^C_{\rm I})f(\bar{v}^B_{\rm II},\bar{u}^B_{\rm II})}{f(\bar{v}^C,\bar{u}^C)f(\bar{v}^B,\bar{u}^B)}\; Z_{a-k,n}(\bar{u}^C_{\rm II},\bar{u}^B_{\rm II}|\bar{v}^C_{\rm I},\bar{v}^B_{\rm I})\; Z_{k,b-n}(\bar{u}^B_{\rm I},\bar{u}^C_{\rm I}|\bar{v}^B_{\rm II},\bar{v}^C_{\rm II}), \tag{35}$$

where $a = \#\bar{u}^C = \#\bar{u}^B$, $b = \#\bar{v}^C = \#\bar{v}^B$, $k = \#\bar{u}^C_{\rm I} = \#\bar{u}^B_{\rm I}$ and $n = \#\bar{v}^C_{\rm I} = \#\bar{v}^B_{\rm I}$. Here the sum runs over all the partitions $\bar{u}^C \Rightarrow \{\bar{u}^C_{\rm I},\bar{u}^C_{\rm II}\}$, $\bar{u}^B \Rightarrow \{\bar{u}^B_{\rm I},\bar{u}^B_{\rm II}\}$, $\bar{v}^C \Rightarrow \{\bar{v}^C_{\rm I},\bar{v}^C_{\rm II}\}$ and $\bar{v}^B \Rightarrow \{\bar{v}^B_{\rm I},\bar{v}^B_{\rm II}\}$ with $\#\bar{u}^C_{\rm I} = \#\bar{u}^B_{\rm I}$ and $\#\bar{v}^C_{\rm I} = \#\bar{v}^B_{\rm I}$.

Here we use notation

$$r_1(z) = \frac{\lambda_1(z)}{\lambda_2(z)}, \quad r_3(z) = \frac{\lambda_3(z)}{\lambda_2(z)}. \tag{36}$$

The function $Z(\bar{s}|\bar{t})$ is called the highest coefficient. We obtaine an expression for $Z_{a,b}$ as the determinant of an $(a+b)\times(a+b)$ matrix

$$Z_{a,b}(\bar{t};\bar{x}|\bar{s};\bar{y}) = h(\bar{w},\bar{t})\Delta_{a+b}(\bar{w})\Delta'_a(\bar{t})\Delta'_b(\bar{y})\;\det_{a+b}\mathcal{J}_{jk}, \tag{37}$$

where $\Delta'_n(\bar{u})$ and $\Delta_n(\bar{v})$ are defined by

$$\Delta'_n(\bar{u}) = \prod_{j<k}^{n} g(u_j,u_k), \qquad \Delta_n(\bar{v}) = \prod_{j>k}^{n} g(v_j,v_k). \tag{38}$$

where $\bar{w} = \{\bar{x},\bar{s}\}$ and the matrix $\mathcal{J}_{jk}$ is defined by

$$\begin{aligned} &\mathcal{J}_{jk} = \frac{g(w_j,t_k)}{h(w_j,t_k)}, && k=1,\dots,a; \\ &\mathcal{J}_{j,k+a} = g(w_j,y_k)\frac{h(w_j,\bar{x})}{h(w_j,\bar{t})}, && k=1,\dots,b; \end{aligned} \qquad j=1,\dots,a+b. \tag{39}$$

Similar highest coefficients were obtained in the $\mathfrak{gl}_2$ case [14] explicitly in the determinant form and in the $\mathfrak{gl}_3$ case [19] as sum.

Also the sum formula for the scalar product was obtained in the $\mathfrak{gl}_2$ case [17] and in the $\mathfrak{gl}_3$ case [9].

6 Norm of eigenvector

We prove that a norm of eigenvector of transfer matrix (7) has a determinant form. For this one should consider the limit $u_j^B \to u_j^C = u_j$, $v_j^C \to v_j^B = v_j$ in scalar product. Then the result has the form

$$\|\mathbb{B}_{a,b}(\bar{u};\bar{v})\|^2 = (-1)^{a+b} \prod_{j=1}^{b}\prod_{k=1}^{a} f(v_j,u_k) \prod_{\substack{j,k=1\\ j\neq k}}^{a} f(u_j,u_k) \prod_{\substack{j,k=1\\ j\neq k}}^{b} g(v_j,v_k) \det_{a+b} \widehat{\mathcal{N}}. \tag{40}$$

Here $\widehat{\mathcal{N}}$ is an $(a+b)\times(a+b)$ block-matrix. The left-upper block is

$$\widehat{\mathcal{N}}_{jk} = \delta_{jk}\left[c\,\frac{r_1'(u_k)}{r_1(u_k)} + \sum_{\ell=1}^{a}\frac{2c^2}{u_{k\ell}^2-c^2} - \sum_{m=1}^{b} t(v_m,u_k)\right] - \frac{2c^2}{u_{kj}^2-c^2}, \quad j,k=1,\dots,a, \tag{41}$$

where $u_{kj} = u_k - u_j$ and $r_1'(u_k)$ means the derivative of the function $r_1(u)$ at the point $u = u_k$. Also we use notation

$$t(x,y) = \frac{c^2}{(x-y)(x-y+c)}. \tag{42}$$

The right-lower block is diagonal

$$\widehat{\mathcal{N}}_{j+a,k+a} = \delta_{jk}\left[c\,\frac{r_3'(v_k)}{r_3(v_k)} + \sum_{\ell=1}^{a} t(v_k,u_\ell)\right], \quad j,k=1,\dots,b, \tag{43}$$

where $r_3'(v_k)$ means the derivative of the function $r_3(v)$ at the point $v = v_k$. The antidiagonal blocks are

$$\widehat{\mathcal{N}}_{j,k+a} = t(v_k,u_j), \quad j=1,\dots,a, \quad k=1,\dots,b, \tag{44}$$

and

$$\widehat{\mathcal{N}}_{j+a,k} = -t(v_j,u_k), \quad j=1,\dots,b, \quad k=1,\dots,a. \tag{45}$$

It is easy to relate the determinant of the matrix $\widehat{\mathcal{N}}$ with the Jacobian of the Bethe equations. Namely, let

$$\begin{aligned} \Phi_j &= \log\Big(\frac{r_1(u_j)}{f(\bar{v},u_j)}\prod_{\substack{k=1\\ k\neq j}}^{a}\frac{f(u_k,u_j)}{f(u_j,u_k)}\Big), \qquad j=1,\dots,a,\\ \Phi_{a+j} &= \log\left(\frac{r_3(v_j)}{f(v_j,\bar{u})}\right), \qquad j=1,\dots,b. \end{aligned} \tag{46}$$

Then the Bethe equations for the sets $\bar{u}$ and $\bar{v}$ take the form

$$\Phi_j = 2\pi i n_j, \qquad j = 1, \ldots, a+b, \tag{47}$$

where n_j are integer numbers. A straightforward calculation shows that

$$\begin{aligned} &\widehat{\mathcal{N}}_{j,k} = c\,\frac{\partial \Phi_j}{\partial u_k} \qquad && k = 1, \ldots, a, \\ &\widehat{\mathcal{N}}_{j,a+k} = c\,\frac{\partial \Phi_j}{\partial v_k}, \qquad && k = 1, \ldots, b, \end{aligned} \qquad j = 1, ..., a+b. \tag{48}$$

This statement generalizes Gaudin formula in $\mathfrak{gl}_2$ case [26] and Reshetikhin result in $\mathfrak{gl}_3$ case [9].

The results of this dissertation are published in four articles:

- A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Multiple Actions of the Monodromy Matrix in $gl(2|1)$-Invariant Integrable Models*, SIGMA 12 (2016), 099
- A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Scalar products of Bethe vectors in models with $gl(2|1)$ symmetry 1. Super-analog of Reshetikhin formula*, J. Phys. A49 (2016) 454005
- A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Scalar products of Bethe vectors in models with $gl(2|1)$ symmetry 2. Determinant representation*, J. Phys. A50 (2017) 034004
- A. Hutsalyuk, A. Liashyk, S. Pakuliak, E. Ragoucy, N. Slavnov, *Form factors of the monodromy matrix entries in $gl(2|1)$-invariant integrable models*, Nucl. Phys. B911 (2016) 902

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