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h-Principle and maps with prescribed singularities

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Introduction

In our work we extend Y. Eliashberg's theorem on maps with fold type singularities to arbitrary Thom-Boardman singularities. Namely, we state a necessary and sufficient condition for a continuous map of smooth manifolds of the same dimension to be homotopic to a generic map with a prescribed Thom-Boardman singularity Σ^I at each point. In dimensions 2 and 3 we rephrase this condition in terms of the homology classes of the given singular loci and the characteristic classes of the manifolds.

All manifolds are assumed to be infinitely smooth and without boundary. Maps between manifolds are assumed to be infinitely smooth unless we explicitly specify otherwise. We fix two manifolds M and N of the same dimension $n > 1$.

1 Thom-Boadman singularities and generic maps

First we recall the Thom-Boardman classification of singularities. Take a map $f : M \to N$. For a sequence of integers $I = i_1, i_2, \dots, i_r$ such that $n \geq i_1 \geq i_2 \geq \ldots \geq i_r \geq 0$ there is a subset $\Sigma^I(f)$ of the set of the critical points of f. One can define $\Sigma^I(f)$ inductively:

$$\begin{aligned}
&\Sigma^{i_1}(f) = \{x \in M \mid \dim(\ker df(x)) = i_1\},\\
&\Sigma^{i_1,i_2}(f) = \{x \in M \mid \dim(\ker df|_{\Sigma^{i_1}(f)}(x)) = i_2\},\\
&\dots\\
&\Sigma^{I}(f) = \{x \in M \mid \dim(\ker df|_{\Sigma^{i_1,\dots,i_{r-1}}(f)}(x)) = i_r\}.
\end{aligned}$$

In this definition it is necessary to assume that the set $\Sigma^{i_1,\dots,i_k}(f) \subset M$ obtained on the k-th step is a submanifold. If this is the case, we can always do the next step.

Also, the set $\Sigma^I(f)$ can be defined as the preimage of a certain submanifold $\Sigma^I(M,N) \subset J^r(M,N)$ of the r-jet space. If the jet extension $j^r(f)$ is transversal to $\Sigma^I(M,N)$, then both definitions coincide (the details can be found in [5], see also [3, §2]). For dimension reasons, we can set $r = n+1$.

For different sequences I of length r the Thom-Boardman submanifolds Σ^I do not intersect. Denote the set of critical r-jets of maps $M \to N$ by $\Sigma(M,N)$. It is well known that the Thom-Boardman decomposition $\Sigma(M,N) = \bigsqcup_{I\neq 0,0,\dots} \Sigma^I(M,N)$ is not a stratification [5, p. 48]. However, since the intersections of $\Sigma^I(M,N) \subset J^r(M,N)$ with fibers of $J^r(M,N) \to M \times N$ are (locally closed) algebraic subsets, one can choose a stratification of $\Sigma(M,N)$ which is a subdivision of the Thom-Boardman decomposition (see [8], [17])

If the jet extension $j^r(f) : M \to J^r(M,N)$ of a map $f : M \to N$ is transversal to all $\Sigma^I(M,N)$, then we call f a *Thom-Boardman map*, and if $j^r(f)$ is transversal to all strata of $\Sigma(M,N)$, then we say that f is *generic*. Clearly, every generic map is Thom-Boardman.

If M is compact, then by the Thom transversality theorem generic maps $M \to N$ form a dense open subset in the set of all C^∞-maps $M \to N$ equipped with the Whitney topology. The set of Thom-Boardman maps is always dense in this space, but it is not open for $n > 3$ (see e. g. [18]).

For a generic map f the set of its critical points $\Sigma(f) = j^r(f)^{-1}(\Sigma(M,N))$ is a stratified subset of M. Every stratum $C \subset \Sigma(f)$ belongs to a certain $\Sigma^I(f)$, and for dimension reasons the sequence I has a zero at the end. Therefore the restriction $f|_C$ is an immersion $C \to N$.

2 Collections of compatible germs

We study the following problem:

Question. *When does there exist a generic map $M \to N$ that has prescribed singularities?*

We will now define what it means for a map to to "prescribed singularities". Recall that if n is sufficiently large, then the Thom-Boardman singularities Σ^I are in general not stable, see e. g. [3, §3.7]. In particular, for a fixed I the germs of a given generic map f at two points $x, x' \in \Sigma^I(f)$ need not be equivalent. Therefore we need to specify the germ of a generic map at each critical point.

More specifically, we fix a closed nonempty subset $S \subset M$. Suppose we are given a collection of open subsets $U_i \subset M$ such that $S \subset \bigcup U_i$, and a collection of n-dimensional manifolds V_i (here $i = 1, 2, \ldots$ is a fixed, possibly infinite set of indices). Suppose we have a collection of generic maps $\varphi_i : U_i \to V_i$ such that $\bigcup \Sigma(\varphi_i) = S$.

The collection $\{\varphi_i\}$ is called *locally compatible* if for every i, j the germs of φ_i and φ_j at every point $x \in U_i \cap U_j$ are L-equivalent (this means that there are neighborhoods $x \in U \subset (U_i \cap U_j)$ and $\varphi_i(U) \subset V \subset V_i$, such that for some imbedding $\beta : V \to V_j$ we have $\beta \circ \varphi_i|_U = \varphi_j|_U$).

Since the maps φ_i are generic, the stratification of S is well defined, therefore it will always be a stratified subset of M.

Fix a locally compatible collection of maps $\{\varphi_i\}$ as above. We will say that a map $f' : M \to N$ has *prescribed singularities*, if $\Sigma(f') = S$ and for every i the germs of f' and φ_i at every point $x \in U_i$ are L-equivalent.

3 Twisted tangent bundle

We construct the following rank n vector bundle $T^{\{\varphi_i\}}M$ over M, called *the twisted tangent bundle*.

First, let $U_0 = V_0 = M \setminus S$ and $\varphi_0 = \mathrm{Id} : U_0 \to V_0$. Then if $\{\varphi_1, \varphi_2, \ldots\}$ is a compatible collection, so will be $\{\varphi_0, \varphi_1, \varphi_2, \ldots\}$.

For every $i = 0, 1, \ldots$ take the vector bundle $E_i \to U_i$ to be $\varphi_i^*(TV_i)$. Since the collection $\{\varphi_i\}$ is locally compatible, for every i, j and every point $x \in U_i \cap U_j$ we have an isomorphism of the fibers $d\beta : E_i|_x \to E_j|_x$, where β set the L-equivalence of the germs of φ_i and φ_j at x. This isomorphism does not depend on β, therefore we have a global isomorphism $\Psi_{i,j} : E_i|_{U_i \cap U_j} \to E_j|_{U_i \cap U_j}$.

The vector bundle $T^{\{\varphi_i\}}M$ is obtained by gluing the vector bundles E_i using $\Psi_{i,j}$, $i, j = 0, 1, \ldots$. We prove that the gluing is well defined.

4 The main theorem

The following theorem generalizes Y. Eliashberg's h-principle for maps with fold type singularities [6]. This theorem is the main result of our work ([14], [15]).

Theorem 1. *A continuous map $f : M \to N$ is homotopic to a generic map f' with prescribed singularities determined by $\{\varphi_i\}$ if and only if the vector bundles $T^{\{\varphi_i\}}M$ and $f^*(TN)$ are isomorphic.*

In order for the proof to work it is essential that S should be a stratified subset of M and for every stratum $C \subset S$ and every i the restriction $\varphi_i|_{C \cap U_i}$ should be an immersion. We also use the relative version of Y. Eliashberg's h-principle [6, Th. 2.2], so it is essential that there should be points of S at which φ_i have folds.

Let us compare Theorem 1 with results by Y. Ando on maps with prescribed singularities. In [2] he proved a condition for an arbitrary map between manifolds to be homotopic to a map that has Thom-Boardman singularities, which are *not worse* (lexicographically) then the previously given one. This condition is formulated in terms of jets.

The difference between Theorem 1 and Ando's result is that we require more data (namely, a given germ at each point instead of a jet section), and also we obtain a map with prescribed singularity *at each point*, while Ando does not directly control even the Thom-Boardman classes of singularities and their loci.

5 Applications in dimension 2

Suppose M, N are compact connected 2-manifolds without boundary. We now state a generalization of the theorem which was proved by Y. Eliashberg for oriented N, see [6, §4]. We deduce this generalization from Theorem 1 by computing the characteristic classes of the vector bundle $T^{\{\varphi_i\}}M$.

Generic maps $M \to N$ can have only fold $\Sigma^{1,0}$ and cusp $\Sigma^{1,1,0}$ singularities. This singularities are called *Morin* and they are stable [12], [16]. Also recall that for every I the set $\Sigma^I \subset J^r(M, N)$ is a submanifold ([5, §6]). Therefore if $f : M \to N$ is generic, then the subset $\Sigma^1(f) \subset M$ is a smooth closed 1-submanifold, and a subset $\Sigma^{1,1}(f) = \Sigma^{1,1,0}(f) \subset \Sigma^1(f)$ is discrete.

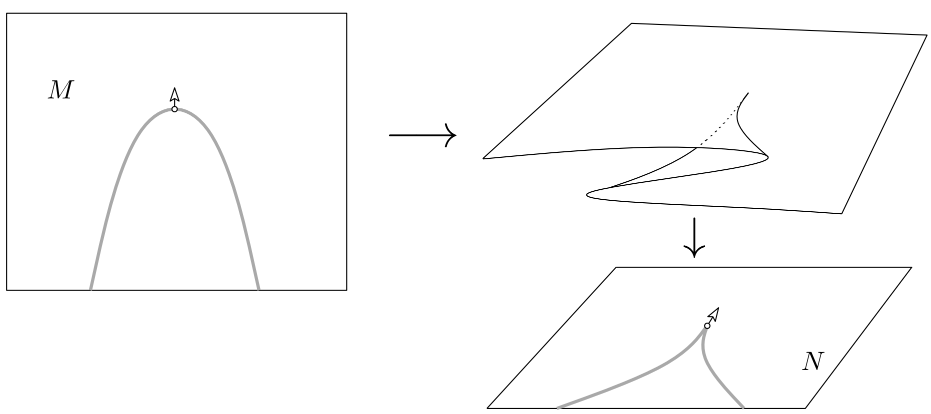

Figure 1: The characteristic vector of a cusp.

Take a closed nonempty 1-submanifold $C \subset M$ and a discrete subset $P \subset C$. At every point $p \in P$ we choose a unit vector normal to C called *the characteristic vector*. The choice of a characteristic vector defines a p-germ with a cusp singularity at p and folds at the points of C near p, up to equivalence (the image of the characteristic vector must be directed "out" of the cusp).

If $[C] = 0$, then C bounds two 2-submanifolds with boundary $M_+, M_- \subset M$. Denote the number of points of P whose characteristic vector is directed outside M_+, respectively outside M_-, by n_+, respectively n_-.

Theorem 2A. *A continuous map $f : M \to N$ is homotopic to a generic map with fold at each point of $C \setminus P$, with cusp at each point of P (which have prescribed characteristic vectors) and with no other critical points if and only if all of the following conditions hold:*

A-1. $[C] = w_1(M) + f^*w_1(N)$*;*

A-2. $[P] = w_2(M) + f^*w_2(N)$*;*

A-3. if $[C] = 0$*, then* $\left|\chi(M_+) - \chi(M_-) - n_+ + n_-\right| = |\deg f \cdot \chi(N)|$.

Here $\deg f$ is the degree of a map of possibly nonorientable manifolds. It can be defined as the degree of the induced map of the top cohomology groups with coefficients in the orientation sheafs. The lather map is well defined if $f^*w_1(N) = w_1(M)$. So if $[C] = 0$, then $\deg f$ is well defined as soon as condition A-1 holds.

We deduce from Theorem 2A the following fact, which answers the question from §2 for $n = 2$.

Theorem 2B. *There is a generic map $f : M \to N$ with folds at the points of $C \setminus P$, with cusps at the points of P (which have prescribed characteristic vectors) and with no other critical points if and only if all of the following conditions hold:*

B-1. $[P] = [C]^2$*;*

B-2. if $w_1(N) = 0$*, then* $[C] = w_1(M)$*;*

B-3. if $[C] = 0$*, then there is* $d \in \mathbb{Z}$ *such that* $\left|\chi(M_+) - \chi(M_-) - n_+ + n_-\right| = |d \cdot \chi(N)|$ *and either* $\chi(M) \leq |d| \cdot \chi(N)$*, or* $d = 0$*; if M is orientable and N is nonorientable, then d must be even;*

B-4. if $[C] \neq 0$, $w_1(M) \neq 0$ *and* $[C]^2 \neq w_2(M)$*, then* $w_2(N) \neq 0$ *and* $\chi(N) > \chi(M)$.

Our theorem generalizes results of [9], [10], [11] and [19] on the combinatorial description of the critical locus of a generic map. The proofs of Theorems 2B and 2A were published in [13].

6 Applications in dimension 3

Now suppose M, N are compact 3-manifolds without boundary. Then the analog of Theorem 2A is even more simple since rank 3 vector bundles over M are classified by their Stiefel-Whitney classes.

Generic maps $M \to N$ can only have folds, cusps and swallowtail points as singularities. These singularities are all stable and they form smooth submanifolds of M which have dimensions 2, 1 and 0 respectively.

Take a closed nonempty 2-submanifold $S \subset M$, a closed 1-submanifold $C \subset S$ and a discrete subset $P \subset C$.

Theorem 3. *A continuous map $f : M \to N$ is homotopic to a generic map f' such that $\Sigma^1(f') = S$, $\Sigma^{1,1}(f') = C$ and $\Sigma^{1,1,1}(f') = P$ if and only if all of the following conditions hold:*

- $[S] = w_1(M) + f^* w_1(N)$;
- $[C] = w_2(M) + w_1(M) \cdot f^* w_1(N)$;
- *for every component $C' \subset C$ we have* $[C'] \cdot [S] \equiv |P \cap C'| \mod 2$.

The third condition is needed in order to construct a characteristic vector field on $C \setminus P$ (the images of the vectors of this field will be directed "outside" the cusp). More precisely, this condition implies that there are exactly two such vector fields, and the generic map f' can be constructed for either of them.

Note that if the required map f' exists, then Theorem 3 implies that $[P] = [S] \cdot [C]$. This fact and the first two conditions of Theorem 3 are consistent with the computation of the Thom polynomials for Morin singularities (see e. g. [4, p. 204])

At present, we are unable to obtain the analog of Theorem 2B for 3-manifolds. However, Theorem 3 can be used to prove some facts about maps with prescribed singularities between given 3-manifolds.

Corollary 1. *Let M, N be homological spheres of dimension 3. Then in every homotopy class of maps $M \to N$ there is a generic map with the a given nonempty submanifold of folds and no other critical points.*

Corollary 2. *Let M, N be closed orientable 3-manifolds and let $S \subset M$ be a nonempty codimension 1 submanifold such that $[S] = 0$. Then in every homotopy class of maps $M \to N$ there is a generic map with folds at all points of S and no other critical points.*

Conclusion

In this work we study the question whether or not there exists a map of manifolds of the same dimension with prescribed generic singularities. We answer this question by proving a kind of h-principle (Theorem 1). It allows one to homotope an arbitrary map into a map with prescribed singularities when certain vector bundles are isomorphic. In dimensions 2 and 3 we rephrase the condition for these vector bundles to be isomorphic in terms of the characteristic classes of the manifolds (Theorems 2A and 3).

Let us mention several further questions.

Question. *Can one prove an analog of Theorem 1 for manifolds of different dimensions? Such a result will generalize the corresponding theorem from [7].*

Question. *Does Theorem 1 remain true if we replace local L-equivalence by local equivalence in the definition of a collection of compatible maps?*

Question. *Is there a combinatorial description of the twisted tangent bundle similar to the one we used in the proof of Theorems 2A and 3 for $n > 2$?*

Question. *How can one approach the problem of finding a map with prescribed critical values? How can one construct invariants of the sets of critical values (even in dimension* 2*, see e. g. [1], [20, §6])?*

Presentations

1. Talk "Maps of surfaces with prescribes critical loci" (Moscow, HSE, math department, seminar "Combinatorics of Vassiliev invariants", May 2018)

2. Talk "Eliashberg's h-principle for maps with Thom-Boardman singularities" (Moscow, HSE, math department and Steklov Mathematical Institute, Geometric topology seminar, February 2019)

3. Talk "Eliashberg's h-principle for maps with Thom-Boardman singularities II" (Moscow, HSE, math department and Steklov Mathematical Institute, Geometric topology seminar, February 2019)

4. Talk "Maps of 3-manifolds with prescribed Thom-Boardman singularities" (Moscow, HSE, math department and Steklov Mathematical Institute, Geometric topology seminar, April 2019)

5. Talk "Maps of 3-manifolds with prescribed Thom-Boardman singularities" (Moscow, HSE, math department and Steklov Mathematical Institute, Geometric topology seminar, May 2019)

6. Talk "Maps of 3-manifolds with prescribed Boardman singularities" (Moscow, HSE, math department and Steklov Mathematical Institute, Geometric topology seminar, March 2020)

7. Talk "h-Principle and maps with prescribed singularities" (Moscow, HSE, math department, seminar "Combinatorics of Vassiliev invariants", June 2020)

Publications

The results of the dissertation were published in two articles.

1. A. Ryabichev. *Eliashberg's h-principle and generic maps of surfaces with prescribed singular locus* // Topology and its Applications. 2020. No. 276. 16 P.

2. A. Ryabichev. *Maps with prescribed Boardman singularities (in Russian)* // Doklady Mathematics. 2020. Vol. 492, No. 1. 62–64.

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