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INVESTIGATING NONLOCALITY OF A MULTIPARTITE QUANTUM STATE ON THE BASIS OF THE LOCAL PROBABILISTIC MODEL

DISSERTATION SUMMARY

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1 General description of the thesis

Relevance of the research topic. The series of articles presented in the present thesis is devoted to the development of a new mathematical approach to the description of nonlocality of a multipartite quantum state of an arbitrary dimension and the consideration within this new approach of various open problems on quantum nonlocality.

Nonlocality of a quantum state – in the sense of violation by this state of Bell inequalities – is now one of the main resources for a wide class of quantum information technologies tasks and research in this field is performed in scientific centers all over the world. Entanglement of a quantum state is necessary but not sufficient for its nonlocality and there are entangled quantum states that do not violate any of Bell inequalities and are, therefore, fully local. Moreover, a larger violation of Bell inequalities can be attained on a quantum state with a low degree of entanglement, so that, for the estimation of a degree of nonlocality of a bipartite pure state, such its characteristic as the entanglement entropy is not suitable in principle.

Recently, due to their importance both from a theoretical point of view and for practical applications, various aspects of nonlocality of a quantum state have been intensively studied in the literature. For quantifying bipartite quantum nonlocality, several "measures" have been proposed, including: by a degree of violation by a bipartite quantum state of Bell inequalities; via the robustness of a quantum state nonlocality to a noise; by the "volume" of bipartite nonlocality. However, most results available in the literature concern nonlocality of specific quantum states, whereas even nonlocality of an arbitrary bipartite quantum state has not been yet fully mathematically studied and quantified.

This is even more true for studies on nonlocality of multipartite quantum states of arbitrary dimensions ($d \geq 2$) even in an N-qubit case ($d = 2$), though specifically N-qubit quantum nonlocality is currently the main resource for creating "unconditionally secure" quantum communications.

In view of noisiness of quantum communication channels in practical applications, it is also necessary to know the critical amount of noise not destroying nonlocality of a multipartite quantum state. However, the stability of nonlocality of an arbitrary multipartite quantum state to noise has been analytically studied in the literature also only for a bipartite case and only in the presence of white noise.

Elaboration degree. A general theory for quantifying nonlocality of an N-partite quantum state of any dimension and an arbitrary $N \geq 2$ and the stability of its nonlocality to a noise is currently in its embryonic form, and studies in this field are of interest not only from a theoretical point of view, but also for a wide class of applications based on quantum nonlocality.

The notion of quantum nonlocality goes back to the seminal paper[1] J. Bell where he analyzed mathematically the consequential conclusion of Einstein, Podolsky and Rosen in their famous article[2] of 1935 that locality of measurements by spartially separated observers of perfectly correlated quantum events points to a possibility of the description of a quantum system state via additional (hidden) variables which would allow to consider quantum randomness on the basis of a local classical probability model. By introducing his famous inequality for

[1] J. S. Bell. *On the Einstein Podolsky Rosen paradox* // Physics **1**, 195-200 (1964).

[2] A. Einstein, B. Podolsky and N. Rosen. *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?* // Phys. Rev. **47**, 777–780 (1935).

correlation functions, called now the original Bell inequality, Bell mathematically proved that the quantum probability description of a quantum correlation scenario where each of two participants, possibly spatially separated, observes perfectly correlated quantum events cannot be reproduced via the local classical probability model, that is, by random variables on a single probability space, each depending only on a setting of the corresponding measurement at the corresponding site. In the physical literature, this model for the probabilistic description of a correlation scenario is called a local hidden variable (LHV) model. Assuming however that, for the description of a quantum correlation scenario, the classical probability model is true while the quantum violation of the original Bell inequality is due to the violation under joint quantum measurements of the physical principle of locality, Bell introduced the notion of quantum nonlocality, manifesting, in his opinion, itself by the fact that each of random variables describing quantum measurements of participants on a single probability space can depend not only on parameters of a measurement described by this random variable but also on parameters of measurements of other participants. As a result, correlation functions and joint probabilities of a quantum correlation scenario can violate the special inequalities, called now Bell inequalities. For each correlation scenario admitting the description within a local hidden variable model these inequalities are satisfied.

To the present moment violation of Bell inequalities under quantum measurements has been confirmed in actual physical experiments and is widely used in practical applications for the discrimination between quantum and classical correlations. And though there is still no a unique conceptual view on the notion of quantum nonlocality conjectured by Bell, in quantum information theory, nonlocality of a multipartite quantum state is now defined purely mathematically – via violation by this state of a Bell inequality and is referred to as Bell nonlocality[3].

Note also that since nonlocality of a multipartite quantum state is determined via its violation of Bell inequalities, which are always true for correlation scenarios admitting a local classical probability model, from a mathematical point of view, the development of a new general mathematical formalism for the description of quantum nonlocality is closely related to the fundamental mathematical problem discussed in the literature ever since the famous works of von Neumann[4] and A. N. Kolmogorov[5] – namely, to the problem on a possibility or impossibility of the mathematical description of all or some class of quantum measurements in terms of a single probability space, that is, in the frame of the classical probability model.

Herewith, depending on conditions and aims of such a mathematical modelling, the following main types of hidden variable models are considered in quantum theory: (a) noncontextual /contextual – for modelling the probabilistic description of all joint von Neumann quantum measurements on an arbitrary Hilbert space and, respectively, (b) local/nonlocal – for modelling the probabilistic description of a correlation scenario on an N-partite quantum state.

From the Cohen–Specker theorem[6] it follows that for a Hilbert space of a dimension more

[3]See in Loubenets, *Bell's Nonlocality in a General Nonsignaling Case: Quantitatively and Conceptually* // Foundations of Physics **47**, 1100–1114 (2017).

[4]J. von Neumann. *Mathematische Grundlagen der Quantenmechanik* (Berlin, 1932).

[5]A. N. Kolmogorov. *Grundbegriffe der Wahrschein- lichkeitsrechnung, in Ergebnisse der Mathematik* (Berlin, 1933).

[6]S. Kochen and E. Specker. *The Problem of Hidden Variables in Quantum Mechanics* // J. Math. Mech. **17**, No. 1, 59 (1967).

than two, there is no a noncontextual hidden variable model where an injective mapping from the set of all quantum observables on a Hilbert space into a set of random variables on some measurable space $(\Omega, \mathcal{F}_\Omega)$ would satisfy the Cohen–Specker functional relation corresponding to some physically justified requirements.

At the same time, A.S. Holevo[7] proved that, for the probabilistic description of all von Neumann quantum measurements on a Hilbert space of an arbitrary dimension, there exists a contextual hidden variable model where the Cohen–Specker functional relation is satisfied and the mathematical expectations of quantum observables in quantum states can be represented as the mathematical expectations in the classical probabilistic model – by random variables and probability measures on a single measurable space $(\Omega, \mathcal{F}_\Omega)$.

Note that, under the probabilistic description of a quantum correlation scenario, the local (noncontextual) model of hidden parameters (LHV model) is formulated explicitly via the representation of all scenario joint probability distributions via conditional probability distributions on a single probability space, with each of these conditional distributions being "local" in the sense of its depending only on parameters of the corresponding quantum measurement at the corresponding site. This representation automatically takes into account all the necessary properties of the mean values and correlations of a scenario considered, so that the setting of a local hidden variable model for a quantum correlation scenario does not contain any additional functional relations. As a result, there are quantum correlation scenarios that admit a local hidden variable model.

Moreover, from the existence of a contextual hidden variable model for all joint von Neumann measurements on an arbitrary quantum state it follows the existence of a nonlocal hidden variable model for any quantum correlation scenario. However, within the latter mathematical model, the study of various aspects of quantum nonlocality is unpromising.

Since, under a probabilistic modelling of a correlation scenario on a specific entangled quantum state, existence or non-existence of a local hidden variable model depends not only on this entangled state but also on a number of quantum observables measured at each scenario site, the following questions arise:

- Does there exist for an arbitrary quantum scenario a local probability model describing the joint probabilities of this scenario in terms of local random variables on a measurable space and reducing to the local classical probability model only in a particular case?

- Does a local probability model of a new type exist for all local von Neumann measurements on an arbitrary multipartite quantum state?

- What are an analytical criterion for nonlocality of a multipartite quantum state and a parameter of a multipartite quantum state on the tensor product of Hilbert spaces specifying the a degree of a quantum state nonlocality?

- What analytical characteristic of a nonlocal quantum state determines stability of its nonlocality to a noise, and what is the analytical relation between this characteristic and the degree of a quantum state nonlocality?

[7] A. S. Holevo. *Covariant measurements and imprimitivity systems* // Lecture Notes in Math. 1055, 153-172 (1984); A. S. Holevo. *The statistical structure of quantum mechanics and hidden variables* (Moscow, Znanie, 1985).

The solution of these mathematical problems has resulted in the development in the presented dissertation of a new local probability model in terms of the general measure theory, the development within this model of a new mathematical techniques for the study of quantum nonlocality and the investigation via the developed formalism of various aspects of quantum nonlocality.

The most analytically studied cases of quantum nonlocality refer to the violation of such well-known Bell inequalities as the Clauser–Horne–Shimony–Holt (CHSH) inequality by a bipartite quantum state and the Mermin–Klyshko inequality by an N-qubit quantum state. It is also well known that the maximal violation by an arbitrary bipartite state of correlation bipartite Bell inequalities cannot exceed the real Grothendieck constant $K_G \in [1.676, 1.783]$, regardless of a dimension of a bipartite state, a number of quantum observables and a number of outcomes at each of two sites. But this restriction is not already true for quantum violation of bipartite Bell inequalities on joint probabilities.

In recent years, the maximal violation by an N-partite quantum state of general Bell inequalities has been intensively studied in the literature by many scientific groups on quantum information via various mathematical methods. The following main results have been derived.

For $N = 2, 3$, the maximal violation of general Bell inequalities by a bipartite quantum state and a three-partite quantum state was analyzed by the scientific groups of professors M. Junge, M. Wolf, D. Perez-Garcia, C. Pazaluelos, T. Vidick within the operator spaces formalism. Most of the analytical estimates obtained within this approach are determined up to unknown universal constants. Also, the approach based on the operator spaces formalism is technically very time-consuming and was not, therefore, applied to studying N-partite quantum nonlocality for $N > 3$.

For an arbitrary $N \geq 2$, the maximal violation of general Bell inequalities by a N-partite quantum state of an arbitrary dimension was also analyzed on the basis of a new mathematical approach developed by the author of this dissertation. The analytical bounds derived within this new approach are precise and expressed both in terms of the analytical characteristics of an N-partite quantum state on the tensor product of Hilbert spaces and in terms of "numerical" parameters $d_n, S_n, n = 1, 2, \ldots, N$, of a correlation scenario.

The stability of nonlocality of an arbitrary N-partite quantum state to a noise was previously studied by other authors only for a bipartite quantum state and only in case of a white noise.

Research goals and objectives. The aim of the present work was to develop a new general mathematical approach to investigating Bell nonlocality of a multipartite quantum state of an arbitrary dimension and the consideration within this new approach of the existing open problems which are important both from the theoretical point of view and for practical quantum applications based on quantum nonlocality. In order to achieve this aim, the following objectives were considered.

- To develop for the description of an arbitrary correlation scenario (with any number of observables and any spectral type of outcomes of each participant) of a general local probability model, formulated in terms of the *general measure theory* and reducing to the local classical probability model only in a particular case. To formulate and to prove criteria for the probabilistic description of an arbitrary correlation scenario within a new local probability model.

- To prove that the developed local probability model exactly reproduces the quantum formalism description of every quantum correlation scenario. To develop a new mathematical formalism for studying various aspects of multipartite quantum nonlocality within the new local probability model and to find via this formalism:

  (i) analytical criteria for nonlocality/locality of a multipartite quantum state, expressed in terms of characteristics of a quantum state on the tensor product of Hilbert spaces;

  (ii) exact analytical bound on the maximal violation by an arbitrary N-partite quantum state ρ of general Bell inequalities, expressed in terms of dimensions $d_n := \dim \mathcal{H}_n$, $n = 1, \ldots, N$ of Hilbert spaces and the "numerical" characteristics of a quantum correlation scenario – a number $N \geq 2$ of scenario participants ("sites") and a number $S_n \geq 1$ of measurements at each n-th "site".

- To develop a new approach to modelling the description of all joint von Neumann measurements on an arbitrary Hilbert space.

- To develop a new approach to describing the stability of nonlocality of an arbitrary N-partite quantum state to a local noise of any type. To find for an arbitrary nonlocal multipartite quantum state, as well as for specific nonlocal quantum states used in quantum technologies, the critical values of noise not destroying their nonlocality.

Research methodology. The development in the dissertation of a new approach to investigating multipartite quantum nonlocality is based on the application of fundamental concepts of general measure theory, the theory of linear operators, quantum measurement theory, quantum probability theory and quantum information theory.

Theoretical and application significance. The proposed general mathematical approach to investigating nonlocality of a multipartite quantum state is new and completely original (without coauthors). The high scientific potential of this approach resulted in the derivation of new important results on nonlocality of a multipartite quantum state, which have no analogues in the literature, in particular: the analytical upper bound on the maximal violation by an arbitrary N-partite quantum state of general Bell inequalities in terms of its Hilbert space characteristics; the analytical sufficient conditions for the full locality and the partial locality of an N-partite quantum state of an arbitrary dimension and for any $N \geq 2$; the precise upper bounds on the maximal violation of general Bell inequalities by an N-partite state for $N \geq 2, d_n \geq 2$ – both for projective and generalized quantum measurements; the proof of the attainability of some of these bounds; the development of new models for the probabilistic description of all joint von Neumann measurements on an arbitrary Hilbert space.

Due to noisiness of quantum information transmission channels in practical applications, the new author results on nonlocality stability of an N-partite quantum state of an arbitrary dimension are of the particular interest. The new approach developed by the author for investigating the nonlocality stability of a multipartite quantum state to noise resulted in finding the new precise analytical relation between the tolerance of a quantum state to any local noise and the maximal violation by this state of general Bell inequalities, also, the derivation via this relation of new analytical bounds on the maximal amount of admissible noise not violating nonlocality of a quantum state.

Note that, for $N > 2$, the bounds on the critical noise values for nonlocality stability of an N-partite state available in the literature and derived by other authors, are based only on the study of the full separability of its noisy state and only in the presence of white noise.

All the new results derived by the author in the presented series of articles are important for a wide range of quantum technologies problems based on quantum nonlocality.

Provisions for the defense. The following new results are presented for the defense:

1. Development [1, 4, 5] of a general approach to the formalization and modelling of the probabilistic description of an arbitrary N-partite correlation scenario ($N \geq 2$) with any number of measurements and any spectral type of outcomes on each of N "sites". Development via this general approach of *a new local probability model* for the description of an arbitrary correlation scenario – the local quasi hidden variable (LqHV) model reducing to the local classical probability (LHV) model only in a particular case. Formulating and proving general criteria for the probabilistic description of an arbitrary correlation scenario within a new local probability model (LqHV model). Proof of the existence of an LqHV model for any quantum correlation scenario. Constructing quantum LqHV models.

2. Constructing [2] a single analytical representation for all general Bell inequalities and its specifications for all correlation Bell inequalities and all Bell inequalities for joint probabilities. Proving that the form of every correlation Bell inequality does not depend on a spectral type of outcomes, in particular, on a number of outcomes, observed by each participant, but is determined only by extreme values of measurement outcomes.

3. Finding [3] a new class of bipartite quantum states of an arbitrary dimension satisfying the original Bell inequality for any three quantum observables without obeying the Bell condition on perfect correlations. Proving that, for a dimension $d \geq 3$, this class of states includes all two-qudit Werner states, separable and non-separable.

4. Development [4] for the description of quantum nonlocality within LqHV modelling of new mathematical notions on the tensor product of Hilbert spaces: source operator of an N-partite quantum state, tensor positivity, covering norm, with proof of their basic mathematical properties.

5. Development [4, 7, 8, 11] on the basis of the new local probability model and the new introduced notions on the tensor product of Hilbert spaces of a new mathematical formalism (LqHV formalism) for the description of multipartite quantum nonlocality and the derivation within this formalism of the following *new* results:

   5.1. Quantum analogues of Bell inequalities;

   5.2. The new analytical bound on the maximal violation by an N-partite quantum state of all general Bell inequalities, expressed in terms of a new characteristic of a N-partite state on the tensor product of Hilbert spaces and having no analogues in the literature;

5.3. The new sufficient analytical conditions for the full locality and the partial locality of an N-partite quantum state, which do not have analogues in the literature;

5.4. The new exact analytical bounds on for the maximal violation of the general Bell inequalities by an arbitrary N-partite quantum state, expressed in terms of the "numerical" parameters of a correlation scenario: a number N of scenario participants, S_n, d_n - a number of quantum measurements and a qudit dimension at each n-th "site";

5.5. Proof of the attainability of some of the new derived estimates for the maximal violation of general Bell inequalities.

6. Formulation and proof [6] of a generalization of the extension theorem for consistent probability measures to the case of consistent operator-valued measures. The proof due to this generalization of the existence for the probabilistic description of all joint von Neumann measurements on an arbitrary Hilbert space of two new qHV models: statistically noncontextual and context-invariant.

7. Introduction [8] for an N-partite quantum state of a new notion — the notion of a single LqHV model for all correlation scenarios performed on this state and with von Neumann measurements at each site, and the proof of the existence of such a model for every N-partite quantum state of an arbitrary dimension.

8. Introduction [9] of a new general concept "Bell nonlocality" and the specification of the mathematical description of Bell nonlocality under an arbitrary "nonsignaling" N-partite scenario.

9. Investigation [10] within the developed new formalism for the description of quantum nonlocality of the nonlocality stability of an N-partite quantum state to the white noise. Finding for the noisy N-qudit GHZ state and an arbitrary noisy N-qudit state the new bounds on the critical values of white noise for their full locality beyond their full separability.

10. Introduction [12] within the developed general formalism of a new approach to the description of the nonlocality stability of an arbitrary N-partite quantum state to any local noise. The derivation for an arbitrary N-partite quantum state of the exact new analytical relation between the tolerance of its nonlocality to any local noise and the maximal violation by this state of all general Bell inequalities. Finding, due to this relation, the new precise analytical bounds on the critical values of noise not violating nonlocality for: (i) an arbitrary nonlocal N-qudit state; (ii) an N-qubit GHZ state with $N \geq 2, d \geq 2$, in particular, an N-qubit GHZ state; (iii) N -qubit Dicke states. Specifying the asymptotics of the new bounds for $N \gg 1$ and $d \gg 1$.

Credibility, novelty and the personal contribution of the author. All the new results presented inthis dissertation: are rigorously mathematically proved within the corresponding theorems, statements and lemmas in the articles, presented for the defense; are completely original; derived by the author of the thesis without coauthors and are published in peer-reviewed international journals in mathematical physics, indexed in the scientific databases Web of Science and Scopus with quartiles Q1, Q2.

Approbation of the derived results. The main results of the thesis were reported at the following international conferences and seminars:

- Invited talk "*New upper bounds on the maximal violation of Bell inequalities by a multipartite quantum state*", International Conference "Quantum information, statistics, probability", Steklov Mathematical Institute of the Russian Academy of Sciences, September 2018, Moscow, Russia;

- Invited talk "*Amount of a local noise tolerated by a nonlocal N-qudit state*", International Conference "Towards Ultimate Quantum Theory, Linnaeus University, June 2018, Sweden.

- Invited talk *"On full locality of a noisy state for $N \geq 3$ nonlocally entangled qudits"*, International Conference "Foundations of Quantum Mechanics and Quantum Technologies", Linnaeus University, June 2017, Sweden.

- Invited talk "*Specifying nonlocality of an N-partite quantum state via its dilation characteristics*", International Conference "Quantum and Beyond", Linnaeus University, June 2016, Sweden.

- Invited talk "*On qHV modelling of quantum randomness and analytical upper bounds on quantum violations of Bell-type inequalities"*, International Conference "Quantum Theory: from Foundations to Technologies", Linnaeus University, June 2015, Sweden.

- Invited talk "*Local quasi hidden variable (LqHV) modelling and violations of Bell-type inequalities by a multipartite quantum state*", Seminar on Quantum Information, Institute of Quantum Information and Matter (IQIM), Caltech, July 2015, USA.

List of articles on the dissertation topic, presented for the defense. New results on modelling multipartite correlation scenarios and the study of quantum nonlocality, presented for the defense, were derived by the author without coauthors and are published in 12 articles in the leading international peer-reviewed scientific journals indexed in Web of Science and Scopus databases, among them 11 articles are published in journals with quartiles Q1, Q2.

1. Elena R. Loubenets. *On the probabilistic description of a multipartite correlation scenario with arbitrary numbers of settings and outcomes per site* // Journal of Physics A: Mathematical and Theoretical **41** (2008), 445303 (23pp)

   http://iopscience.iop.org/article/10.1088/1751-8113/41/44/445303

2. Elena R. Loubenets. *Multipartite Bell-type inequalities for arbitrary numbers of settings and outcomes per site* // Journal of Physics A: Mathematical and Theoretical **41** (2008), 445304 (18pp)

   http://iopscience.iop.org/article/10.1088/1751-8113/41/44/445304

3. Elena R. Loubenets. *Local hidden variable modelling, classicality, quantum separability, and the original Bell inequality* // Journal of Physics A: Mathematical and Theoretical **44** (2011) 035305 (16pp)

   https://iopscience.iop.org/article/10.1088/1751-8113/44/3/035305

4. Elena R. Loubenets *Loca quas hidden variable modelling and violations of Bell-type inequalities by a multipartite quantum state* // Journal of Mathematical Physics **53**, 022201 (2012)

   https://aip.scitation.org/doi/10.1063/1.3681905

5. Elena R. Loubenets. *Nonsignaling as the consistency condition for local quasi- classical probability modeling of a general multipartite correlation scenario* // Journal of Physics A: Mathematical and Theoretical **45** (2012), 185306 (10pp)

   http://iopscience.iop.org/article/10.1088/1751-8113/45/18/185306

6. Elena R. Loubenets. *Context-invariant quasi hidden variable (qHV) modelling of all joint von Neumann measurements for an arbitrary Hilbert space* // Journal of Mathematical Physics **56**, 032201 (2015)

   https://aip.scitation.org/doi/10.1063/1.4913864

7. Elena R. Loubenets. *Context-invariant and Local Quasi Hidden Variable (qHV) Modelling Versus Contextual and Nonlocal HV Modelling* // Foundations of Physics **45**, 840–850 (2015)

   https://link.springer.com/article/10.1007%2Fs10701-015-9903-8

8. Elena R. Loubenets. *On the existence of a local quasi hidden variable (LqHV) model for each N-qudit state and the maximal violation of Bell inequalities* // International Journal of Quantum Information 14 (2016), 1640010 (15pages)

   https://doi.org/10.1142/S0219749916400104

9. Elena R. Loubenets. *Bell's Nonlocality in a General Nonsignaling Case: Quantitatively and Conceptually* // Foundations of Physics **47**, 1100–1114 (2017)

   https://link.springer.com/article/10.1007%2Fs10701-017-0077-4

10. Elena R. Loubenets. *Full Bell locality of a noisy state for $N \geq 3$ nonlocally entangled qudits* // Journal of Physics A: Mathematical and Theoretical **50** (2017), 405305 (16pp)

    http://iopscience.iop.org/article/10.1088/17518121/aa84e8

11. Elena R. Loubenets. *New concise upper bounds on quantum violation of general multipartite Bell inequalities* // Journal of Mathematical Physics **58**, 052202 (2017)

    https://aip.scitation.org/doi/10.1063/1.4982961

12. Elena R. Loubenets. *Quantifying Tolerance of a Nonlocal Multi-Qudit State to Any Local Noise* // Entropy **20** (2018), 217 (13pp)

    http://www.mdpi.com/1099-4300/20/4/217

2 Summary of the Main Results

The main new results of the dissertation are sequentially presented below by seventeen (17) theorems and 16 (sixteen) propositions, rigorously mathematically proven by the author in the articles presented for the defense.

2.1 Modelling of the probabilistic description of an arbitrary correlation scenario

A general approach to the formalization and modelling of the probabilistic description of an arbitrary correlation scenario was developed by the author in articles [1, 4, 5]. Consider a scenario with N participants (players), possibly spatially separated from each other, where each n-th participant performs S_n different measurements, indexed by $s_n = 1, ..., S_n$ and with outcomes $\lambda_n \in \Lambda_n$. Denote by $\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}(\cdot)$ the probability distribution of outcomes $(\lambda_1, ..., \lambda_N) \in \Lambda_1 \times \cdots \times \Lambda_N$ under the joint measurement defined by an N-tuple $(s_1, ..., s_N)$ of measurement settings where each n-th participant performs a measurement s_n at his site. The complete probabilistic description of such a correlation scenario is given by the family

$$\begin{aligned} \mathcal{P}_{S,\Lambda} : &= \left\{ \mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)} \mid s_n = 1, ..., S_n, \quad n = 1, ..., N \right\}, \\ \Lambda &= \Lambda_1 \times \cdots \times \Lambda_N, \end{aligned} \tag{1}$$

of all $S = S_1 \times \cdots \times S_N$ joint probability measures specifying this scenario. The notation

$$\langle \phi(\lambda_1, ..., \lambda_N) \rangle_{\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}} := \int\limits_{\Lambda} \phi(\lambda_1, ..., \lambda_N)\ \mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}(\mathrm{d}\lambda_1 \times ... \times \mathrm{d}\lambda_N) \tag{2}$$

means the mathematical expectation of the value of a bounded measurable function $\phi : \Lambda \to \mathbb{R}$ under a joint measurement $(s_1, ..., s_N)$. Depending on a choice of a function ϕ and a type of outcome sets Λ_n, expression (2) can constitute both the probability of some event and the mathematical expectation (mean) of the product $\lambda_{n_1} \cdot ... \cdot \lambda_{n_M}$ of outcomes measured by $M \leq N$ participants if $\Lambda_n \subset \mathbb{R}$, $n = 1, ..., N$. In quantum information, expectation $\langle \lambda_{n_1} \cdot ... \cdot \lambda_{n_M} \rangle_{P^{(\Lambda)}_{(s_1,...,s_N)}}$ is called a correlation function or correlation, for short. If in (2) $M = N$, then the correlation function $\langle \lambda_1 \cdot ... \cdot \lambda_N \rangle_{P^{(\Lambda)}_{(s_1,...,s_N)}}$ is called full.

Remark 1 *In what follows, instead of the term "for each participant", we use, for short, the expression "at each site" accepted in quantum information.*

Proposition 1 (Lemma 1 in [1]). *For a correlation scenario with two outcomes $\Lambda_n = \{-1, 1\}$, $n = 1, ..., N$, per site, the joint probability $\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}(\lambda_1, ..., \lambda_N) := \mathrm{P}^{\{-1,1\}^N}_{(s_1,...,s_N)}(\{\lambda_1\} \times ... \times \{\lambda_N\})$ of outcomes $\lambda_n = \pm 1$ at all N sites and the correlation functions $\left\langle \lambda_{n_1} \cdot ... \cdot \lambda_{n_{N-k}} \right\rangle_{\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}}$, full and partial, satisfy the relation*

$$\begin{aligned} 2\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}(\lambda_1, ..., \lambda_N) &= 1 + \sum_{\substack{1 \leq n_1 < ... < n_{N-k} \leq N, \\ k=0,...,N-1}} \xi(\lambda_{n_1}) \cdot ... \cdot \xi(\lambda_{n_{N-k}}) \left\langle \lambda_{n_1} \cdot ... \cdot \lambda_{n_{N-k}} \right\rangle_{\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}}, \\ \lambda_n &= \pm 1, \quad \xi(\pm 1) = \pm 1. \end{aligned}$$

This relation is important for such an N-partite correlation scenario since it establishes the connection between Bell inequalities on joint probabilities and Bell inequalities on correlation functions.

Definition 1 (Definition 1 in [1]). *If, for all joint probability distributions of a correlation scenario (1) describing joint measurements* $(s_1, ..., s_N)$, $(s'_1, ..., s'_N)$ *with* $M < N$ *common settings* $s_{n_1}, ..., s_{n_M}$ *at arbitrary sites* $1 \leq n_1 < ... < n_M \leq N$, *the marginal probability distributions at* $n_1, \ldots, n_M$ *coincide:*

$$\begin{aligned} &\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_N) \\ &= \mathrm{P}^{(\Lambda)}_{(s'_1,..,s'_N)}(\Lambda_1 \times \cdots \times \Lambda_{n_1-1} \times d\lambda_{n_1} \times \Lambda_{n_1+1} \times \cdots \times \Lambda_{n_M-1} \times d\lambda_{n_M} \times \Lambda_{n_M+1} \times \cdots \times \Lambda_N), \end{aligned} \tag{3}$$

then this correlation scenario is called "nonsignaling".

In Section 3 of article [1], we discuss the relationship between "nonsignaling" and locality as formulated by Einstein–Podolsky–Rosen in their seminal 1935 paper (see footnote 2).

2.1.1 New local probabilistic model

For the probabilistic description of an *arbitrary* correlation scenario (1), we introduce in [4] the following new probabilistic model.

Definition 2 (Definition 5 in [4]). *An N-partite correlation scenario (1) admits a local quasi hidden variable (LqHV) model, if each of the probability distributions* $\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}$ *in (1) is represented*

$$\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}(d\lambda_1 \times \cdots \times d\lambda_N) = \int_\Omega \mathrm{P}_1^{(s_1)}(d\lambda_1|\omega) \cdot ... \cdot \mathrm{P}_N^{(s_N)}(d\lambda_N|\omega)\ \mu_{\mathcal{P}_{S,\Lambda}}(d\omega) \tag{4}$$

in terms of a measure space $(\Omega, \mathcal{F}_\Omega, \mu_{\mathcal{P}_{S,\Lambda}})$ *with a normalized bounded real-values measure* $\mu_{\mathcal{P}_{S,\Lambda}}$: $\mathcal{F}_\Omega \to R$, $\mu_{\mathcal{P}_{S,\Lambda}}(\Omega) = 1$ *and conditional probability distributions* $\mathrm{P}_n^{(s_n)}(\cdot|\omega)$, $\omega \in \Omega$, $n = 1, ..., N$, *each being "local" in the sense that it depends only on a setting of the corresponding* s_n*-th measurement at* n*-th "site".*

We stress that, in a general case, measure $\mu_{\mathcal{P}_{S,\Lambda}}$ in the LqHV representation (4) may depend on parameters of a correlation scenario, in particular, on measurement settings at all scenario "sites".

Definition 3 (Definition 2 in [5]). *LqHV model (4) is called deterministic if each of its conditional probability distributions in (4) has the form*

$$\mathrm{P}_n^{(s_n)}(F_n|\omega) := \chi_{f^{-1}_{n,s_n}(F_n)}(\omega), \quad F_n \in \mathcal{F}_{\Lambda_n}, \quad s_n = 1, ..., S_n, \quad n = 1, ..., N, \tag{5}$$

$\mu_{\mathcal{P}_{S,\Lambda}}$*-almost everywhere on* Ω, *where* $f_{n,s_n} : \Omega \to \Lambda_n$ *are random variables on* $(\Omega, \mathcal{F}_\Omega)$ *with values in* $(\Lambda_n, \mathcal{F}_{\Lambda_n})$, *each being "local" in the sense that it depends only a measurement setting* s_n *at* n*-th site and* $\chi_D(\omega) = 1$, $\omega \in D$, *and* $\chi_D(\omega) = 0$, $\omega \notin D$.

In a deterministic LqHV model (5), a real-valued measure $\mu_{\mu_{\mathcal{P}_{S,\Lambda}}}$ and random variables f_{n,s_n} are such that the values of measure $\mu_{\mathcal{P}_{S,\Lambda}}$ on sets $\bigcap\limits_{n=1,...,N} f_{n,s_n}^{-1}(F_n)$ are non-negative for all $F_n \in \mathcal{F}_{\Lambda_n}$, $s_n = 1, ... S_n$, and determine the joint probabilities

$$\mathrm{P}_{(s_1,...,s_N)}^{(\Lambda)}(F_1 \times \cdots \times F_N) = \mu_{\mathcal{P}_{S,\Lambda}}\left(f_{1,s_1}^{-1}(F_1) \cap \cdots \cap f_{N,s_N}^{-1}(F_N)\right) \geq 0 \tag{6}$$

of events $F_n \in \mathcal{F}_{\Lambda_n}$ observed at N sites. In addition, for arbitrary bounded measurable functions $\varphi_n : \Lambda_n \to \mathbb{R}$, $n = 1, ..., N$, the product expectation (2) takes the form

$$\langle \varphi_1(\lambda_1) \cdot ... \cdot \varphi_N(\lambda_N)\rangle_{P_{(s_1,...,s_N)}^{(\Lambda)}} = \int (\varphi_1 \circ f_{1,s_1})(\omega) \cdot ... \cdot (\varphi_N \circ f_{N,s_N})(\omega)\ \mu_{\mathcal{P}_{S,\Lambda}}(\mathrm{d}\omega), \tag{7}$$

which differs from the similar representation in a local classical probability model only by the fact that in a deterministic LqHV model (5) a normalized measure $\mu_{\mathcal{P}_{S,\Lambda}}$ is real-valued.

We have the following important statement.

Theorem 1 (Theorem 1 in [4]; Theorem 1 and Propositions 1, 2 of [5]). *For family (1) of joint probability distributions* $\mathrm{P}_{(s_1,...,s_N)}^{(\Lambda)}$, *describing an N-partite correlation scenario, the following statements are equivalent:*
(a) *there exists a LqHV model (4);*
(b) *there exists a deterministic LqHV model (5);*
(c) *correlation scenario is "nonsignaling";*
(d) *there exists a normalized real-valued joint measure*

$$\tau_{\mathcal{P}_{S,\Lambda}}\left(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(S_N)}\right), \tag{8}$$

$$\tau_{\mathcal{P}_{S,\Lambda}}\left(\Lambda_1^{S_1} \times \cdots \times \Lambda_N^{S_N}\right) = 1, \quad \lambda_n^{(s_n)} \in \Lambda_n, \quad s_n = 1, ..., S_n, \quad n = 1, ..., N,$$

returning all scenario probability distributions $\mathrm{P}_{(s_1,...,s_N)}^{(\Lambda)}$ *as the corresponding marginals.*

Note that the affine model considered in the physical literature for the probabilistic description of a correlation scenario is a special case of a LqHV model (4). Moreover, the rigorous proof of the existence of an affine model for a "nonsignaling" correlation scenario is presented in the literature by other authors only for a scenario with two measurement outcomes at each site.

Definition 4 [4]. *If, in representation (4), a normalized real-valued measure* $\nu_{\mathcal{P}_{S,\Lambda}}$ *is positive, then this LqHV model reduces to a general LHV model (local hidden variable model), formulated by us in [1].*

Unlike the original LHV model by Bell, in a general LHV model formulated in [1], the probability measure $\nu_{\mathcal{P}_{S,\Lambda}}$ may depend on all/some settings of scenario measurements. As we prove in a general case in [1], the existence for an arbitrary correlation scenario (1) of a general LHV model is equivalent to the existence for this scenario of a deterministic LHV model and a joint probability measure for which all scenario joint probability distributions constitute marginals. The "nonsignaling" condition (3), which is necessary and sufficient for

LqHV modeling of an arbitrary correlation scenario (see Theorem 1), is *necessary, but not sufficient* for its description within a LHV model.

Recall that on the vector space of bounded real-valued measures on a measurable space $(\Omega,\,\mathcal{F}_\Omega)$, the norm is given by the full variation $\|\nu\|_{var}$ of a measure $\nu(\cdot)$. If a measure $\nu(\cdot)$ is normalized $\nu(\Omega)=1$, then the norm $\|\nu\|_{var}\geq 1$ and $\|\nu\|_{var}=1$ iff $\nu(\cdot)$ is positive.

Let a correlation scenario be *"nonsignaling"* and described by a family $\mathcal{P}_{S,\Lambda}^{n\text{-}sig}$ of joint probability measures (1). Then, by Theorem 1, the probabilistic description of this scenario allows LqHV modelling, and in order to characterize a "nonsignaling" scenario, we introduce the following LqHV parameter [4]

$$\gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig}) := \inf_{\mu_{\mathcal{P}_{S,\Lambda}^{n\text{-}sig}}} \left\| \mu_{\mathcal{P}_{S,\Lambda}^{n\text{-}sig}} \right\|_{var} \geq 1, \tag{9}$$

where the infimum is taken over measures $\mu_{\mathcal{P}_{S,\Lambda}^{n\text{-}sig}}$ of all possible LqHV models (4) describing this *"nonsignaling"*scenario.

Proposition 2 (Lemma 2 in [4]) *"Nonsignaling" correlation scenario admits LHV modelling iff* $\gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})=1$.

We formulate and prove the following general theorem.

Theorem 2 (Theorem 2 in [1]). *The necessary and sufficient condition for the LHV modelling a bipartite correlation scenario* $\{\mathrm{P}_{(s_1,s_2)}^{(\Lambda)} \mid s_n=1,...,S_n,\ n=1,2,\ \Lambda=\Lambda_1\times\Lambda_2\}$ *is the existence of joint probability distributions*

$$\nu_{s_1}(\mathrm{d}\lambda_1^{(s_1)}\times \mathrm{d}\lambda_2^{(1)}\times\cdots\times \mathrm{d}\lambda_2^{(S_2)}),\ \ s_1=1,...,S_1,$$

such that, for each $\nu_{s_1}(\cdot)$, *all scenario probability distributions* $\mathrm{P}_{(s_1,s_2)}^{(\Lambda)}(\cdot)$, $s_2=1,...,S_2$, *constitute marginals and the relation*

$$\nu_{s_1}(\Lambda_1\times \mathrm{d}\lambda_2^{(1)}\times\cdots\times \mathrm{d}\lambda_2^{(S_2)}) = \nu_{s_1'}(\Lambda_1\times \mathrm{d}\lambda_2^{(1)}\times\cdots\times \mathrm{d}\lambda_2^{(S_2)})$$

holds for any $s_1,s_1'\in\{1,...,S_1\}$. *Similarly – the existence of joint probability distributions* $\nu_{s_2}(\mathrm{d}\lambda_1^{(1)}\times...\times \mathrm{d}\lambda_1^{(S_1)}\times \mathrm{d}\lambda_2^{(s_2)})$, $s_2=1,...,S_2$, *for which all distributions* $\mathrm{P}_{(s_1,s_2)}^{(\Lambda)}(\cdot)$, $s_1=1,...,S_1$, *are marginals and the condition*

$$\nu_{s_2}(\mathrm{d}\lambda_1^{(1)}\times\cdots\times \mathrm{d}\lambda_1^{(S_1)}\times \mathrm{d}\lambda_2^{(s_2)}) = \nu_{s_2'}(\mathrm{d}\lambda_1^{(1)}\times\cdots\times \mathrm{d}\lambda_1^{(S_1)}\times \mathrm{d}\lambda_2^{(s_2')})$$

is fulfilled for any $s_2,s_2'\in\{1,...,S_2\}$.

Note that the description of a correlation scenario within a *deterministic LqHV model (5)* where a normalized bounded real-valued measure $\mu_{\mathcal{P}_{S,\Lambda}}:=\mu$ is independent of measurement settings, is equivalent to its modelling via the local version of *the quasi-classical probability model, introduced by us for a general case in [5] and including the classical probabilistic model only as a special case.*

Within the *general quasi-classical probability model* [5] – equivalently, the qHV model determined by a space $(\Omega,\mathcal{F}_\Omega,\mu)$ with a normalized bounded real-valued measure μ, observables with values in a set Λ are described by such random variables $f : \Omega \to \Lambda$ on a measurable space $(\Omega, \mathcal{F}_\Omega)$, for which $\mu(f^{-1}(F)) \geq 0$ for all $F \in \mathcal{F}_\Lambda$, and the joint measurement of two such observables f_1, f_2 is possible if and only if $\nu(f_1^{-1}(F_1) \cap f_2^{-1}(F_2)) \geq 0$ for all events $F_n \in \mathcal{F}_\Lambda$.

In practical applications, for the specification of cases where a correlation scenario does not admit the description within the local classical probability model (LHV model), the violation by scenario mathematical expectations and joint probabilities of special inequalities, called as Bell inequalities after Bell is used.

2.1.2 General Bell inequalities

For an N-partite correlation scenario described by a family (1) of joint probability distributions $\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}$, consider a linear combination

$$\mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}) := \sum_{s_1,...,s_N} \langle \phi_{s_1,...,s_N}(\lambda_1, ..., \lambda_N) \rangle_{\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}} \tag{10}$$

of its mathematical expectations (2) of an arbitrary form, specified by a family

$$\begin{aligned} \Phi_{S,\Lambda} &= \{\phi_{s_1,...,s_N} : \Lambda \to \mathbb{R} \mid s_n = 1, ..., S_n; \quad n = 1, ..., N\}, \\ \Lambda &= \Lambda_1 \times \cdots \times \Lambda_N, \quad S = S_1 \times \cdots \times S_N, \end{aligned} \tag{11}$$

of bounded measurable functions on $(\Lambda, \mathcal{F}_\Lambda)$. We denote by $\mathfrak{G}^{lhv}_{S,\Lambda}$ the set of all families (1) of joint probability distributions describing N-partite correlation scenarios with parameters S, Λ, and admitting the LHV modelling.

Theorem 3 (Theorem 1 in [2]). *If an $S_1 \times \ldots \times S_N$-setting correlation scenario (1) admits the LHV modelling, then each linear combination (10) of its mathematical expectations (2) of a general form satisfies the "tight" LHV constraints*

$$\begin{aligned} \mathcal{B}^{\inf}_{\Phi_{S,\Lambda}} &\leq \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda})\big|_{lhv} \leq \mathcal{B}^{\sup}_{\Phi_{S,\Lambda}}, \\ \left| \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda})\big|_{lhv} \right. &\leq \mathcal{B}^{lhv}_{\Phi_{S,\Lambda}} := \max\left\{ \left|\mathcal{B}^{\sup}_{\Phi_{S,\Lambda}}\right|, \left|\mathcal{B}^{\inf}_{\Phi_{S,\Lambda}}\right| \right\}, \end{aligned} \tag{12}$$

where

$$\mathcal{B}^{\sup}_{\Phi_{S,\Lambda}} := \sup_{\mathcal{P}_{S,\Lambda} \in \mathfrak{G}^{lhv}_{S,\Lambda}} \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}) = \sup_{\lambda_n^{(s_n)} \in \Lambda_n, \forall s_n, \forall n} \sum_{s_1,...,s_N} \phi_{s_1,...,s_N}(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}), \tag{13}$$

$$\mathcal{B}^{\inf}_{\Phi_{S,\Lambda}} := \inf_{\mathcal{P}_{S,\Lambda} \in \mathfrak{G}^{lhv}_{S,\Lambda}} \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}) = \inf_{\lambda_n^{(s_n)} \in \Lambda_n, \forall s_n, \forall n} \sum_{s_1,...,s_N} \phi_{s_1,...,s_N}(\lambda_1^{(s_1)}, \ldots, \lambda_N^{(s_N)}). \tag{14}$$

Depending on a form of the functional (10) determined by a family $\Phi_{S,\Lambda}$ of functions (11), some of the LHV relations in (12) can hold for a wider (than LHV) class of correlation scenarios, some can be simply trivial, i.e. fulfilled for all scenarios.

Definition 5 [2] *Each of the "tight" constraints in (12), that is violated under a non-LHV correlation scenario is called a general Bell inequality.*

The general forms of *Bell inequalities* on scenario correlation functions (full and partial) – *correlation Bell inequalities*, and Bell inequalities on joint probabilities were found by us in [2].

Proposition 3 (Corollary 1 in [2]). *For an N-partite correlation scenario (1) with S_n settings and outcomes $\lambda_n \in \Lambda_n \subseteq [-1,1]$, $\sup \Lambda_n = 1$, $\inf \Lambda_n = -1$, $n = 1, ..., N$, of an arbitrary spectral type (discrete or continuous) at each n-th site, every correlation Bell inequality has the form*

$$
\begin{aligned}
&\min_{\eta_n \in \{-1,1\}^{S_n},\ n=1,...,N} \sum_{\substack{1 \leq n_1 < ... < n_M \leq N, \\ M=1,...,N}} F_M^{(\gamma)}(\eta_{n_1}, ..., \eta_{n_M}) \\
\leq &\sum_{\substack{1 \leq n_1 < ... < n_M \leq N, \\ M=1,...,N}} \sum_{s_{n_1}, ..., s_{n_M}} \gamma_{(s_{n_1}, ..., s_{n_M})} \left\langle \lambda_{n_1}^{(s_{n_1})} \cdot ... \cdot \lambda_{n_M}^{(s_{n_M})} \right\rangle_{\mathrm{P}^{(\Lambda)}_{(s_1,...,s_N)}} \Big|_{lhv} \\
\leq &\max_{\eta_n \in \{-1,1\}^{S_n},\ n=1,...,N} \sum_{\substack{1 \leq n_1 < ... < n_M \leq N, \\ M=1,...,N}} F_M^{(\gamma)}(\eta_{n_1}, ..., \eta_{n_M}),
\end{aligned}
\tag{15}
$$

where:
(i) *extremums are taken over all $2^{S_1+...+S_N}$ vertices of hypercube $[-1,1]^{S_1+...+S_N} \subset \mathbb{R}^{S_1+...+S_N}$;*
(ii) $\gamma_{(s_{n_1}, ..., s_{n_M})}$ *– real coefficients and*

$$
F_M(\eta_{n_1}, ..., \eta_{n_M}) = \sum_{s_{n_1}, ..., s_{n_M}} \gamma_{(s_{n_1}, ..., s_{n_M})}\ \eta_{n_1}^{(s_{n_1})} \cdot ... \cdot \eta_{n_M}^{(s_{n_M})}
$$

– an M-linear form of vectors $\eta_{n_1} = (\eta_{n_1}^{(1)}, ..., \eta_{n_1}^{(S_{n_1})}) \in \mathbb{R}^{S_{n_1}}, ..., \eta_{n_M} = (\eta_{n_M}^{(1)}, ..., \eta_{n_M}^{(S_{n_M})}) \in \mathbb{R}^{S_{n_M}}$.

Proposition 3 implies.

Corollary 1 [2] *The form of every correlation Bell inequality does not depend on a spectral type of outcomes, in particular, on a number of outcomes, observed at each "site", but is determined only by the extreme values of measurement outcomes.*

For a Bell inequality on full correlation functions ($M = N$), which is characterized in (15) by the family of functions

$$
\widetilde{\Phi}_{S,\Lambda} = \{\phi_{s_1,...,s_N}(\lambda_1, ..., \lambda_N) = \gamma_{(s_1,...,s_N)} \lambda_1 \cdot ... \cdot \lambda_N^N\},
\tag{16}
$$

the LHV constants (13), (14) obey the relation $\mathcal{B}^{\sup}_{\widetilde{\Phi}_{S,\Lambda}} = -\mathcal{B}^{\inf}_{\widetilde{\Phi}_{S,\Lambda}}$ and the following general statement holds.

Proposition 4 (Corollary 2 in [2]). *For an N-partite correlation scenario with S_n settings and outcomes $\lambda_n \in \Lambda_n \subseteq [-1,1]$, $\sup \Lambda_n = 1$, $\inf \Lambda_n = -1$, $n = 1, ..., N$, of an arbitrary spectral type (discrete or continuous) at each n-th site, Bell inequalities for the full correlation functions take the form*

$$\left| \sum_{s_1,...,s_N} \gamma_{(s_1,...,s_N)} \left\langle \lambda_1^{(s_1)} \cdot ... \cdot \lambda_N^{(s_N)} \right\rangle_{P^{(\Lambda)}_{(s_1,...,s_N)}} \right|_{lhv} \leq \max_{\eta_n \in \{-1,1\}^{S_n},\, n=1,...N} |F_N(\eta_1, ..., \eta_N)|, \quad (17)$$

where

$$F_N(\eta_1, ..., \eta_N) = \sum_{s_1,...,s_N} \gamma_{(s_1,...,s_N)} \eta_1^{(s_1)} \cdot ... \cdot \eta_N^{(s_N)}$$

– an N-linear form of vectors $\eta_n = (\eta_n^{(1)}, ..., \eta_n^{(S_n)}) \in \mathbb{R}^{S_n}$, $n = 1, ..., N$.

From Theorem 3 it also follows that all Bell inequalities on joint probabilities are given by the following single representation.

Proposition 5 (Corollary 3 in [2]). *For an N-partite correlation scenario with not less than $(Q_n + 1)$ outcomes at each n-th site, any Bell inequality on joint probabilities takes the form:*

$$\begin{aligned}
&\min_{\eta_n \in \Xi_n,\, n=1,...,N} \sum_{\substack{1\leq n_1,...,n_M \leq N,\\ M=1,...,N}} F_M^{(\gamma)}(\eta_{n_1}, ..., \eta_{n_M}) \\
\leq &\sum_{\substack{1\leq n_1<...<n_M \leq N,\\ M=1,...,N}} \sum_{\substack{s_{n_1},...,s_{n_M},\\ q_{n_1},...,q_{n_M}}} \gamma^{(q_{n_1},...,q_{n_M})}_{(s_{n_1},...,s_{n_M})} \mathrm{P}^{(\Lambda)}_{(s_{n_1},...,s_{n_M})}(D_{n_1}^{(s_{n_1},q_{n_1})} \times \cdots \times D_{n_M}^{(s_{n_M},q_{n_M})})\big|_{lhv} \quad (18)\\
\leq &\max_{\eta_n \in \Xi_n,\, n=1,...,N} \sum_{\substack{1\leq n_1,...,n_M \leq N,\\ M=1,...,N}} F_M^{(\gamma)}(\eta_{n_1}, ..., \eta_{n_M}),
\end{aligned}$$

where: (i) $\gamma^{(q_{n_1},...,q_{n_M})}_{(s_{n_1},...,s_{n_M})}$ *– real coefficients;*
(ii) $D_n^{(s_n,q_n)} \subset \Lambda_n^{(s_n)}$, $D_n^{(s_n,q_n)} \neq \varnothing$, $q_n = 1, ..., Q_n$ *– events, observed under s_n-th measurement at n-th site and such, that, for $Q_n \geq 2$, these events are incompatible:* $D_n^{(s_n,q_n)} \cap D_n^{(s_n,q_n')} = \varnothing$, $\forall q_n \neq q_n'$, *and* $\cup_{q_n=1,...,Q_n} D_n^{(s_n,q_n)} \neq \Lambda_n^{(s_n)}$;
(iii) $F_M^{(\gamma)}(\eta_{n_1}, ..., \eta_{n_M}) = \sum_{\substack{s_{n_1},...,s_{n_M},\\ q_{n_1},...,q_{n_M}}} \gamma^{(q_{n_1},...,q_{n_M})}_{(s_{n_1},...,s_{n_M})} \eta_{n_1}^{(s_{n_1},q_{n_1})} \cdot ... \cdot \eta_{n_M}^{(s_{n_M},q_{n_M})}$ *is an M-linear form of vectors* $\eta_n \in \mathbb{R}^{S_n Q_n}$, *with components* $\eta_n^{(s_n,q_n)}$, $s_n \in \{1, ..., S_n\}$, $q_n \in \{1, ..., Q_n\}$;
(iv) $\Xi_n := \{\eta_n \in \{0,1\}^{S_n Q_n} \mid \sum_{q_n=1,...,Q_n} \eta_n^{(s_n,q_n)} \in \{0,1\}, \forall s_n \in \{1, ..., S_n\}$ *for all* $n = 1, ..., N$.

Analytical representations (15), (17), (18) allows us [2] not only to derive within the unified approach all the known Bell inequalities developed earlier on the basis of other various methods, but also to expand the ranges of their validity. Moreover, the analytic representation (18) includes all Bell inequalities on joint probabilities obtained earlier by Collins and Gisin[8] via numerical methods.

[8]Daniel Collins and Nicolas Gisin. *A relevant two qubit Bell inequality inequivalent to the CHSH inequality* // J. Phys. A: Math. Gen. **37**, 1775 (2004)

2.1.3 Bell nonlocality

At the conceptual level, the notion of quantum nonlocality, conjectured by Bell, has longly been viewed in the literature as violation of the physical principle of locality. This problem is discussed by us in Introductions of our articles [7, 9], also in our earlier work[9] on quantum nonlocality.

However, mathematically quantum nonlocality manifests itself only via violation by a quantum state of Bell inequalities and this violation is possible not only under a quantum correlation scenario, but also, for a wider class of correlation scenarios, which do not admit the LHV modelling.

Definition 6 (Section 4 in [9]). *If, for an N-partite correlation scenario $\mathcal{P}_{S,\Lambda}$, one of general Bell inequalities in (12) is violated, then we refer to this scenario as Bell nonlocal.*

Consider analogs of the LHV constraints (12) for an arbitrary "nonsignaling" $S_1\times\ldots\times S_N$-correlation scenario (1).

Proposition 6 (Eqs. 15-17 in [8]; Lemma 3 in [4]). *For each "nonsignaling" correlation scenario $\mathcal{P}_{S,\Lambda}^{n\text{-}sig}$ every linear combination (10) of its mathematical expectations (2) of a general form satisfies the "tight" LqHV constraints*

$$\begin{aligned} &\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}-\frac{\gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})-1}{2}(\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup}-\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}) \\ &\leq \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{n\text{-}sig}) \\ &\leq \mathcal{B}_{\Phi_{S,\Lambda}}^{\sup}+\frac{\gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})-1}{2}(\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup}-\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}) \end{aligned} \tag{19}$$

and

$$\begin{aligned} \left|\ \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{nonsig})\right| &\leq \gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})\ \mathcal{B}_{\Phi_{S,\Lambda}}^{lhv} \\ \mathcal{B}_{\Phi_{S,\Lambda}}^{lhv} &= \max\left\{\left|\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup}\right|,\left|\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}\right|\right\}, \end{aligned} \tag{20}$$

where $\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup}$, $\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}$ – the LHV constants (13), (14), and the LqHV parameter

$$\gamma(\mathcal{P}_{S,\Lambda}^{n-sig})=\inf_{\mu_{\mathcal{P}_{S,\Lambda}^{n\text{-}sig}}}\left\|\mu_{\mathcal{P}_{S,\Lambda}^{n\text{-}sig}}\right\|_{var}, \tag{21}$$

defined by (9), is equal to

$$\gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})=\sup_{\Psi_{S,\Lambda},\ \mathcal{B}_{\Phi_{S,\Lambda}}^{lhv}\neq 0}\frac{\left|\mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{n\text{-}sig}\right|}{\mathcal{B}_{\Phi_{S,\Lambda}}^{lhv}}. \tag{22}$$

In (22), supremum is taken over all families $\Psi_{S,\Lambda}=\{\psi_{(s_1,\ldots,s_N)}\}$ of functions (11), determining a functional $\mathcal{B}_{\Phi_{S,\Lambda}}(\cdot)$.

Since for a "nonsignaling" scenario $\mathcal{P}_{S,\Lambda}^{n\text{-}sig}$, relation $\left|\mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})\right|>\mathcal{B}_{\Phi_{S,\Lambda}}^{lhv}$ holds only for those $\Phi_{S,\Lambda}$, which specify Bell inequalities, from Proposition 6 it follows that the LqHV parameter $\gamma(\mathcal{P}_{S,\Lambda}^{n\text{-}sig})$, introduced in (9), constitutes *the maximal violation* under a scenario $\mathcal{P}_{S,\Lambda}^{n\text{-}sig}$ of all general Bell inequalities for $S_1\times\ldots\times S_N$-setting correlation scenarios.

[9] Elena R. Loubenets. *Local Realism, Bell's Theorem and Quantum Locally Realistic Inequalities // Foundations of Physics* **35**, No. 12, 2051–2072 (2005).

Corollary 2 [4, 8] *For each "nonsignaling" correlation scenario $\mathcal{P}_{S,\Lambda}^{n\text{-}sig}$ the following statements are equivalent:*
(i) scenario $\mathcal{P}_{S,\Lambda}^{n\text{-}sig}$ admits the LHV modelling;
(ii) *all general Bell inequalities hold.*

2.2 LqHV modelling of quantum correlation scenarios

Consider an N-partite correlation scenario where each n-th of N participants performs at his "site" S_n local measurements on a N-partite quantum state ρ on a Hilbert space $\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_N$. For this case, the joint measurement $(s_1, ..., s_N)$ is described by a probability distribution

$$\operatorname{tr}[\rho\{\mathrm{M}_1^{(s_1)}(\mathrm{d}\lambda_1) \otimes \cdots \otimes \mathrm{M}_N^{(s_N)}(\mathrm{d}\lambda_N)\}], \tag{23}$$

where $\mathrm{M}_n^{(s_n)}(\cdot)$, $\mathrm{M}_n^{(s_n)}(\Lambda_n) = \mathbb{I}_{\mathcal{H}_n}$ is a normalized positive operator-valued (POV) measure, specifying s_n-th quantum measurement at n-th site. If s_n-th quantum measurement at n-th "site" is projective, then the corresponding POV measure $\mathrm{M}_n^{(s_n)}$ is projection- valued.

The probabilistic description of the whole quantum correlation scenario on a state ρ on $\mathcal{H}_1 \otimes ... \otimes \mathcal{H}_N$ is described by the family

$$\begin{aligned} \mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}} \quad &: \quad = \{\operatorname{tr}[\rho\{\mathrm{M}_1^{(s_1)}(\mathrm{d}\lambda_1) \otimes \cdots \otimes \mathrm{M}_N^{(s_N)}(\mathrm{d}\lambda_N)\}]\}, \\ s_n \quad &= \quad 1, ..., S_n, \quad n = 1, ..., N, \end{aligned} \tag{24}$$

of all scenario joint probability distributions (23) where

$$\mathfrak{m}_{S,\Lambda} := \left\{\mathrm{M}_n^{(s_n)}(\mathrm{d}\lambda_n),\ \lambda_i \in \Lambda_n \ \mid \ s_n = 1, ..., S_n,\ n \in 1, ..., N\right\} \tag{25}$$

is a set of all POV measures, specifying under this quantum scenario all local measurements at each of N "sites".

Theorem 4 (Theorem 2 in [4]). *Every correlation scenario* (24) *on an arbitrary quantum state ρ on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ admits the probabilistic description in the frame of the LqHV model* (4).

For a quantum scenario (24) the general LqHV constraints (19) take the form

$$\begin{aligned} &\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf} - \frac{\gamma(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}) - 1}{2}(\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} - \mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}) \\ &\leq \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}) \\ &\leq \mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} + \frac{\gamma(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}) - 1}{2}(\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} - \mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}), \end{aligned} \tag{26}$$

where

$$\begin{aligned} 1 \quad \leq \quad \gamma(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}) &:= \inf_{\mu_{\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}}} \left\| \mu_{\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}} \right\|_{var} \\ &= \sup_{\Psi_{S,\Lambda},\ \mathcal{B}_{\Phi_{S,\Lambda}}^{lhv} \neq 0} \frac{\left|\mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}})\right|}{\mathcal{B}_{\Phi_{S,\Lambda}}^{lhv}}. \end{aligned} \tag{27}$$

Maximizing the right-hand sites of relations in (26) over families $\mathfrak{m}_{S,\Lambda}$ of all scenario POV measures and all outcomes sets Λ_n in $\Lambda = \Lambda_n \otimes \cdots \otimes \Lambda_N$, we come to the following new result.

Proposition 7 (Lemma 3, Eqs. 46-49, Lemma 3 in [4]). *For every quantum correlation scenario $\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}$ on an N-partite quantum state ρ, each linear combination (10) of its mathematical expectations (2) of a general form satisfies the "tight" quantum constraints:*

$$\begin{aligned} &\mathcal{B}_{\Phi_{S,\Lambda}}^{\inf} - \frac{\Upsilon_{S_1\times\cdots\times S_N}^{(\rho)} - 1}{2}(\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} - \mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}) \\ &\leq \mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}) \\ &\leq \mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} + \frac{\Upsilon_{S_1\times\cdots\times S_N}^{(\rho)} - 1}{2}(\mathcal{B}_{\Phi_{S,\Lambda}}^{\sup} - \mathcal{B}_{\Phi_{S,\Lambda}}^{\inf}), \end{aligned} \tag{28}$$

with the quantum parameter

$$1 \leq \Upsilon_{S_1\times\cdots\times S_N}^{(\rho)} := \sup_{\Lambda,\mathfrak{m}_{S,\Lambda}} \left(\inf_{\mu_{\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}}} \left\| \mu_{\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}} \right\|_{var} \right) = \sup_{\Lambda,\ \mathfrak{m}_{S,\Lambda},\Psi_{S,\Lambda},\ \mathcal{B}_{\Phi_{S,\Lambda}}^{lhv}\neq 0} \frac{\left|\mathcal{B}_{\Phi_{S,\Lambda}}(\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}})\right|}{\mathcal{B}_{\Phi_{S,\Lambda}}^{lhv}} \tag{29}$$

determining the maximal violation by a state ρ of all general Bell inequalities for N-partite correlation scenarios with $L_n \leq S_n$ measurements at each n-th "site"

Introduce also the parameter

$$\Upsilon_\rho := \sup_{S_1,\ldots,S_N} \Upsilon_{S_1\times\cdots\times S_N}^{(\rho)} \geq 1. \tag{30}$$

Proposition 8 (Proposition 6 in [4]). *For an N-partite quantum state ρ:*
(i) *parameter $\Upsilon_{S_1\times\cdots\times S_N}^{(\rho)} = 1$ iff any correlation scenario (24) performed on this state and with $L_n \leq S_n$ measurements at each n-th "site" admits the LHV modelling, equivalently, all general Bell inequality for such scenarios hold;*
(ii) *parameter $\Upsilon_\rho = 1$ iff every correlation scenario on state ρ admits the LHV modelling, equivalently, all general Bell inequality hold.*

Note that the condition in item (ii) of Proposition 8 does not imply, that, for all correlation scenarios on a state ρ there exists a unique LqHV model.

2.2.1 Quantum nonlocality

In a quantum case, to Bell nonlocality – the notion introduced in section 2.1.3, there corresponds the term "quantum nonlocality", having, however, in the physical literature quite another conceptual meaning (see Introduction in our article [9]).

Let us introduce the notions of full locality and partial locality.

Definition 7 (Section 5 in [9, 10]). *An N-partite quantum state ρ :*
(i) *$S_1 \times \cdots \times S_N$-setting Bell local if it does not violate any of general Bell inequalities for $L_n \leq S_n$ settings and any type of outcomes at each n-th "site" – otherwise, $S_1 \times \cdots \times S_N$-setting Bell nonlocal;*
(ii) *fully Bell local if does not violate any of N-partite general Bell inequalities for any number of settings and any type outcomes at each n-th "site" – otherwise, overall Bell nonlocal.*

From Definition 7 and Proposition 8 it follows.

Proposition 9 (Section 5 in [9], Eqs. 19–21 in [10]). *An N-partite quantum state ρ :*
(i) *$S_1 \times \cdots \times S_N$-setting Bell local iff $\Upsilon^{(\rho)}_{S_1 \times \cdots \times S_N} = 1$ and $S_1 \times \cdots \times S_N$-setting Bell nonlocal iff $\Upsilon^{(\rho)}_{S_1 \times \cdots \times S_N} > 1$;*
(ii) *fully Bell local iff $\Upsilon_\rho = 1$ and overall Bell nonlocal iff $\Upsilon_\rho > 1$.*

2.2.2 Constructing quantum LqHV models

For finding specific LqHV models for an arbitrary quantum correlation scenario (24) on an N-partite quantum state ρ, we introduce in [4] the following new notion, summarizing our results in earlier papers[10].

Definition 8 (Definition 1 in [4]) *Let ρ be a quantum state on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and $S_1, ..., S_N \geq 1$ be arbitrary integers. The self-adjoint trace class operator $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ on the Hilbert space $\mathcal{H}_1^{\otimes S_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes S_N}$, satisfying the relation*

$$\begin{aligned} &\operatorname{tr}\left[T^{(\rho)}_{S_1 \times \cdots \times S_N}\left\{\mathbb{I}_{\mathcal{H}_1^{\otimes k_1}} \otimes X_1 \otimes \mathbb{I}_{\mathcal{H}_1^{\otimes (S_1-1-k_1)}} \otimes \cdots \otimes \mathbb{I}_{\mathcal{H}_N^{\otimes k_N}} \otimes X_N \otimes \mathbb{I}_{\mathcal{H}_1^{\otimes (S_N-1-k_N)}}\right\}\right] \\ &= \operatorname{tr}\left[\rho\left\{X_1 \otimes \cdots \otimes X_N\right\}\right], \quad k_n = 0, ..., (S_n - 1), \quad n = 1, ...N, \end{aligned} \tag{31}$$

for all bounded linear operators X_n on $\mathcal{H}_n$, is called a source operator for a state ρ. In (31):

$$I_{\mathcal{H}_n^{\otimes m}} \otimes X_n \mid_{m=0} = X_n \otimes I_{\mathcal{H}_n^{\otimes m}} \mid_{m=0} := X_n \quad \text{and} \quad T^{(\rho)}_{1 \times \cdots \times 1} := \rho.$$

From Definition 7 it follows that a source operator $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ of a state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ constitutes a self-adjoint extension of ρ to space $\mathcal{H}_1^{\otimes S_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes S_N}$ with $\operatorname{tr}[T^{(\rho)}_{S_1 \times \cdots \times S_N}] = 1$.

Theorem 5 *(Proposition 1 in [4]). For every N-partite quantum state ρ and arbitrary integers $S_1, ..., S_N \geq 1$, a source operator $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ exists.*

If $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ is a source operator for a state ρ, then each of its reduced on a Hilbert space $\mathcal{H}_1^{\otimes L_1} \otimes \cdots \otimes \mathcal{H}_N^{\otimes L_N}$, where $1 \leq L_n < S_n$, constitutes a source operator $T^{(\rho)}_{L_1 \times \cdots \times L_N}$ for state ρ.

Examples of specific source operators for various N-partite quantum states, pure and mixed, are constructed by us in [1, 3, 4, 10, 11].

Let $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ be a source operator of a quantum state ρ on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. Then, for the real-valued measure

$$\begin{aligned} &\tau^{(\mathfrak{m}_{S,\Lambda})}_{T^{(\rho)}_{S_1 \times \cdots \times S_N}}\left(\mathrm{d}\lambda_1^{(1)} \times \cdots \times \mathrm{d}\lambda_1^{(S_1)} \times \cdots \times \mathrm{d}\lambda_N^{(1)} \times \cdots \times \mathrm{d}\lambda_N^{(S_N)}\right) \\ &:= \operatorname{tr}\Big[T^{(\rho)}_{S_1 \times \cdots \times S_N}\ \{\mathrm{M}_1^{(1)}(\mathrm{d}\lambda_1^{(1)}) \otimes \cdots \otimes \mathrm{M}_1^{(S_1)}(\mathrm{d}\lambda_1^{(S_1)}) \\ &\otimes \cdots \otimes \mathrm{M}_N^{(1)}(\mathrm{d}\lambda_N^{(1)}) \otimes \cdots \otimes \mathrm{M}_N^{(S_N)}(\mathrm{d}\lambda_N^{(S_N)})\}\Big], \end{aligned} \tag{32}$$

[10]Loubenets: *Banach Center Publ.* **73,** 325–337 (2006); *J. Physics A: Math. Gen.* **38**, L653–L658 (2005); *J. Physics A: Math. Gen.* **39**, 5115–5123 (2006).

all joint probability distributions (23) of a quantum scenario (24) on a state ρ constitute its marginals. Therefore, by Theorem 1 a quantum correlation scenario $\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}$ admits the description within the LqHV model with parameters:

$$\mu_{\mathcal{P}_{S,\Lambda}^{\rho,\mathfrak{m}_{S,\Lambda}}} = \tau_{T_{S_1\times\cdots\times S_N}^{(\rho)}}^{(\mathfrak{m}_{S,\Lambda})}, \qquad \mathrm{P}_n^{(s_n)}(D_n|\omega) = \chi_{D_n}(\lambda_n^{(s_n)}), \qquad D_n \subseteq \Lambda_n, \tag{33}$$
$$\omega = \left(\lambda_1^{(1)}, ..., \lambda_1^{(S_1)}, ..., \lambda_N^{(1)}, ..., \lambda_N^{(S_N)}\right), \quad \Omega = \Lambda_1^{S_1} \times ... \times \Lambda_N^{S_N}.$$

If, for a state ρ, there exists a positive source operator $T_{S_1\times\cdots\times S_N}^{(\rho)} \geq 0$, then the measure (32) is positive for any $S_1 \times \cdots \times S_N$-setting correlation scenario and the corresponding LqHV model (33) is a LHV model. For such a state ρ, parameter $\Upsilon_{S_1\times\cdots\times S_N}^{(\rho)}$ defined in (29) is equal to 1, so that, according to Proposition 8, a state ρ is $S_1 \times \ldots \times S_N$-setting Bell local. If, for a state ρ, positive source operators exist for all $S_1, \ldots, S_N$, then this quantum state is fully Bell local.

This condition is, in particular, fulfilled for any separable N-partite quantum state

$$\rho_{sep} = \sum \alpha_i \rho_1^{(i)} \otimes \cdots \otimes \rho_N^{(i)}, \quad \alpha_i > 0, \quad \sum \alpha_i = 1,$$

since the operator

$$T_{S_1\times\cdots\times S_N}^{(\rho_{sep})} := \sum \alpha_i \left(\rho_1^{(i)}\right)^{\otimes S_1} \otimes \cdots \otimes \left(\rho_N^{(i)}\right)^{\otimes S_N} \geq 0$$

is a positive source operator of ρ_{sep} for all integers $S_1, \ldots, S_N \geq 1$.

In the next section, we specify a weaker condition on a source operator $T_{S_1\times\cdots\times S_N}^{(\rho)}$ of a state ρ, introduced by us in [4], under which measure (32) is also positive while the quantum LqHV model (33) becomes a LHV model.

2.2.3 Tensor positivity, the covering norm

Denote by $\mathcal{L}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}$ the space of all bounded linear operators on an arbitrary complex Hilbert space $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$, $m \geq 1$, by $\mathcal{L}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}^{(s)} \subset \mathcal{L}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}$ the subspace of self-adjoint operators and by $\mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}^{(s)} \subset \mathcal{T}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}$ – the subspace of all self-adjoint trace class operators on $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$.

We introduce the following new general notions.

Definition 9 (Definition 2 in [4]). *If an operator $Z \in \mathcal{L}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}$ satisfies the relation*

$$(\psi_1 \otimes \cdots \otimes \psi_m, Z\ \psi_1 \otimes \cdots \otimes \psi_m) \geq 0 \tag{34}$$

for all $\psi_1 \in \mathcal{G}_1, ..., \psi_m \in \mathcal{G}_m$, then we call operator Z as tensor positive and denote this by $Z \overset{\otimes}{\geq} 0$.

If $m = 1$, then tensor positivity is equivalent to positivity. If $m \geq 2$, then positivity of an operator $Z \in \mathcal{L}_{\mathcal{G}_1\otimes\cdots\otimes\mathcal{G}_m}$ implies its tensor positivity, but not vice versa. For example, the flip operator $V(\psi_1 \otimes \psi_2) := \psi_2 \otimes \psi_1$ on $\mathcal{H} \otimes \mathcal{H}$ is tensor positive, but not positive.

For any tensor positive trace class operator $W \overset{\otimes}{\geq} 0$ on $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$, relation $\operatorname{tr}[W\{X_1 \otimes \cdots \otimes X_m\}] \geq 0$ holds for all positive $X_j \in \mathcal{L}_{\mathcal{G}_j}$. In particular, $\operatorname{tr}[W] \geq 0$. If a trace class operator $W \in \mathcal{T}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$ is tensor positive, then all operators reduced from Z are also tensor positive.

Definition 10 (Definition 3 in [4]). *Let $Z \in \mathcal{L}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$. We call a tensor positive operator $Z_{cov} \in \mathcal{L}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$, $Z \overset{\otimes}{\geq} 0$, satisfying the relation $Z_{cov} \pm Z \overset{\otimes}{\geq} 0$, as a covering of operator Z.*

If an operator $Z \in \mathcal{L}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$ is tensor positive, then itself is one of its coverings. If Z_{cov} is a covering of $Z \in \mathcal{L}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$, then

$$Z = \frac{1}{2}(Z_{cov} + Z) - \frac{1}{2}(Z_{cov} - Z), \tag{35}$$

where the terms $Z_{cov} \pm Z \overset{\otimes}{\geq} 0$.

Proposition 10 (Proposition 2 in [4]). *For every self-adjoint trace class operator $W \in \mathcal{T}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$ a trace class covering $W_{cov} \in \mathcal{T}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$ exists and satisfies the relation* $\operatorname{tr}[W_{cov}] \geq |\operatorname{tr}[W]| \geq 0$.

As we prove in [4], the function

$$f(W) = \inf_{W_{cov} \in \mathcal{T}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}} \operatorname{tr}[W_{cov}], \qquad W \in \mathcal{T}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}, \tag{36}$$

constitutes a norm on space $\mathcal{T}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}$ of self-adjoint trace class operators.

Definition 11 *(Definition 4 in [4]) We refer to norm (36) as the covering norm and denote it by*

$$\|W\|_{cov} := \inf_{W_{cov} \in \mathcal{T}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}} \operatorname{tr}[W_{cov}], \qquad W \in \mathcal{T}^{(s)}_{\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m}. \tag{37}$$

The following statement holds.

Theorem 6 (Lemma 1 in [4]). *For a self-adjoint trace class operator W on a Hilbert space $\mathcal{G}_1 \otimes \cdots \otimes \mathcal{G}_m$:*
(a) $|\operatorname{tr}[W]| \leq \sup |\operatorname{tr}[W\{X_1 \otimes \cdots \otimes X_m\}]| \leq \|W\|_{cov} \leq \|W\|_1$,
where $\|\cdot\|_1$ is the trace norm and supremum is taken over all self-adjoint operators $X_j \in \mathcal{L}_{\mathcal{G}_j}$ with the operator norm $\|X_j\|_0 = 1$;
(b) *if $W \overset{\otimes}{\geq} 0$, then $\|W\|_{cov} = \operatorname{tr}[W]$;*
(c) $|\operatorname{tr}[W]| \leq \|W_{red}\|_{cov} \leq \|W\|_{cov}$ *for every operator W_{red}, reduced from W.*

For example, for the flip operator $V(\psi_1 \otimes \psi_2) = \psi_2 \otimes \psi_1$ on $\mathbb{C}^d \otimes \mathbb{C}^d$, which is tensor positive, the covering norm is $\|V\|_{cov} = d$, while the trace norm is $\|V\|_1 = d^2$.

For a source operator $T^{(\rho)}_{S_1 \times \cdots \times S_N}$ of a state ρ from Theorem 6 it follows

$$1 \leq \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_{cov} \leq \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_1, \qquad T^{(\rho)}_{S_1 \times \cdots \times S_N} \overset{\otimes}{\geq} 0 \;\Rightarrow\; \left\| T^{(\rho)}_{S_1 \times \cdots \times S_N} \right\|_{cov} = 1, \tag{38}$$

$$1 \leq \left\| \left(T^{(\rho)}_{S_1\times\cdots\times S_N} \right)_{red} \right\|_{cov} \leq \left\| T^{(\rho)}_{S_1\times\cdots\times S_N} \right\|_{cov}. \tag{39}$$

Therefore, for any source operator $T^{(\rho)}_{S_1\times\cdots\times S_N}$, the covering norm $||T^{(\rho)}_{S_1\times\cdots\times S_N}||_{cov} \geq 1$ and $||T^{(\rho)}_{S_1\times\cdots\times S_N}||_{cov} = 1$, if $T^{(\rho)}_{S_1\times\cdots\times S_N}$ is tensor positive.

2.2.4 Analytical bound on an N-partite quantum state nonlocality

Based on the new general notions introduced by us in [4] and described in section 2.2.3 – a source operator of an N-partite quantum state ρ, tensor positivity and the covering norm, we derive the following analytical bounds on parameter $\Upsilon^{(\rho)}_{S_1\times\cdots\times S_N}$, determining the maximal violation by a state ρ of all general $S_1 \times \cdots \times S_N$ -setting Bell inequalities and characterizing according to Proposition 9, *the "extent" of nonlocality* of an N-partite quantum state ρ.

Theorem 7 (Proposition 4 in [4]). *For parameter* $\Upsilon^{(\rho)}_{S_1\times\cdots\times S_N} \geq 1$ *characterizing the "extent" of nonlocality of an N-partite quantum state* ρ *under correlation scenarios with* $L_n \leq S_n$ *measurements at each n-th site, the following bounds are true:*

$$\begin{aligned} 1 \quad &\leq \quad \Upsilon^{(\rho)}_{S_1\times\cdots\times S_N} \leq \inf_{T^{(\rho)}_{S_1\times\cdots\times \underset{n_0}{\underset{\uparrow}{1}} \times\cdots\times S_N},\ \forall n_0} ||T^{(\rho)}_{S_1\times\cdots\times \underset{n_0}{\underset{\uparrow}{1}} \times\cdots\times S_N}||_{cov} \\ &\leq \quad \inf_{T^{(\rho)}_{S_1\times\cdots\times S_N}} ||T^{(\rho)}_{S_1\times\cdots\times S_N}||_{cov}, \end{aligned} \tag{40}$$

where (i) *infimum is taken over all source operators* $T^{(\rho)}_{S_1\times\cdots\times \underset{n_0}{\underset{\uparrow}{1}} \times\cdots\times S_N}$ *of a state* ρ *with* $S_{n_0} = 1$ *for some* n_0 *and over all* $n_0 \in \{1, ..., N\}$; (ii) *notation* $||\cdot||_{cov}$ *means the covering norm.*

Note that, besides our analytical bound (40), in the literature there are no upper bounds on the maximal violation $\Upsilon^{(\rho)}_{S_1\times\cdots\times S_N}$, formulated *in terms of Hilbert space characteristics* of an N-partite quantum state.

2.2.5 Sufficient conditions for an N-partite quantum state locality

The analytical bound (40) implies the following sufficient conditions for Bell locality, full and partial, of an N-partite quantum state.

Theorem 8 (Propositions 5, 6 in [4] and Theorem 1 in [4, 9]). *If for a quantum state* ρ *on* $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ *there exist:*

(a) *a tensor positive source operator* $T^{(\rho)}_{S_1\times\cdots\times S_N} \overset{\otimes}{\geq} 0$, *then state* ρ *is* $S_1 \times \cdots \times S_N$*-setting Bell local;*

(b) *a tensor positive source operator* $T^{(\rho)}_{S_1\times\cdots\times \underset{n_0}{\underset{\uparrow}{1}} \times\cdots\times S_N} \overset{\otimes}{\geq} 0$ *for some* $n_0 = 1, ..., N$, *then state* ρ *is* $S_1 \times \cdots \times \widetilde{S}_{n_0} \times \cdots \times S_N$*-setting Bell local for all numbers* $S_{n_0} \geq 1$ *of measurements at* n_0*-the "site";*

(c) *a tensor positive source operator* $T^{(\rho)}_{S_1\times\cdots\times S_N} \overset{\otimes}{\geq} 0$ *for all integers* $S_1, ..., S_N \geq 1$, *then an* N*-partite quantum state* ρ *is fully Bell local.*

Examples of nonseparable N-partite quantum states for which the sufficient conditions in Theorem 8 are fulfilled are presented by us in [1, 3, 4, 10]. For instance, in [3] we prove that, for any nonseparable Werner state[11]

$$\rho_W(\Phi) = \frac{1+\Phi}{2}\frac{P_d^{(+)}}{r_d^{(+)}} + \frac{1-\Phi}{2}\frac{P_d^{(-)}}{r_d^{(-)}}, \qquad \Phi \in [-1, 0), \tag{41}$$

on $\mathbb{C}^d \otimes \mathbb{C}^d$ with $d \geq 3$, there exist tensor positive source operators $T_{2\times 1}^{(\rho_W)} \overset{\otimes}{\geq} 0$ and $T_{1\times 2}^{(\rho_W)} \overset{\otimes}{\geq} 0$. Therefore, by Theorem 8, for each $d \geq 3$ and every $\Phi \in [-1, 0)$, the Werner nonseparable state $\rho_W(\Phi)$ is $2 \times S_2$-setting Bell local for all $S_2 \geq 2$ and $S_1 \times 2$-setting Bell local for all $S_1 \geq 2$.

2.2.6 Analytical "numerical" bounds on quantum nonlocality

Based on our general analytical bound (40) on $S_1 \times \cdots \times S_N$-nonlocality of an N-partite state ρ, we also derived the new exact analytical bounds in terms of the following "numerical" characteristics of a quantum correlation scenario – a number $N \geq 2$ of scenario participants ("sites") and a number $S_n \geq 1$ of measurements and a dimension $d_n := \dim \mathcal{H}_n$ of a Hilbert spaces $\mathcal{H}_n$ at each n-th of N sites.

For the most well-known N-partite quantum states used in tasks of quantum information processing and quantum communications, the new derived bounds on the maximal violation of general Bell inequalities have the forms.

Proposition 11 (Eqs. (54)–(59) in [4]). *For quantum correlation scenarios with any type, generalized or projective, of measurements at each of N "sites" the following precise upper bounds are true:*

(1) *for the maximal violation by the two-qubit singlet $|\beta_{11}\rangle = \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle - |1\rangle \otimes |0\rangle)$ of all general Bell inequalities with settings $S_1 \times 2$ and $2 \times S_2$:*

$$\begin{aligned} \Upsilon_{S_1\times 2}^{(|\beta_{11}\rangle)} &\leq \sqrt{3}, \quad \Upsilon_{2\times S_2}^{(|\beta_{11}\rangle)} \leq \sqrt{3}, \\ S_1 &\geq 2, \quad S_2 \geq 2; \end{aligned} \tag{42}$$

(Recall that the maximal violation by singlet $|\beta_{11}\rangle$ of correlation Bell inequalities is upper bounded by the real Grothendieck constant $K_G(3) \in [\sqrt{2}, 1.5164]$.)

(2) *for the maximal violation by the generalized N-qubit GHZ state $|\psi_{d,N}^{ghz}(\varphi)\rangle := \sin\varphi\ |0\rangle^{\otimes N} + \cos\varphi\ |1\rangle^{\otimes N}$ of all general Bell inequalities:*

$$\Upsilon_{|\psi_{d,N}^{(ghz)}(\varphi)\rangle} \leq 1 + 2^{N-1}\ |\sin 2\varphi|\,; \tag{43}$$

(3) *for the maximal violation by the N-qudit GHZ state $\rho_{d,N}^{ghz}$ of all general Bell inequalities:*

$$\Upsilon_{\rho_{d,N}^{(ghz)}} \leq 1 + 2^{N-1}\,(d-1)\,. \tag{44}$$

[11] In the expression (41), the operators $P_d^{(\pm)}$ are orthogonal projections onto the symmetric (plus sign) and antisymmetric (minus sign) subspaces $\mathbb{C}^d \otimes \mathbb{C}^d$, with dimensions $\frac{d(d\pm 1)}{2}$. For any $d \geq 2$, the Werner state is separable for $\Phi \in [0, 1]$, otherwise nonseparable.

Note that, for the maximal violation $\Upsilon_{\rho_{d,2}^{(ghz)}}$ of Bell inequalities by the two-qubit GHZ state $\rho_{d,2}^{ghz}$, the known "numerical" bound by other authors[12] is derived within the operator space formalism and has the form $Cd/\sqrt{\ln d}$ and is determined up to an unknown constant C which is independent of d.

Theorem 9 (Theorem 4 in [4] and Theorem 1 in [11]). *For an arbitrary N-partite quantum state ρ and any type of quantum measurements at each n-th "site" the following general upper bound on the maximal violation of all $S_1 \times \cdots \times S_N$-setting general Bell inequalities holds:*

$$\begin{aligned} \Upsilon_{S_1\times\cdots\times S_N}^{(\rho)} &\leq \min\left\{\xi_N(d_1,\ldots,d_N),\ \theta_N(S_1,...,S_N)\ \right\} \\ &\leq 1+2^{N-1}\left[\min\left\{\frac{d_1\cdot\ldots\cdot d_N}{\max_n d_n},\ \frac{S_1\cdot\ldots\cdot S_N}{\max_n S_n}\right\}-1\right], \end{aligned} \tag{45}$$

where $d_n := \dim \mathcal{H}_n$, $n = 1,...,N$,

$$\begin{aligned} \xi_N(d_1,\ldots,d_N) &= 1+2^{N-1}\left(\frac{d_1\cdot\ldots\cdot d_N}{\max_n d_n}-1\right), \\ \theta_N(S_1,...,S_N) &= (-1)^{N-1}+\min_{\substack{\{n_1,...,n_{N-1}\}\\ \subset\{1,...,N\}}}\sum_{k=0}^{N-2}(-1)^k\ 2^{N-1-k}\sum_{\substack{n_{j_1}\neq\ldots\neq n_{j_{N-1-k}},\\ n_j\in\{n_1,...,n_{N-1}\}}} S_{n_{j_1}}\cdot\ldots\cdot S_{n_{j_{N-1-k}}}, \end{aligned} \tag{46}$$

in particular, for $S_n = S$, $d_n = d$, $n = 1,...,N$,

$$\Upsilon_{S\times\cdots\times S}^{(\rho_d)} \leq (2\min\{d,S\}-1)^{N-1}. \tag{47}$$

From Theorem 9 it follows that, for any N-partite quantum state ρ, the violation of an arbitrary general Bell inequality with S settings per site cannot exceed the value $(2S-1)^{N-1}$ even for a state ρ on an infinite-dimensional tensor product Hilbert space.

Similarly, for an N-qudit state, the violation of every general Bell inequality with an arbitrary number of measurements and any type of outcomes at each of "sites" cannot exceed the value $(2d-1\ )^{N-1}$.

Theorem 9, in particular, implies: (i) for an arbitrary bipartite quantum state ρ and all integers $S_1, S_2 \geq 1$,

$$\Upsilon_{S_1\times S_2}^{(\rho)} \leq 2\min\{S_1,S_2,d_1,d_2\}-1, \tag{48}$$

(ii) for every three-partite quantum state ρ and all $S_1, S_2, S_3 \geq 1$, $d_1, d_2, d_3 \geq 2$,

$$\begin{aligned} \Upsilon_{S_1\times S_2\times S_3}^{(\rho)} &\leq \min\left\{\min_{\{n_1,n_2\}\subset\{1,2,3\}}(4S_{n_1}S_{n_2}-2(S_{n_1}+S_{n_2})+1),\ \ 4\frac{d_1d_2d_3}{\max_n d_n}-3\right\} \\ &\leq \min\left\{\min_{\{n_1,n_2\}\subset\{1,2,3\}}(4S_{n_1}S_{n_2}-2(S_{n_1}+S_{n_2})+1),\ \ 4\frac{d_1d_2d_3}{\max_n d_n}-3\right\}, \end{aligned} \tag{49}$$

[12] C. Palazuelos. *On the largest Bell violation attainable by a quantum state* // J. Funct. Analysis **267**, 1959–1985 (2014).

in particular,

$$\Upsilon^{(\rho)}_{S\times S\times S} \leq \min\left\{(2S-1)^2,\ 4\frac{d_1d_2d_3}{\max_n d_n}-3\right\}. \tag{50}$$

For von Neumann measurements at each scenario site, based on Eq. (19) in [7]; Eq. (19) in [8] and Corollary 1 in [11], we have proved the following general bounds.

Theorem 10 [7, 8, 11]. *For correlation scenarios, performed on an N-qudit quantum state $\rho_{d,N}$ and with von Neumann measurements at each of scenario "sites":*

$$\Upsilon^{(\rho_{d,N})}_{2\times\cdots\times 2} \leq \min\left\{d^{\frac{N-1}{2}},\ 3^{N-1}\right\}, \quad S=2, \tag{51}$$

$$\Upsilon^{(\rho_{d,N})}_{S\times\cdots\times S} \leq \min\left\{\ d^{\frac{S(N-1)}{2}}, (\ 2\min\{d,S\}-1)^{N-1}\ \right\}, \quad S\geq 3. \tag{52}$$

The attainability of the new bounds derived in Theorems 9, 10 is discussed by us in articles [8, 11].

For $d=S=2$ and $N\geq 2$, the bound (51) is equal to $2^{\frac{N-1}{2}}$. For $N=2$, this upper bound on the maximal violation $\Upsilon^{(\rho_{d,N})}_{2\times\cdots\times 2}$ is attained on the Clauser–Horne–Shimoni–Holt inequality (CHSH) by Bell states, for $N\geq 3$ – on the Mermin–Klyshko inequality by the N-qubit GHZ state. This, in particular, proves that, in case of two dichotomic quantum observables at each site, these correlation Bell inequalities give the maximal quantum violation not only among the *correlation* Bell inequalities, but also *among all general Bell inequalities*.

For $S=N=2$ and a dimension $d\geq 9$, the upper bound (51) on the maximum violation $\Upsilon^{(\rho_{d,2})}_{2\times 2}$ is equal to "3", and, as we prove in [11], this upper bound is attained on the Bell inequality introduced earlier[13] by Zohren and Gill for joint probabilities and by bipartite states $\tau_{d,2}$, $d\to\infty$, that they describe numerically.

For comparison, we recall that for a bipartite state of an arbitrary dimension, the violation of the Clauser–Horne–Shimony–Holt (CHSH) inequality cannot exceed $\sqrt{2}$ (*Tsirelsson bound*) while the quantum violation *of every correlation Bell inequality* cannot exceed the real Grothendieck constant $K_G\in[1.676, 1.783]$.

2.2.7 Discussion

"Numerical" analytical bounds on the maximum violation of general Bell inequalities were also analyzed in articles of researchers from various international scientific groups studying the problem of quantifying on quantum nonlocality. Our results in Theorems 9, 10, 11 indicate:

1. For an arbitrary bipartite quantum state ρ our precise upper bound $(2\min\{S_1,S_2,d_1,d_2\}-1)$ in (48) significantly improves the precise bounds:

$$\begin{aligned}&\Upsilon^{(\rho_2)}_{S_1\times S_2} \leq 2K_G+1, \quad d_1=d_2=2,\\ &\Upsilon^{(\rho_2)}_{S_1\times S_2} \leq 2d_1d_2(K_G+1)-1, \quad \forall d_1,d_2\geq 2,\\ &\quad K_G\in[1.676, 1.783],\end{aligned} \tag{53}$$

[13] S. Zohren and R. D. Gill. *Maximal Violation of the Collins-Gisin-Linden-Massar-Popescu Inequality for Infinite Dimensional States* // Phys. Rev. Lett. **100**, 120406 (2008).

derived for arbitrary $S_1, S_2 \geq 1$ by Degorre&et al [*Lecture Notes in Computer Science* (2009) 5734, 270], also the approximate estimates

$$\Upsilon_{S\times S}^{(\rho_{d,2})} \preceq \min\{d, S\}, \qquad \Upsilon_{S\times S}^{(\rho_2)} \preceq \frac{d}{\ln d} \tag{54}$$

and the precise bound $\Upsilon_{S\times S}^{(\rho_{d,2})} \leq 2d$, presented in the whole series of papers by other authors within the operator spaces formalism, see, for example, in Junge and Palazuelos [*Comm. Math. Phys.* **306,** 695 (2011)]; Palazuelos [*J. Funct. Anal.* **267,** 1959 (2016)]; Palazuelos and Vidick [*J. Math. Phys.* (2016) **57**, 015220 (2016)].

2. For an arbitrary three-partite quantum state, the comparison of our upper bound $(2S-1)^2$ in (50) with the approximate bound $\succeq \sqrt{d}$, obtained by Junge, Palazuelos, Wolf [*Commun. Math. Phys.* **279**, 455 (2008)] for the state on $\mathbb{C}^d \otimes \mathbb{C}^D \otimes \mathbb{C}^D$, $d \geq 1$, shows that their approximate estimate $\succeq \sqrt{d}$ can be true only for a Bell inequality with a number S measurements at each site, satisfying the condition $(2S-1)^2 \succeq \sqrt{d}$.

3. For an arbitrary N-qudit state $\rho_{d,N}$, our upper bound $(2d-1)^{(N-1)}$ in (47) is better than the bound $\Upsilon_{S\times\cdots\times S}^{(\rho_{d,N})} \leq (2d)^{(N-1)}$, presented by Palazuelos and Vidick [*J. Math. Phys.* (2016) **57**, 015220] due to the operator spaces formalism.

To our knowledge, in the literature, there are no analogues of our general estimates (45), (47), (51), (52).

2.3 Modelling the probabilistic description of all joint von Neumann measurements

Let ρ be an arbitrary quantum state on a complex Hilbert space $\mathcal{H}$. Denote by $\mathfrak{X}_{\mathcal{H}}$ the set of all quantum observables on $\mathcal{H}$ and by $\mathcal{L}_{\mathcal{H}}^{(s)} \subseteq \mathfrak{X}_{\mathcal{H}}$ – the linear space of all self-adjoint bounded operators on $\mathcal{H}$.

Recall that the joint von Neumann measurement[14] of several quantum observables $X_1, \ldots, X_n \in \mathfrak{X}_{\mathcal{H}}$ with spectra $\mathrm{sp}X_j \subseteq \mathbb{R}$, $j = 1, \ldots, n$, is possible iff all values of the spectral measures $\mathrm{E}_{X_i}(\cdot)$ of self-adjoint operators X_i, $i = 1, \ldots, n$, mutually commute

$$[\mathrm{E}_{X_{i_1}}(B_{i_1}), \mathrm{E}_{X_{i_2}}(B_{i_2})] = 0, \quad B_i \in \mathcal{B}_{\mathrm{sp}X_i}, \quad i = 1, ..., n, \tag{55}$$

and is described in this case by the normalized projection-valued measure

$$\mathrm{E}_{(X_1,\ldots,X_n)}(B) = \int\limits_{(x_1,\ldots,x_n)\in B} \mathrm{E}_{X_1}(\mathrm{d}x_1) \cdot \ldots \cdot \mathrm{E}_{X_n}(\mathrm{d}x_n), \tag{56}$$

$$B \in \mathcal{B}_{\mathrm{sp}X_1\times\cdots\times\mathrm{sp}X_n}, \quad x_j \in \mathrm{sp}X_j, \quad j = 1, ..., n.$$

For bounded self-adjoint operators $X_1, ..., X_n$, condition (55) is equivalent to their mutual commutativity $[X_j, X_j] = 0$, $i, j = 1, ..., n$. Therefore, for short, we further refer to quantum

[14]According to the accepted terminology, this is a joint measurement of quantum observables described by their joint spectral measure.

observables $X_1, ..., X_n$, bounded or unbounded, satisfying the condition (55), as mutually commuting. The measure $\mathrm{E}_{(X_1,...,X_n)}(\cdot)$ is called the joint spectral measure of mutually commuting observable $X_1, ..., X_n$. The expression

$$\mathrm{tr}[\rho \mathrm{E}_{(X_1,...,X_n)}(B)] \tag{57}$$

gives the probability that, under the joint von Neumann measurement of mutually commuting quantum observables $X_1, ..., X_n$ in a state ρ, their observed values $(x_1, ..., x_n) \in B \in \mathcal{B}_{\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n}$.

In view of the general quasi-hidden variable (qHV) model developed by us in [5] and the proven existence (see Theorem 4) of a LqHV model (4) for every quantum correlation scenario, in article [6], we formulate and then analyze the answers to the following two questions.

1. Is it possible to describe all joint Neumann measurements on a quantum state ρ on an arbitrary Hilbert space $\mathcal{H}$ within a single qHV model $(\Omega, \mathcal{F}_\Omega, \mu_\rho)$ with a normalized real-valued measure μ_ρ, where, to each quantum observable $X \in \mathfrak{X}_\mathcal{H}$, there put into a one-to-one correspondence $X \overset{\Phi}{\mapsto} f_X$ a random variable $f_X : \Omega \to \mathbb{R}$, satisfying the condition $f_X(\Omega) = \mathrm{sp}X$, but the von Neumann linearity condition and the Cohen–Specker functional relations[15] on a mapping Φ are satisfied only in average, that is[16]:

$$\begin{aligned} \langle \varphi \circ X \rangle_\rho &= \langle \varphi \circ f_X \rangle_{qHV} && (58)\\ \langle X_1 + \cdots + X_n \rangle_\rho &= \langle f_{X_1} + \cdots + f_{X_n} \rangle_{qHV} && (59)\\ \langle X_1 \cdot \ldots \cdot X_n \rangle_\rho &= \langle f_{X_1} \cdot \ldots \cdot f_{X_n} \rangle_{qHV}, \quad n \in \mathbb{N}, && (60) \end{aligned}$$

where equality (58) holds for all quantum observables X and all bounded Borel functions $\varphi : \mathbb{R} \to \mathbb{R}$; equality (59) – for any finite set $X_1, \ldots, X_n$ of bounded quantum observables, and the relation (60) – for mutually commuting bounded quantum observables $X_1, \ldots, X_n$?

2. Does there exist a single qHV model $(\Omega, \mathcal{F}_\Omega, \mu_\rho)$ that correctly reproduces all joint von Neumann measurements on a state ρ and where each quantum observable X can be represented by a whole family $\{f_X^{(\theta)}, \theta \in \Theta_X\}$ of random variables on $(\Omega, \mathcal{F}_\Omega)$ satisfying the Cohen–Specker functional relation

$$\begin{aligned} \Psi(\varphi \circ f_X^{(\theta)}) &= \varphi \circ \Psi(f_X^{(\theta)}) && (61)\\ &= \varphi \circ X, \quad \forall X, \quad \forall \varphi : \mathbb{R} \to \mathbb{R}, \end{aligned}$$

but each of these random variables *equivalently represents* a quantum observable X in all joint von Neumann measurements on a state ρ – that is, regardless of a context of a joint von Neumann measurement, condition (60) is fulfilled under the substitution into its right-hand side of any of random variables $f_{X_n}^{(\theta_n)}$, $\theta_n \in \Theta_{X_n}$ on $(\Omega, \mathcal{F}_\Omega)$, modelling

[15]These are the corresponding conditions in the setting of the (no-going) von Neumann theorem and in the setting of the (no-going) Cohen–Specker theorem, for details, see sections Introduction and Preliminaries in our article [6].

[16]Here, $\langle X \rangle_\rho := \mathrm{tr}\,[\rho X]$ and $\langle f_X \rangle_{qHV} := \int_\Omega f_X(\omega)\mu_\rho(\mathrm{d}\omega)$.

quantum observables X_n on $\mathcal{H}$?

Based on the generalization of Kolmogorov's extension theorem of consistent probability measures to the case of consistent operator-valued measures, formulated and proved by us in the article [6], we have proved [6] that the answers to both of the above questions are positive – there do exist quasi hidden variable (qHV) models: *statistically noncontextual and context-invariant*, described above in items (1), (2), respectively.

2.3.1 The extension theorem

In article [6] we consider the case of an arbitrary a complex separable Hilbert space $\mathcal{H}$, finite-dimensional or infinite-dimensional. In this section, for short of notation and presentation, we assume $\mathcal{H}$ to be finite-dimensional.

For each tuple $(X_1, ..., X_n)$ of mutually different quantum observables on $\mathcal{H}$ with spectra spX_1,...,spX_n and spectral measures $\mathrm{E}_{X_j}(\cdot)$, $j = 1, ..., n$, we define on the measurable space $(\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n,\ \mathcal{B}_{\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n})$ the operator-valued measure

$$\begin{aligned} \mathcal{P}_{(X_1,...,X_n)}(B) \quad &: \quad = \frac{1}{n!} \sum\nolimits_{(x_1,...,x_n) \in B} \{\mathrm{E}_{X_1}(x_1) \cdot \ldots \cdot \mathrm{E}_{X_n}(x_n)\}_{sym}\,, \\ \mathcal{P}_{(X_1,...,X_n)}(\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n) \quad &= \quad \mathbb{I}_{\mathcal{H}} \in \mathfrak{X}_{\mathcal{H}}, \quad B \in \mathcal{B}_{\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n}, \end{aligned} \tag{62}$$

with values in $\mathcal{L}_{\mathcal{H}}^{(s)}$. Denote by

$$\mathcal{P}_{sym}^{(\mathcal{H})} : = \left\{ \mathcal{P}_{(X_1,...,X_n)} \mid \{X_1, ..., X_n\} \subset \mathfrak{X}_{\mathcal{H}}, \quad n \in \mathbb{N} \right\} \tag{63}$$

the family of all measures (62). If observables $X_1, ..., X_n$ mutually commute, then measure $\mathcal{P}_{(X_1,...,X_n)}$ constitutes the joint spectral measure $\mathrm{E}_{(X_1,...,X_n)}$ of this set of self-adjoint operators and is projection-valued.

Proposition 12 (Lemma 1 in [6]). *For every tuple* $\{X_1, ..., X_n\} \subset X_{\mathcal{H}}$, $n \in N$, *of quantum observables on* $\mathcal{H}$: (i) *the relation*

$$\begin{aligned} \mathcal{P}_{(X_1,...,X_n)}(B_1 \times \cdots \times B_n) &= \mathcal{P}_{(X_{i_1},...,X_{i_n})}(B_{i_1} \times \cdots \times B_{i_n}), \\ B_1 \in \mathcal{B}_{\mathrm{sp}X_1}, ..., B_n &\in \mathcal{B}_{\mathrm{sp}X_n}, \end{aligned} \tag{64}$$

holds for all permutations $\binom{1,...,n}{i_1,...,i_n}$ *indices;* (ii) *the relation*

$$\begin{aligned} &\mathcal{P}_{(X_1,...,X_n)}\left(\{\ (x_1, ..., x_n) \in \mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n \mid (x_{i_1}, ..., x_{i_k}) \in B\}\right) \\ = \ &\mathcal{P}_{(X_{i_1},...,X_{i_k})}\left(F\right), \qquad B \in \mathcal{B}_{\mathrm{sp}X_{i_1} \times \cdots \times \mathrm{sp}X_{i_k}}, \end{aligned} \tag{65}$$

is true for all $\{X_{i_1}, ..., X_{i_k}\} \subseteq \{X_1, ..., X_n\}$.

For family (63) of the operator-valued measures (62), relations (64), (65) are similar by their form to Kolmogorov's conditions on the consistency[17] of probability measures

$$\{\mu_{(t_1,...,t_n)} : \mathcal{B}_{\mathbb{R}^\tau} \to [0, 1] \ \mid \tau = (t_1, ..., t_n), \quad t_i \in \mathrm{T}, \quad n \in \mathbb{N}\}, \tag{66}$$

[17] See, for example, in A. N. Shiryaev. *Probability* (Berlin: Springer-Verlag, 1996).

each specified by a tuple $\tau = (t_1, ..., t_n)$ of mutually different elements of a set T of indices.

Consider the Cartesian product $\Lambda := \prod_{X \in \mathfrak{X}_{\mathcal{H}}} \mathrm{sp}X$ of the spectra of all quantum observables on $\mathcal{H}$. By its definition[18] Λ is the set of all functions $\lambda : \mathfrak{X}_{\mathcal{H}} \to \cup_{X \in \mathfrak{X}_{\mathcal{H}}} \mathrm{sp}X$ with values $\lambda(X) \equiv \lambda_X \in \mathrm{sp}X$.

Let the random variable $\pi_{(X_1,...,X_n)} : \Lambda \to \mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n$ be the canonical projection on Λ:

$$\begin{aligned} \pi_{(X_1,...,X_n)}(\lambda) \ &: = (\pi_{X_1}(\lambda), ..., \pi_{X_n}(\lambda)), \\ \pi_{X_i}(\lambda) \ &: = \lambda_{X_i} \in \mathrm{sp}X_i. \end{aligned} \tag{67}$$

The set

$$\mathfrak{A}_\Lambda = \left\{ \pi^{-1}_{(X_1,...,X_n)}(B) \subseteq \Lambda \mid B \in \mathcal{B}_{\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n}, \ \ \{X_1, ..., X_n\} \subset \mathfrak{X}_{\mathcal{H}}, \ \ n \in \mathbb{N} \right\} \tag{68}$$

of all cylindrical sets of the set Λ of the form

$$\pi^{-1}_{(X_1,...,X_n)}(B) := \{\lambda \in \Lambda \mid (\pi_{X_1}(\lambda), ..., \pi_{X_n}(\lambda)) \in B\}, \tag{69}$$

constitutes an algebra on Λ.

A generalization of Kolmogorov's extension theorem of consistent probability measures in (66) to a general case of consistent $\mathcal{L}^{(s)}_{\mathcal{H}}$- valued measures is formulated and proved by us in Lemma 3 in [6]. For consistent $\mathcal{L}^{(s)}_{\mathcal{H}}$-valued measures of family (63), this generalization results in the following statement.

Theorem 11 (Theorem 1 and Lemma 3 in [6]). *For family (63) of $\mathcal{L}^{(s)}_{\mathcal{H}}$-valued measures, there exists a unique normalized finitely additive $\mathcal{L}^{(s)}_{\mathcal{H}}$-valued measure*

$$\mathbb{M} : \mathfrak{A}_\Lambda \to \mathcal{L}^{(s)}_{\mathcal{H}}, \qquad \mathbb{M}(\Lambda) = \mathbb{I}_{\mathcal{H}}, \tag{70}$$

on $(\Lambda, \mathfrak{A}_\Lambda)$ such that

$$\mathcal{P}_{(X_1,...,X_n)}(B) = \mathbb{M}\left(\pi^{-1}_{(X_1,...,X_n)}(B)\right), \quad B \in \mathcal{B}_{\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n}, \tag{71}$$

in particular,

$$\begin{aligned} &\frac{1}{n!}\{\mathrm{E}_{X_1}(B_1) \cdot \ldots \cdot \mathrm{E}_{X_n}(B_n)\}_{\mathrm{sym}} = \mathbb{M}(\pi^{-1}_{X_1}(B_1) \cap \cdots \cap \pi^{-1}_{X_n}(B_n)), \\ &\qquad B_i \in \mathcal{B}_{\mathrm{sp}X_i}, \quad i = 1, ..., n, \end{aligned} \tag{72}$$

for all subsets $\{X_1, ..., X_n\} \subset \mathfrak{X}_{\mathcal{H}}$, $n \in \mathbb{N}$, of quantum observables on $\mathcal{H}$ with spectral measures $\mathrm{E}_{X_1}, \ldots, \mathrm{E}_{X_n}$.

If all quantum observables X_j in a tuple $(X_1, ..., X_n)$ mutually commute, then measure $\mathcal{P}_{(X_1,...,X_n)}(\cdot)$ coincides with the joint spectral measure $\mathrm{E}_{(X_1,...,X_n)}(\cdot)$ of these self-adjoint operators, and therefore, according to (71), for this tuple $(X_1, ..., X_n)$ of quantum observables, the value $\mathbb{M}\left(\pi^{-1}_{(X_1,...,X_n)}(B)\right)$ of measure (70) constitutes an orthogonal projection on $\mathcal{H}$.

[18] N. Dunford and J. T. Schwartz (1957). *Linear Operators. Part I: General theory* (New York: Interscience).

For an arbitrary quantum state ρ on $\mathcal{H}$, Theorem 11 allows us to express the real-valued measures

$$\operatorname{tr}[\rho\mathcal{P}_{(X_1,...,X_n)}(\cdot)], \qquad \mathcal{P}_{(X_1,...,X_n)} \in \mathcal{P}_{sym}^{(\mathcal{H})}, \tag{73}$$

via a unique real-valued measure on $(\Lambda, \mathfrak{A}_\Lambda)$.

Theorem 12 (Proposition 1 in [6]). *Let $\{\mathcal{P}_{(X_1,...,X_n)}\}$ be the family (63) of $\mathcal{L}_{\mathcal{H}}^{(s)}$-valued measures. To each quantum state ρ on a complex separable Hilbert space $\mathcal{H}$ there corresponds $(\rho \mapsto \mu_\rho)$ a unique normalized finitely additive real-valued measure*

$$\mu_\rho : \mathfrak{A}_\Lambda \to \mathbb{R}, \quad \mu_\rho(\Lambda) = 1, \tag{74}$$

on $(\Lambda, \mathfrak{A}_\Lambda)$ such that

$$\operatorname{tr}[\rho\mathcal{P}_{(X_1,...,X_n)}(B)] = \mu_\rho\left(\pi^{-1}_{(X_1,...,X_n)}(B)\right), \quad B \in \mathcal{B}_{\mathrm{sp}X_1\times\cdots\times\mathrm{sp}X_n}, \tag{75}$$

in particular,

$$\frac{1}{n!}\operatorname{tr}[\rho\{\ \mathrm{E}_{X_1}(B_1)\cdot\ldots\cdot\mathrm{E}_{X_n}(B_n)\}_{\mathrm{sym}}] = \mu_\rho\left(\pi^{-1}_{X_1}(B_1)\cap\cdots\cap\pi^{-1}_{X_n}(B_n)\right), \tag{76}$$
$$B_i \in \mathcal{B}_{\mathrm{sp}X_i}, \quad i = 1, ..., n,$$

for all collections $\{X_1, ..., X_n\} \subset \mathfrak{X}_{\mathcal{H}}$, $n \in \mathbb{N}$, of quantum observables on $\mathcal{H}$ with spectral measures $\mathrm{E}_{X_1}, \ldots, \mathrm{E}_{X_n}$.

From Theorem 12 it follows that the representation

$$\operatorname{tr}[\rho\{X_1\cdot\ldots\cdot X_n\}] = \int_\Lambda \pi_{X_1}(\lambda)\cdot\ldots\cdot\pi_{X_n}(\lambda)\ \mu_\rho(\mathrm{d}\lambda) \tag{77}$$

holds for any finite set $\{X_1, \ldots, X_n\}$ of bounded mutually commuting quantum observables on $\mathcal{H}$.

2.3.2 Statistically noncontextual qHV model

Recall that in case of mutually commuting quantum observables $X_1, ..., X_n$, for any Borel function $\psi : \mathbb{R}^n \to \mathbb{R}$ the quantum observable $\psi(X_1, ..., X_n)$ is defined by

$$\psi(X_1, ..., X_n) := \int_{\mathbb{R}^n} \psi(x_1, ..., x_n)\ \mathrm{E}_{(X_1,...,X_n)}(\mathrm{d}x_1 \times\cdots\times \mathrm{d}x_n)$$

and, for any Borel function $\varphi : \mathbb{R} \to \mathbb{R}$, the spectral measure $\mathrm{E}_{\varphi(X)}(B) = \mathrm{E}_X(\varphi^{-1}(B))$, $B \in \mathcal{B}_{\mathrm{sp}\varphi(X)}$.

The positive answer to the question, posed in item (1) of Section 2.3, is given by the below statements based on the application of Theorems 11, 12.

Theorem 13 (Theorem 2 and Proposition 2 in [6]). *Let ρ be an arbitrary quantum state on a Hilbert space $\mathcal{H}$. There exist:*
(i) *a measurable space* $(\Omega, \mathcal{F}_\Omega)$;
(ii) *a normalized real-valued measure* ν_ρ *on* $(\Omega, \mathcal{F}_\Omega)$;
(iii) *a set* $\{f_X : \Omega \to \mathrm{sp}X \mid f_X(\Omega) = \mathrm{sp}X,\ X \in \mathfrak{X}_\mathcal{H}\}$ *of random variables on* $(\Omega, \mathcal{F}_\Omega)$ *one-to-one corresponding to the set* $\mathfrak{X}_\mathcal{H}$ *of all quantum observables on* $\mathcal{H}$;
such that for all collections $\{X_1, ..., X_n\}$, $n \in \mathbb{N}$, *of mutually commuting quantum observables on* $\mathcal{H}$, *the probability measures (57) describing the joint von Neumann measurements on a state* ρ *admit the qHV representation*

$$\begin{aligned} \mathrm{tr}[\rho \mathrm{E}_{(X_1,...,X_n)}(B)] &= \nu_\rho\left(f^{-1}_{(X_1,...X_n)}(B)\right), \quad B \in \mathcal{B}_{\mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n}, \\ f_{(X_1,...,X_n)} &: = (f_{X_1}, ..., f_{X_n}), \end{aligned} \tag{78}$$

in particular,

$$\begin{aligned} \mathrm{tr}[\ \rho\{\mathrm{E}_{X_1}(B_1) \cdot \ldots \cdot \mathrm{E}_{X_n}(B_n)\}] &= \nu_\rho\left(\ f^{-1}_{X_1}(B_1) \cap \cdots \cap f^{-1}_{X_n}(B_n)\right) \\ &\equiv \int_\Omega \chi_{f^{-1}_{X_1}(B_1)}(\omega) \cdot \ldots \cdot \chi_{f^{-1}_{X_n}(B_n)}(\omega) \nu_\rho(d\omega), \end{aligned} \tag{79}$$

for all $B_i \in \mathcal{B}_{\mathrm{sp}X_i}$, $i = 1, ..., n$.

From Theorem 14 it follows that the relation

$$\begin{aligned} \langle X_1 + ... + X_n \rangle_\rho &: = \mathrm{tr}[\rho(X_1 + ... + X_n)] \\ &= \int_\Omega (f_{X_1}(\omega) + \cdots + f_{X_n}(\omega)) \nu_\rho (\mathrm{d}\omega) \\ &: = \langle f_{X_1} + \cdots + f_{X_n} \rangle_{qHV} \end{aligned} \tag{80}$$

holds for each collection $\{X_1, ..., X_n\}$ of bounded quantum observables on $\mathcal{H}$.

Proposition 13 (Corollary 2 in [6]). *In the setting of Theorem 14:*
(i) *the relation*

$$\langle \psi \circ X \rangle_\rho := \mathrm{tr}[\rho\psi(X)] = \int_\Omega (\psi \circ f_X)(\omega) \nu_\rho(\mathrm{d}\omega) := \langle \psi \circ f_X \rangle_{qHV} \tag{81}$$

holds for any bounded Borel function $\psi : R \to R$;
(ii) *the noncontextual qHV representation*

$$\begin{aligned} \langle X_1 \cdot \cdots \cdot X_n \rangle_\rho &: = \mathrm{tr}[\rho(X_1 \cdot \ldots \cdot X_n)] = \int_\Omega f_{X_1}(\omega) \cdot \ldots \cdot f_{X_n}(\omega) \nu_\rho(\mathrm{d}\omega) \\ &: = \langle f_{X_1} \cdot \ldots \cdot f_{X_n} \rangle_{qHV} \end{aligned} \tag{82}$$

is true for every finite collection $\{X_1, ..., X_n\}$ *of bounded mutually commuting quantum observables on* $\mathcal{H}$.

Theorem 13 and relations (80)–(82) prove [6] the possibility of simulating *all* joint von Neumann measurements performed on a quantum state ρ on an arbitrary Hilbert space $\mathcal{H}$ within *a statistically noncontextual qHV model* formulated above in item (1) of Section 2.3.

In a statistically noncontextual qHV model, each quantum observable X on a Hilbert space $\mathcal{H}$ is represented by only one random variable f_X on some measurable space $(\Omega, \mathcal{F}_\Omega)$, modelling X in all joint von Neumann measurements; the Cohen–Specker functional relation on a one-to-one mapping $X \mapsto f_X$ does not need to be fulfilled, moreover, by Cohen–Specker theorem it cannot hold if $\dim \mathcal{H} \geq 3$.

However, this new probability qHV model correctly reproduces via representation (78) the probabilities of all joint von Neumann measurements on a state ρ on $\mathcal{H}$ and the Cohen–Specker functional relation holds in this model *in average* – in the sense of relations (81), (82).

2.3.3 Context-invariant qHV model

Based on Theorem 12, we also formulate and prove in [6] the following new general statements.

Theorem 14 (Theorem 3 in [6]). *For all joint von Neumann measurements performed on a quantum state ρ on a complex Hilbert space $\mathcal{H}$ there exist:*
(i) *a measurable space* $(\Omega, \mathcal{F}_\Omega)$;
(ii) *a normalized real-valued measure ν_ρ on* $(\Omega, \mathcal{F}_\Omega)$;
(iii) *a mapping $\Psi : \mathfrak{F}_{\mathfrak{X}_\mathcal{H}} \rightarrow \mathfrak{X}_\mathcal{H}$ of a set $\mathfrak{F}_{\mathfrak{X}_\mathcal{H}}$ of random variables $g(\omega)$ on $(\Omega, \mathcal{F}_\Omega)$ onto the set $\mathfrak{X}_\mathcal{H}$ of all quantum observables X on $\mathcal{H}$, satisfying the spectral rule $g(\Omega) = spX$ for each random variable $g \in \Psi^{-1}(\{X\})$ and the functional relation*

$$\begin{aligned} \varphi \circ g &\in \mathfrak{F}_{\mathfrak{X}_\mathcal{H}}, \quad \Psi(\varphi \circ g) = \varphi \circ \Psi(g) = \varphi \circ X, \\ \forall g &\in \Psi^{-1}(\{X\}), \quad X \in \mathfrak{X}_\mathcal{H}, \end{aligned} \tag{83}$$

for all Borel functions $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ and all quantum observables $X \in \mathfrak{X}_\mathcal{H}$;
such that the context-invariant qHV representation

$$\begin{aligned} &\mathrm{tr}[\rho\{\mathrm{E}_{X_1}(B_1) \cdot \ldots \cdot \mathrm{E}_{X_n}(B_n)\}] = \nu_\rho\left(g_1^{-1}(B_1) \cap \cdots \cap g_n^{-1}(B_n)\right), \\ &\quad \forall g_i \in \Psi^{-1}(\{X_i\}), \end{aligned} \tag{84}$$

holds for all $B_i \in \mathcal{B}_{\mathrm{sp}X_i}$, $i = 1, ..., n$, and all finite collections $\{X_1, ..., X_n\} \subset \mathfrak{X}_\mathcal{H}$, $n \in \mathbb{N}$, of mutually commuting quantum observables on $\mathcal{H}$.

Proposition 14 (Corollary 3 in [6]). *In the setting of Theorem 15, for all finite collections $\{X_1, ..., X_n\} \subset \mathfrak{X}_\mathcal{H}$ of mutually commuting quantum observables on $\mathcal{H}$ the context-invariant qHV representation*

$$\begin{aligned} \mathrm{tr}[\rho\psi(X_1, ..., X_n)] &= \int_\Omega \psi(g_1(\omega), ..., g_n(\omega))\nu_\rho(\mathrm{d}\omega), \\ \forall g_i &\in \Psi^{-1}(\{X_i\}), \quad i = 1, ..., n, \end{aligned} \tag{85}$$

holds for any Borel function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}$ and the context-invariant qHV representation

$$\begin{aligned} \mathrm{tr}[\rho(X_1 \cdot \ldots \cdot X_n)] &= \int_\Omega g_1(\omega) \cdot \ldots \cdot g_n(\omega)\nu_\rho(\mathrm{d}\omega), \\ \forall g_i &\in \Psi^{-1}(\{X_i\}), \quad i = 1, ..., n, \end{aligned} \tag{86}$$

is true if mutually commuting quantum observables $X_1, ..., X_n$ are bounded.

Theorem 14 and Proposition 14 prove [6] the possibility of modelling all joint von Neumann measurements on a quantum state ρ on an arbitrary Hilbert space $\mathcal{H}$ within a *context-invariant qHV model,* formulated above in item (2) of Section 2.3.

Our term "context-invariant" in Theorem 14 means that *regardless of the context* of a joint von Neumann measurement of mutually commuting quantum observables $X_1, \dots, X_n$, into the right-hand sides of qHV representations (84)–(86) of this model, each of random variables $g_i \in \Psi^{-1}(\{X_i\})$, modelling on $(\Omega, \mathcal{F}_\Omega)$ quantum observables X_i, $i = 1, ..., n$, can be equivalently substituted.

For a dimension $\dim \mathcal{H} \geq 3$, the context-invariant qHV model cannot be noncontextual, for $\dim \mathcal{H} = 2$ it does not need to be noncontextual.

2.4 LqHV model for an N-partite quantum state

The representation (79) is valid for all finite collections $\{X_1, ..., X_n\}$ of mutually commuting quantum observables, in particular, for observables

$$X_1 = Z_1 \otimes \mathbb{I}_{\mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N}, ..., X_N := \mathbb{I}_{\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{(N-1)}} \otimes Z_N, \tag{87}$$

on Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. Therefore, for all correlation scenarios, performed on an N-partite quantum state ρ and with an arbitrary number $S_n \geq 1$ of von Neumann measurements on each n-th site, we have the following statement.

Theorem 15 (Proposition 3 in [6]). *For all correlation scenarios (24), performed on an N-partite quantum state on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and with an arbitrary number of von Neumann measurements at each of N sites, there exists a unified LqHV model, where for every tuple $(Z_1, ..., Z_N)$ of quantum observables, each on a space $\mathcal{H}_n$, respectively, all joint von Neumann probability distributions*

$$\mathrm{tr}[\rho\{\mathrm{E}_{Z_1}(\cdot) \otimes \cdots \otimes \mathrm{E}_{Z_N}(\cdot)\}]$$

admit the LqHV representation

$$\mathrm{tr}[\rho\{\mathrm{E}_{Z_1}(B_1) \otimes \cdots \otimes \mathrm{E}_{Z_N}(B_N)\}] = \int\limits_{\Omega} \mathrm{P}_{Z_1}(B_1 \mid \omega) \cdot \ldots \cdot \mathrm{P}_{Z_N}(B_N \mid \omega) \nu_\rho(d\omega), \tag{88}$$

$$B_i \in \mathcal{B}_{\mathrm{sp} X_i}, \quad i = 1, ..., n,$$

in terms of a single measure space $(\Omega, \mathcal{F}_\Omega, \nu_\rho)$ with a normalized real-valued measure ν_ρ and conditional probability distributions $\mathrm{P}_{Z_n}(\cdot|\omega)$ each being "local" in the sense that it depends only on the corresponding observable Z_n on a Hilbert space $\mathcal{H}_n$.

Similarly to the notion of a LHV model for an N-partite quantum state introduced by Werner[19], we call (88) as a LqHV model for an N-partite quantum state ρ.

Theorem 15 proves that, *for every N-partite quantum state, an LqHV model does exist.*

[19] R.F. Werner. *Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden variable model* // Phys. Rev. A **40**, 4277 (1989).

2.5 Stability of quantum nonlocality to noise

On the basis of the new developed formalism for the description of quantum nonlocality, we also investigate in articles [10, 12] the stability of nonlocality of an N-partite quantum state to a local noise of any type. Let ρ be an arbitrary nonlocal state on a finite-dimensional Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$. In the presence of a quantum noise, its noisy state is described by the mixture

$$\tau_{\rho_{d,N}}^{(\beta)}(\zeta_{loc}) := (1-\beta)\,\zeta_{loc} \;+ \beta\rho_{d,N}, \quad \beta \in [0,1], \tag{89}$$

where ζ_{loc} is a fully Bell local quantum state on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$, describing a quantum noise.

Based on the sufficient conditions for the Bell locality of a quantum state found in Theorem 8, we derived [10] the following new bounds on full Bell locality of an N-qudit state $\rho_{d,N}$ in the presence of the white noise $\zeta_{w\text{-}n} = \mathbb{I}_d^{\otimes N}/d^N$.

Theorem 16 (Propositions 2, 3 in [10]). *In the presence of the white noise:*
(i) *the noisy state (89) of the N-qudit GHZ state $\rho_{d,N}^{ghz}$, $d \geq 2$, $N \geq 3$, is fully Bell local for all*

$$\beta \leq \frac{1}{1+2d^{N-1}(d-1)^{N-1}}; \tag{90}$$

(ii) *the noisy state (89) for an arbitrary nonlocal N-qudit state $\rho_{d,N}$, $d \geq 2$, $N \geq 3$, is fully Bell local for all $\beta \leq \beta_{\rho_{d,N}}^{(loc)}$, where*

$$\frac{1}{d^N(2d-1)^{N-1} - d^N + 1} \leq \beta_{\rho_{d,N}}^{(loc)} \leq \frac{1}{d(2d-1)^{N-1} - d + 1}. \tag{91}$$

For $N = 2$, the bounds in Theorem 16 are also true, but fall into the known ranges of the full separability of the considered noisy states and are not, therefore, interesting.

Theorem 16 implies that the noisy N-qudit GHZ state $\tau_{\rho_{d,N}^{ghz}}^{(\beta)}(\zeta_{w\text{-}n})$ is fully Bell local for all $\beta \leq \frac{1}{1+2d^{N-1}(d-1)^{N-1}}$. At the same time, it is known that, for all $N \geq 3$ and non-prime $d > 3$, this state is fully separable for $\beta \leq \beta_{\rho_{d,N}^{ghz}}^{(sep)}$, where

$$\frac{1}{1+d^{2N-1}} \leq \beta_{\rho_{d,N}^{ghz}}^{(sep)} \leq \frac{1}{1+d^{N-1}}. \tag{92}$$

Since the full separability of a state implies its full Bell locality and since $\frac{1}{1+d^{2N-1}} < \frac{1}{1+2d^{N-1}(d-1)^{N-1}}$, then, for all $N \geq 3$ and non-prime $d > 3$, our bound (90) for the full Bell locality of the noisy GHZ state $\tau_{\rho_{d,N}^{GHZ}}^{(\beta)}(\zeta_{w\text{-}n})$ does not fall into the exactly known range $\beta \leq \frac{1}{1+d^{2N-1}}$ for its full separability and is, therefore, new. However, the following question remains to be open – for $N \geq 3$ and non-prime $d > 3$, is the noisy GHZ state $\tau_{\rho_{d,N}^{ghz}}^{(\beta)}(\zeta_{w\text{-}n})$ for

$$\frac{1}{1+d^{2N-1}} < \beta \leq \frac{1}{1+2d^{N-1}(d-1)^{N-1}} \tag{93}$$

fully separable or fully nonseparable. It is only known with certainty that for a non-prime $d > 3$ and $N \geq 3$, the noisy N-qudit GHZ state $\tau_{\rho_{d,N}^{ghz}}^{(\beta)}(\zeta_{w\text{-}n})$ is fully nonseparable for all $\beta > \frac{1}{1+d^{N-1}}$.

For an arbitrary N-qudit state $\rho_{d,N}$, the lower bound in (91) falls into the known range $\beta \leq \beta_{\rho_{2,3}}^{(sep)} = \frac{1}{d^{2N-1}+1}$ for the full separability of its noisy state $\tau_{\rho_{d,N}}^{(\beta)}(\zeta_{w\text{-}n})$, however, for all $d \geq 2$,

$N \geq 3$, the upper bound in (91) is significantly beyond this range. For an N-qudit state $\rho_{d,N}$, this leads to the significant gap

$$\Delta_{\rho_{d,N}}^{\max} = \frac{1}{d(2d-1)^{N-1} - d + 1} - \frac{1}{d^{2N-1}+1}, \qquad d \geq 2, \quad N \geq 3. \tag{94}$$

between our new bound (91) on its full Bell locality and the critical value $\frac{1}{d^{2N-1}+1}$ for its full separability. For example, for an arbitrary three-qubit state $\rho_{2,3}$, this gap is equal to $0.94\beta_{\rho_{2,3}}^{(sep)}$, so that, for a state $\rho_{2,3}$, our upper bound in (91) on the full Bell locality of its noisy state can be almost twice more than the well-known critical value $\beta_{\rho_{2,3}}^{(sep)}$ on the full separability of this noisy state.

2.5.1 Tolerance to any local quantum noise

Let $\beta_{S_1\times\cdots\times S_N}^{(\rho)}(\zeta_{loc}) \in (0,1]$ be the *threshold "visibility"* for $S_1 \times \cdots \times S_N$-setting nonlocality of a quantum state ρ in the presence of a local quantum noise ζ_{loc}. In other words, the noisy state $\tau_{\rho_{d,N}}^{(\beta)}(\zeta_{loc})$ is $S_1 \times \cdots \times S_N$-setting Bell nonlocal iff $\beta \in (\beta_{S_1\times\cdots\times S_N}^{(\rho)}(\zeta_{loc}), 1\,]$, and $S_1 \times \cdots \times S_N$-setting Bell local iff $\beta \in \ [0, \beta_{S_1\times\cdots\times S_N}^{(\rho)}(\zeta_{loc})\ ]$.

Denote by $\mathfrak{X}_{S_1\times\cdots\times S_N}^{(n\text{-}loc)}$ the set of all $S_1 \times \cdots \times S_N$-setting Bell nonlocal N-partite states on a Hilbert space $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$ and by $\mathfrak{X}_N^{(n\text{-}loc)} \supset \mathfrak{X}_{S_1\times\cdots\times S_N}^{(n\text{-}loc)}$ – the set of all (*overall*) *Bell* nonlocal N-partite quantum states on $\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_N$.

Definition 12 *(*Definition 2 in [12]*). For a Bell nonlocal N-partite quantum state $\rho \in \mathfrak{X}_N^{(n\text{-}loc)}$, we call the parameter*

$$\begin{aligned} \mathfrak{T}_{S_1\times\cdots\times S_N}^{(\rho)} &:= \sup_{\zeta_{loc}} \beta_{S_1\times\cdots\times S_N}^{(\rho)}(\zeta_{loc}) \\ &= \inf\left\{\beta \in [0,1] \mid (1-\beta)\,\zeta_{loc} + \beta\rho \in \mathfrak{X}_{S_1\times\cdots\times S_N}^{(n\text{-}loc)},\ \forall \zeta_{loc}\right\} \end{aligned} \tag{95}$$

as the tolerance to any local noise under all $S_1 \times \cdots \times S_N$-setting correlation scenarios on this state, for short, the $S_1 \times \cdots \times S_N$-tolerance.

For a state $\rho \in \mathfrak{X}_N^{(n\text{-}loc)}$, its noisy state (89) is $S_1 \times \cdots \times S_N$-setting Bell nonlocal in the presence of any noise ζ_{loc} iff $\beta \in (\mathfrak{T}_{S_1\times\cdots\times S_N}^{(\rho)}, 1]$.

Definition 13 *(*Definition 3 in [12]*). For a Bell nonlocal N-partite quantum state $\rho \in \mathfrak{X}_N^{(n\text{-}loc)}$, we call the parameter*

$$\mathfrak{T}_\rho := \inf_{S_1,\ldots,S_N} \mathfrak{T}_{S_1\times\cdots\times S_N}^{(\rho)} \tag{96}$$

as its tolerance to any local noise.

In this terminology, for any overall Bell nonlocal quantum state $\rho \in \mathfrak{X}_N^{(n\text{-}loc)}$, for all $\beta \in (\mathfrak{T}_\rho, 1]$, the noisy state $\tau_\rho^{(\beta)}(\zeta_{loc})$ is overall Bell nonlocal for any local noise ζ_{loc}.

The smaller is the value of the noise tolerance $\mathfrak{T}_\rho$, the greater is the maximal amount $\mathfrak{M}_\rho = 1 - \mathfrak{T}_\rho$ of a local noise of any type tolerated by this nonlocal state, therefore, the greater is the robustness of nonlocality of a state ρ to any local noise.

Theorem 17 (Proposition 1 in [12]). *For an arbitrary N-partite quantum state ρ,*

$$\mathfrak{T}^{(\rho)}_{S_1\times\cdots\times S_N} = \frac{2}{1+\Upsilon^{(\rho)}_{S_1\times\cdots\times S_N}}, \tag{97}$$

$$\mathfrak{T}_\rho = \frac{2}{1+\Upsilon_\rho}, \tag{98}$$

where $\Upsilon^{(\rho)}_{S_1\times\cdots\times S_N}$ – the maximal Bell violation by a state ρ of all general $S_1 \times \cdots \times S_N$-setting Bell inequalities and Υ_ρ – the maximal violation of all general Bell inequalities.

2.5.2 General estimates

In view of the bounds derived by us in Theorems 9, 10 on the maximal violation by a state ρ of general Bell inequalities and the relations (97), (98), we have the following bounds on the stability of an N-partite state nonlocality to a noise.

Proposition 15 (Eqs. 42–44 in [12]). *For an arbitrary nonlocal N-qudit quantum state $\rho_{d,N}$, $d \geq 2$, $N \geq 2$, the tolerances $\mathfrak{T}^{(\rho)}_{S_1\times\cdots\times S_N}$ and $\mathfrak{T}_\rho$ admit the bounds:*
(i) *in case of von Neumann measurements at each "site"*

$$\mathfrak{T}^{(\rho_{d,N})}_{2\times\cdots\times 2} \geq \frac{2}{1+\min\left\{d^{\frac{N-1}{2}},\ 3^{N-1}\right\}}, \tag{99}$$

$$\mathfrak{T}^{(\rho_{d,N})}_{S\times\cdots\times S} \geq \frac{2}{1+\min\left\{d^{\frac{S(N-1)}{2}}, (\ 2\min\{d,S\}-1)^{N-1}\right\}}, \quad S \geq 3;$$

(ii) *for any type of quantum measurements at each of "sites"*

$$\mathfrak{T}^{(\rho_{d,N})}_{S\times\cdots\times S} \geq \frac{2}{1+(\ 2\min\{d,S\}-1)^{N-1}}, \quad S \geq 2, \tag{100}$$

$$\mathfrak{T}_{\rho_{d,N}} \geq \frac{2}{1+(\ 2d-1)^{N-1}}.$$

For the most well-known N-partite quantum states, used in quantum information processing, we derive the following new bounds on the stability of their nonlocality to an arbitrary quantum noise.

Proposition 16 *[12]*
(1) *For the N-qubit GHZ state $\rho_{2,N}^{ghz}$, $N \geq 2$ (Eq. 65 in [12])*:

$$\frac{1}{1+2^{N-2}} \leq \mathfrak{T}_{\rho_{2,N}^{GHZ}} \leq \frac{2}{1+2^{\frac{N-1}{2}}}. \tag{101}$$

(2) *For the N-qudit GHZ state $\rho_{d,N}^{ghz}$, $d \geq 2$, $N \geq 2$ (Eqs. 55, 56 in [12]):*

$$\begin{aligned} \mathfrak{T}_{S\times\cdots\times S}^{(\rho_{d,N}^{GHZ})} &\geq \frac{2}{1+\min\left\{(2S-1)^{N-1}, 1+2^{N-1}(d-1)\right\}}, \quad S \geq 2, \\ \mathfrak{T}_{\rho_{d,N}^{GHZ}} &\geq \frac{1}{1+2^{N-1}(d-1)}. \end{aligned} \tag{102}$$

(3) *For the N-qubit Dicke state $|D_N^{(k)}\rangle$, $k = 1, 2, ..., (N-1)$, $N \geq 2$ (Eq. 72 in [12, 1]):*

$$\frac{2}{1+3^{N-1}} \leq \mathfrak{T}_{D_N^{(k)}} \leq \frac{1}{1+\frac{2^{N-2}(\sqrt{2}-1)}{\binom{N}{k}}}, \tag{103}$$

in particular, for the N-qubit W state $|W_N\rangle := |D_N^{(1)}\rangle$ (Eq. 75 in [12]):

$$\frac{2}{1+3^{N-1}} \leq \mathfrak{T}_{W_N} \leq \frac{N}{N+2^{N-2}(\sqrt{2}-1)}, \quad N \geq 2. \tag{104}$$

To our knowledge, in the literature there are no analogues of our results on stability of nonlocality of an N-partite quantum state to noise, derived in Theorem 17 and Propositions 15, 16.

3 Conclusions

In the dissertation, we present a fundamentally new mathematical approach, developed by the author for the description of nonlocality of an N-partite quantum state of an arbitrary dimension and the investigation in the frame of this new approach of multipartite quantum nonlocality. A new approach to modelling of the probabilistic description of all joint von Neumann measurements on a Hilbert space of an arbitrary dimension has also been developed.

The new results presented by the author on various aspects of quantum nonlocality are important both for further theoretical research on this topic and for a wide range of practical problems in quantum technologies and quantum communications.

The twelve articles, presented by the author on the dissertation topic, contain 195 journal pages, are written without coauthors and are published in peer-reviewed journals of leading international publishing houses, indexed in citation databases Web of Science and Scopus.

The following new results, shortly summarized above within seventeen (17) Theorems and sixteen (16) Propositions and rigorously mathematically proved by the author in the corresponding published articles, are presented for the defense.

1. Development of a general approach to the formalization and modelling of the probabilistic description of an arbitrary N-partite correlation scenario ($N \geq 2$) with any number of measurements and any spectral type of outcomes on each of N "sites". Development via this general approach of *a new local probability model* for the description of an arbitrary correlation scenario – the local quasi hidden variable (LqHV) model reducing to the local classical probability (LHV) model only in a particular case. Formulating and proving general criteria for the probabilistic description of an arbitrary correlation scenario within a new local probability model (LqHV model). Proof of the existence of an LqHV model for any quantum correlation scenario. Constructing quantum LqHV models.

2. Constructing a single analytical representation for all general Bell inequalities and its specifications for all correlation Bell inequalities and all Bell inequalities for joint probabilities. Proving that the form of every correlation Bell inequality does not depend on a spectral type of outcomes, in particular, on a number of outcomes, observed by each participant, but is determined only by extreme values of measurement outcomes.

3. Finding a new class of bipartite quantum states of an arbitrary dimension satisfying the original Bell inequality for any three quantum observables without obeying the Bell condition on perfect correlations. Proving that, for a dimension $d \geq 3$, this class of states includes all two-qudit Werner states, separable and non-separable.

4. Development for the description of quantum nonlocality within LqHV modelling of new mathematical notions on the tensor product of Hilbert spaces: source operator of an N-partite quantum state, tensor positivity, covering norm, with proof of their basic mathematical properties.

5. Development on the basis of the new local probability model and the new introduced notions on the tensor product of Hilbert spaces of a new mathematical formalism (LqHV formalism) for the description of multipartite quantum nonlocality and the derivation within this formalism of the following *new* results:

   5.1. Quantum analogues of Bell inequalities;

   5.2. The new analytical bound on the maximal violation by an N-partite quantum state of all general Bell inequalities, expressed in terms of a new characteristic of a N-partite state on the tensor product of Hilbert spaces and having no analogues in the literature;

   5.3. The new sufficient analytical conditions for the full locality and the partial locality of an N-partite quantum state, which do not have analogues in the literature;

   5.4. The new exact analytical bounds on for the maximal violation of the general Bell inequalities by an arbitrary N-partite quantum state, expressed in terms of the "numerical" parameters of a correlation scenario: a number N of scenario participants, S_n, d_n - a number of quantum measurements and a qudit dimension at each n-th "site";

   5.5. Proof of the attainability of some of the new derived estimates for the maximal violation of general Bell inequalities.

6. Formulation and proof of a generalization of the extension theorem for consistent probability measures to the case of consistent operator-valued measures. The proof due to this generalization of the existence for the probabilistic description of all joint von Neumann measurements on an arbitrary Hilbert space of two new qHV models: statistically noncontextual and context-invariant.

7. Introduction of the notion of an LqHV model for an N-partite quantum state of a new notion — the notion of a single LqHV model for all correlation scenarios performed on this state and with von Neumann measurements at each site, and the proof of the existence of such a model for every N-partite quantum state of an arbitrary dimension.

8. Introduction of a new general concept "Bell nonlocality" and the specification of the mathematical description of Bell nonlocality under an arbitrary "nonsignaling" N-partite scenario.

9. Investigation within the developed new formalism for the description of quantum nonlocality of the nonlocality stability of an N-partite quantum state to the white noise. Finding for the noisy N-qudit GHZ state and an arbitrary noisy N-qudit state the new bounds on the critical values of white noise for their full locality beyond their full separability.

10. Introduction within the developed general formalism of a new approach to the description of the nonlocality stability of an arbitrary N-partite quantum state to any local noise. The derivation for an arbitrary N-partite quantum state of the exact new analytical relation between the tolerance of its nonlocality to any local noise and the maximal violation by this state of all general Bell inequalities. Finding, due to this relation, the new precise analytical bounds on the critical values of noise not violating nonlocality for: (i) an arbitrary nonlocal N-qudit state; (ii) an N-qubit GHZ state with $N \geq 2, d \geq 2$, in particular, an N-qubit GHZ state; (iii) N -qubit Dicke states. Specifying the asymptotics of the new bounds for $N \gg 1$ and $d \gg 1$.

List of articles on the dissertation topic, presented for the defense

[1] Elena R. Loubenets. *On the probabilistic description of a multipartite correlation scenario with arbitrary numbers of settings and outcomes per site* // Journal of Physics A: Mathematical and Theoretical **41** (2008), 445303 (23pp)

http://iopscience.iop.org/article/10.1088/1751-8113/41/44/445303

[2] Elena R. Loubenets. *Multipartite Bell-type inequalities for arbitrary numbers of settings and outcomes per site* // Journal of Physics A: Mathematical and Theoretical **41** (2008), 445304 (18pp)

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[3] Elena R. Loubenets. *Local hidden variable modelling, classicality, quantum separability, and the original Bell inequality* // Journal of Physics A: Mathematical and Theoretical **44** (2011) 035305 (16pp)

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[6] Elena R. Loubenets. *Context-invariant quasi hidden variable (qHV) modelling of all joint von Neumann measurements for an arbitrary Hilbert space* // Journal of Mathematical Physics **56**, 032201 (2015)

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https://link.springer.com/article/10.1007%2Fs10701-015-9903-8

[8] Elena R. Loubenets. *On the existence of a local quasi hidden variable (LqHV) model for each N-qudit state and the maximal violation of Bell inequalities* // International Journal of Quantum Information 14 (2016), 1640010 (15pages)

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[10] Elena R. Loubenets. *Full Bell locality of a noisy state for $N \geq 3$ nonlocally entangled qudits* // Journal of Physics A: Mathematical and Theoretical **50** (2017), 405305 (16pp)

http://iopscience.iop.org/article/10.1088/17518121/aa84e8

[11] Elena R. Loubenets. *New concise upper bounds on quantum violation of general multipartite Bell inequalities* // Journal of Mathematical Physics **58**, 052202 (2017)

https://aip.scitation.org/doi/10.1063/1.4982961

[12] Elena R. Loubenets. *Quantifying Tolerance of a Nonlocal Multi-Qudit State to Any Local Noise* // Entropy **20** (2018), 217 (13pp)

http://www.mdpi.com/1099-4300/20/4/217