National Research University Higher School of Economics

Faculty of Mathematics

as a manuscript

Yakov Kononov

Generalized Khovanov invariants of knots in rectangular representations

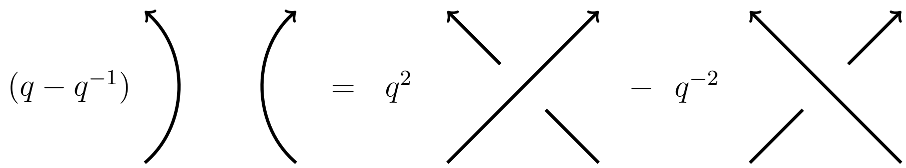
Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Phylosophy in Mathematics

Academic supervisor:
Boris Feigin
Doctor of Science, professor

Moscow-2020

Introduction

The first quantum invariant for knots and links was invented by Jones [17] in 1984. The invariant was constructed in terms of a two-dimensional projection of a knot using *skein relations* which had a representation-theoretic origin.



Later Reshetikhin and Turaev [18] generalized the construction to the case of knots in 3-manifolds with arbitrary groups and representations instead of the fundamental $sl(2)$. It was a challenging problem to give a three-dimensional contruction of the RT invariant. Witten [21] in 1989 figured out that the RT invariant has to do with the Chern-Simons theory [19]. CS is a gauge theory with the action

$$\mathcal{S} = \int AdA + \frac{2}{3}A^3,$$

where A is a connection in a principal G-bundle over a three-manifold for any compact Lie group G. Witten interpreted the skein relation as that certain conformal block has dimension two.

HOMFLY polynomials are defined physically as the averages of holonomies of the connection ("Wilson loops") in any given representation of G with the measure $e^{-\mathcal{S}}$. The most interesting example is when $G = SU(N)$. It turns out that the HOMFLY polynomial is indeed a *polynomial* in variables A and q where

$$q^2 = \exp\left(\frac{2\pi i}{g+N}\right), \ A = q^N, \ g = \text{the coupling constant.}$$

Since the invariants are polynomials with integer coefficients, a natural question was to interpret them as *graded characters* of some vector spaces associated to knots and links. Khovanov [6] in 1999 invented the so-called Khovanov homology, which provided categorifications of the Jones polynomial.

1 Calculation of colored HOMFLY polynomials

1.1 Sums over paths

By Artin's theorem, every knot can be representated as a closure of a braid. The simplest and straightforward way to compute colored HOMFLY polynomials of knots is to use the R-matrix approach [**?**].

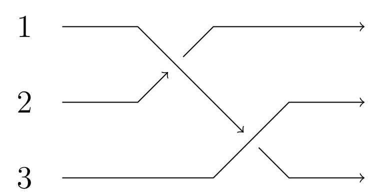


Figure 1: Knot represented as a closure of a braid. It corresponds to a product $R_{12}R_{23}$

We associate a representation λ to each strand, and their tensor products to unions of strands. The HOMFLY polynomial is equal to trace of a certain product of R-matrices $R_{i,i+1}$ for each crossing in subsequent strands. These R-matrices are simpler to describe when $\lambda = \square$ is the vector representation of $SL(N)$. The more general case of arbitrary representation can be reduced to the case of the vector representation. The tensor product has the following decomposition:

$$\square^{\otimes n} = \bigoplus_{|\lambda|=n} [\lambda]^{|SYT(\lambda)|}$$

where multiplicities are the numbers of standard Young tableaux with a given shape. Denote v_T the element of the basis corresponding to the tableau T. Let us describe the action of $R_{i,i+1}$ in this basis. Consider the tableau T' obtained by applying permutation $(i, i+1)$ to T. There are two cases: when it is a SYT, and when it is not.

1. If it is not a SYT, then v_T is an eigenvector for $R_{i,i+1}$:

$$R_{i,i+1}v_T = \begin{cases} qv_T, & i, i+1 \text{ are in the same row of } \lambda \\ -q^{-1}v_T, & i, i+1 \text{ are in the same column of } \lambda \end{cases}$$

2. If T' is a SYT, we have a 2 by two block:

$$R_{i,i+1} = \begin{pmatrix} -\frac{q^{-n}}{[n]} & \sqrt{1-\frac{1}{[n]^2}} \\ \sqrt{1-\frac{1}{[n]^2}} & \frac{q^n}{[n]} \end{pmatrix},$$

where n is the difference of contents of boxes i and $i+1$.

These formulas are q-deformations of the formulas for the action of the generators of the symmetric group in Young basis obtained by A.Okounkov and A.Vershik in [1]. In particular, the operators $R_{1,2}$ act always diagonally in this basis, since "1" cannot be permuted with any other box.

For arbitrary representation λ with n boxes, one can replace each strand by n strands, in accordance with standard fusion theory for R-matrices.

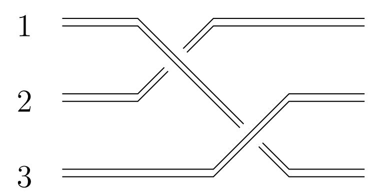


Figure 2: Doubling

1.2 Rosso-Jones formula

One of the simplest kinds of knots are torus knots $T(n,m)$. They are parametrized by pairs of positive relatively coprime integer numbers.

$$H_\lambda(T(n,m)) = \left(q^{\frac{2n}{m}\widehat{W}_2}\Psi_m s_\lambda\right)\Big|_{p_i=\frac{\{A^i\}}{\{q^i\}}},$$

where s_λ are schur polynomials,

$$\Psi_m: \; p_k \mapsto p_{mk}$$

is the Adams operation, and $\widehat{W}_2$ is the cut-and-join operator, acting diagonally in the basis of schur polynomials:

$$\widehat{W}_2 s_\lambda = \left(\sum_{\Box\in\lambda}(a'(\Box)-l'(\Box))\right)s_\lambda$$

1.3 DAHA superpolynomials

There is a one-parametric deformation of the Rosso-Jones formula [3]. Instead of symmetric function one has to consider K-theory of the Hilbert scheme of points in $\mathbb{C}^2$, which is identified with the Fock space by [15, 16].

$$\mathcal{P}^{\lambda}_{n,m} = \left\langle \bigwedge\nolimits^{\bullet}(A\mathscr{U}) \middle| P^{\lambda}_{n,m} \middle| \mathscr{O}_{\mathrm{vir}} \right\rangle_{K(\mathrm{Hilb}(\mathbb{C}^2, m|\lambda|))},$$

where $P^{\lambda}_{n,m}$ is the Macdonald polynomials written in terms of the heisenberg algebra of the slope n/m in the quantum group $U_{\hbar}(\widehat{\widehat{\mathfrak{gl}}}(1))$ acting on the K-theory of the Hilbert scheme.

2 Factorization properties

2.1 Limit when $q \to 1$

The central identity is the factorization property at $q = 1$:

$$H_{\lambda}(A, q = 1) = \Big(H_{\square}(A, q = 1)\Big)^{|\lambda|} \tag{1}$$

It has infinitely many generalizations. From the point of view of matrix models, it corresponds to the limit

$$q = e^{\hbar}, \quad \hbar \to 0, \quad N\hbar = const,$$

and has deep connections with Hurwitz theory. Note that for superpolynomials one has weaker factorization properties

$$P_{S^r}(q = 1, t, A) = \Big(P_{\square}(q = 1, t, A)\Big)^r$$

$$P_{\Lambda^r}(q = 1, t, A) = \Big(P_{\square}(q = 1, t, A)\Big)^r$$

2.2 Limit to other roots of unity

At other roots of unity, factorization properties are more interesting. When $q^{2m} = 1$, for symmetric representations we have the following property:

$$H_{S^{n+m}} = H_{S^n} \cdot H_{S^m}.$$

For more general representations, the conjecture is that

$$H_\mu = H_\lambda \cdot H_{S^m},$$

if μ is obtained by gluing a skew hook of length m to λ. In other words, μ/λ is a connected skew Young diagram of width 1. These are exact diagrams that appear in the right hand side of the Murnaghan-Nakayama product rule

$$p_m \cdot s_\lambda = \sum (-1)^{\mathrm{length}(\mu/\lambda)-1} s_\mu.$$

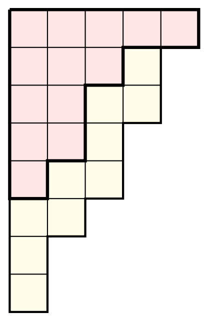


Figure 3: Example of factorization $H_{[5,4,4,3,3,2,1,1]} = H_{[5,3,2,2,1]} \cdot H_{[10]}$ when $q = \exp\left(\frac{\pi i}{10}\right)$

I expect that after proper generalization of these properties there can be a breakthrough in understanding A-polynomials – recurrent difference relations on colored knot polynomials.

2.3 Alexander polynomial

Alexander polynomial is defined as

$$\mathrm{Al}_\lambda(q) = H_\lambda(A=1, q).$$

For representations with one hook

$$\mathrm{Al}_\lambda(q) = \Big(\mathrm{Al}_\square(q)\Big)^{|\lambda|} \tag{2}$$

2.4 Kashaev polynomial

Kashaev polynomials are defined as the values of colored HOMFLY polynomials at primitive roots of unity:

$$K_\lambda(A) = H_\lambda\left(q = \exp\left(\frac{\pi i}{|\lambda|}\right), A\right).$$

When λ is a diagram with one hook,

$$K_\lambda(A) = K_\Box\left(A^{|\lambda|}\right).$$

Together with (2) this indicates some kind of A-q duality which is some nonlinear generalization of the level-rank duality.

3 Differential expansion and defect

For symmetric representations $[r]$ (corresponding to a Young diagram with one row of length r) HOMFLY polynomials possess the following expansion

$$H_{[r]} = 1 + \sum_{s=1}^{r} \begin{bmatrix} r \\ s \end{bmatrix} G_s(A,q)\{A/q\}\prod_{j=0}^{s-1}\{Aq^{r+j}\}.$$

This structure provides a q-deformation of the property (1):

$$H_\lambda\big|_{q=1} = \left(H_{[1]}\right)^{|\lambda|}\Big|_{q=1}.$$

This expansion follows from the following three properties of HOMFLY polynomials:

1. $$H_{\lambda^t}(A,q) = H_\lambda(A,-1/q),$$

2. When $A = q^{r+s}$,
$$H_{[1^r]} = H_{[1^s]}.$$

3. When $A = q^{-1}$
$$H_{[1^r]} = 1.$$

It turns out that for any given knot if G_s are divisible by some factors $\{Aq^k\}$, the same is true for G'_s with $s' > s$. Moreover, G_s are divisible by some factors $\{Aq^k\}$ in a very regular fashion which is described by a single integer number $\delta \geq 0$ for each knot. Namely, G_s is divisible by

$$\prod_{j=0}^{[\frac{s-1}{\delta}]} \{Aq^j\},$$

It turns out that the *defect* δ is a very simple quantity – it is equal to the degree of the Alexander polynomial

$$\delta = \deg_{q^2} H(A,q)\Big|_{A=1}.$$

What is interesting, there are knots with trivial Alexander polynomial, and for them all G_s are divisible by just $\{A\}$.

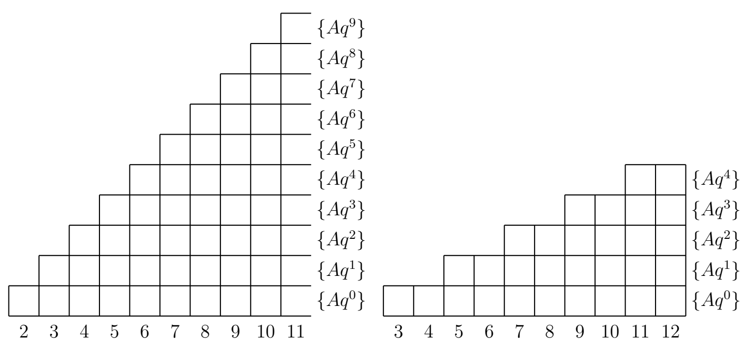


Figure 4: Divisibility of G_s for $\delta = 1$ on the left and $\delta = 2$ on the right.

A very interesting consequence of these properties is that for any given $p \geq 0$, if we substitute $A = q^{-p}$, then the sequence of coefficients in

$$H(A,q)\Big|_{A=q^{-p}}$$

stabilizes. To illustrate this phenomenon, consider the knot 6_2 whose Alexander polynomial is

$$Al(6_2) = -3 - q^{-4} + 3\,q^{-2} + 3\,q^2 - q^4.$$

H_1	1
H_2	$1-3\,q^2+q^4+5\,q^6-8\,q^8+3\,q^{10}+10\,q^{12}-10\,q^{14}-4\,q^{16}+9\,q^{18}-q^{20}-3\,q^{22}+q^{24}$
H_3	$1-3\,q^4-3\,q^6+4\,q^8+9\,q^{10}-2\,q^{12}-12\,q^{14}-6\,q^{16}+11\,q^{18}+14\,q^{20}-2\,q^{22}-12\,q^{24}-$ $-8\,q^{26}+4\,q^{28}+9\,q^{30}+2\,q^{32}-3\,q^{34}-3\,q^{36}+q^{40}$
H_4	$1-3\,q^6-3\,q^8+4\,q^{12}+9\,q^{14}+2\,q^{16}-4\,q^{18}-12\,q^{20}-8\,q^{22}+2\,q^{24}+11\,q^{26}+14\,q^{28}+$ $+2\,q^{30}-4\,q^{32}-12\,q^{34}-8\,q^{36}+4\,q^{40}+9\,q^{42}+2\,q^{44}-3\,q^{48}-3\,q^{50}+q^{56}$
H_5	$1-3\,q^8-3\,q^{10}+4\,q^{16}+9\,q^{18}+2\,q^{20}-4\,q^{24}-12\,q^{26}-8\,q^{28}+2\,q^{32}+11\,q^{34}+14\,q^{36}+$ $+2\,q^{38}-4\,q^{42}-12\,q^{44}-8\,q^{46}+4\,q^{52}+9\,q^{54}+2\,q^{56}-3\,q^{62}-3\,q^{64}+q^{72}$
H_6	$1-3\,q^{10}-3\,q^{12}+4\,q^{20}+9\,q^{22}+2\,q^{24}-4\,q^{30}-12\,q^{32}-8\,q^{34}+2\,q^{40}+11\,q^{42}+14\,q^{44}+$ $+2\,q^{46}-4\,q^{52}-12\,q^{54}-8\,q^{56}+4\,q^{64}+9\,q^{66}+2\,q^{68}-3\,q^{76}-3\,q^{78}+q^{88}$

Thus, if one specialize fixed $A = q^{-k}$, the coefficients of invariants in symmetric representations stabilize and do not have additional topological information.

4 Generalized Khovanov invariants of twist knots

4.1 Localization

Suppose X is a smooth projective variety with an action of an algebraic torus T, so that there is finite number of fixed points. Then the euler characteristic of an equivariant sheaf $\mathscr{F}$ can be computed as

$$\chi(X,\mathscr{F}) = \chi(X^T, \mathscr{F}\otimes N^\vee_{X/X^T}),$$

where X^T is the subvariety of fixed points, and $N^\vee_{X/X^T}$ is its normal bundle.

The most interesting example for us is the case of the Hilber scheme of points in $\mathbb{C}^2$. It is the Nakajima quiver variety corresponding to the Jordan quiver (figure 5).

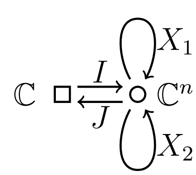


Figure 5: Jordan quiver

Explicitly, $\mathrm{Hilb}(\mathbb{C}^2, n)$ is defined as the set of ideals $\mathscr{I} \subset \mathbb{C}[x, y]$ of finite codimension n. There is a natural torus action

$$T: \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} t_1 & \\ & t_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Fixed points correspond to Young diagrams of size n. There is universal

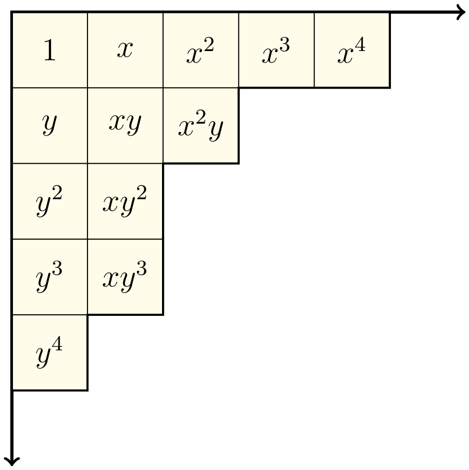
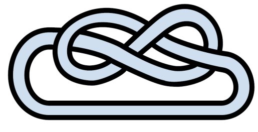


Figure 6: Fixed points in $\mathrm{Hilb}(\mathbb{C}^2)$. The diagram shown on the picture corresponds to the ideal generated by $x^5, x^3y, x^2y^2, xy^4, y^5$.

sheaf $\mathscr{U}$ with the fiber $\mathbb{C}[x, y]/cI$ over ideal $\mathscr{I}$. The tangent space to a fixed point has the following expression in terms of arms and legs:

$$\mathscr{T}_\lambda = \sum_{\square \in \lambda} \left(t_1^{a(\square)+1} t_2^{-l(\square)} + t_1^{-a(\square)} t_2^{l(\square)+1} \right).$$

4.2 Figure-eight knot



For the figure-eight knot 4_1 in rectangular representations the following formula was obtained in [12]:

$$\sum_\lambda \prod_{\square \in \lambda} \frac{\{xq^{a'-l'}\}\{yq^{l'-a'}\}}{\{q^{a+l+1}\}^2} \{Axq^{l'-a'}\}\{Ay^{-1}q^{l'-a'}\},$$

where

$$x = q^r, \quad y = q^s.$$

Because of the presense of $\{xq^{a'-l'}\}\{yq^{l'-a'}\}$ in the numerator, the sum is restricted to those partitions which fit inside the rectangle $r \times s$.

This formula has a clear meaning in the sense of enumerative geometry: it is the localization computation of the euler characteristic of the following canonical sheaf on $\mathrm{Hilb}(\mathbb{C}^2)$ with weights $t_1 = q^2, t_2 = q^{-2}$:

$$\bigwedge^{\diamond}\left(x\left(\mathscr{U} + A\mathscr{U}^{\vee}\right)\right) \otimes \bigwedge^{\diamond}\left(y^{-1}\left(\mathscr{U} + A^{-1}\mathscr{U}^{\vee}\right)\right) \otimes \mathscr{K}_{\mathrm{vir}}^{1/2}.$$

Here $\bigwedge^{\diamond}$ is a symmetrized version of the exterior algebra defined as

$$\bigwedge^{\diamond}(V) = \bigwedge^{\bullet}(V) \otimes \det(V)^{-1/2},$$

and $\mathscr{K}_{\mathrm{vir}} = \det(\mathscr{T}_{\mathrm{vir}})^{-1}$ is the virtual canonical sheaf.

Motivated by this geometry, we deform the construction and consider equivariant euler characteristic of the same sheaf on $\mathrm{Hilb}(\mathbb{C}^2)$ with weights $t_1 = q^2, t_2 = t^{-2}$.

For the trefoil knot, we need a slight modification of the sheaf

$$\left(-A^2 q/t\right)^{\chi} \otimes \det(\mathscr{U})^{-1} \otimes \bigwedge^{\diamond}\left(x\left(\mathscr{U} + A\mathscr{U}^{\vee}\right)\right) \otimes \bigwedge^{\diamond}\left(y^{-1}\left(\mathscr{U} + A^{-1}\mathscr{U}^{\vee}\right)\right) \otimes \mathscr{K}_{\mathrm{vir}}^{1/2}.$$

4.3 Other twist knots

Twist knots are parametrized by an integer number $m \in \mathbb{Z}$, such that $m = 1$ corresponds to the figure-eight knot, $m = -1$ to the trefoil, and $m = 0$ to the unknot. We obtained formulas for HOMFLY for arbitrary m for rectangular representations with 2 rows, and they are much more complicated and involve skew schur functions. Later our construction was reformulated generalized in [22, 23] for arbitrary rectangular diagrams.

4.4 Comparison of our approach to Generalized Khovanov invariants with others

These results opened a new page of research. It is the first example of construction of Generalized Khovanov invariants of a non torus-iterated knot in a large class of nontrivial representations. The only intersection of the families of twist knots and torus knots is the trefoil knot. One can compare two proposals for Generalized Khovanov invariants for the trefoil: our expression as for twist knot in terms of differential expansion and as for torus knot in terms of quantum group $U_\hbar(\widehat{\widehat{\mathfrak{gl}}}(1))$. In all computed examples they give the same answers. In both approaches there is appearance of the Hilbert scheme, but in totally different ways.

The results of the thesis are published in six papers:

1. Yakov Kononov, Alexei Morozov "On the defect and stability of differential expansion", Journal of Experimental and Theoretical Physics Letters, 2015, 101:12, 8
2. Yakov Kononov, Alexei Morozov "Factorization of colored knot polynomials at roots of unity", Physics Letters B, Volume 747, 30 July 2015, Pages 500-510
3. Yakov Kononov, Alexei Morozov "Colored HOMFLY and generalized Mandelbrot set", Journal of High Energy Physics, volume 2015, Article number: 151 (2015)
4. Yakov Kononov, Alexei Morozov "On factorization of generalized Macdonald polynomials", Eur. Phys. J. C (2016) 76:424
5. Yakov Kononov, Alexei Morozov "Rectangular superpolynomials for the figure-eight knot 4_1", Theoretical and Mathematical Physics, volume 193, pages 1630–1646(2017)
6. Yakov Kononov, Alexei Morozov "On rectangular HOMFLY for twist knots", Mod.Phys.Lett. A Vol. 31, No. 38 (2016) 1650223

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