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Jacobi forms in several variables and their applications

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Reflection groups play a special role in the theory of discrete groups. The simplest examples of these groups are finite reflection groups or, as they are also called, Coxeter groups. These groups include both the Weyl groups of all simple Lie algebras and some exceptional examples. But each of them can be represented as a group of linear reflections of a finite dimensional real affine space, the fundamental domain in this case will be a cone. The study of discrete groups is closely related with the theory of invariants, i.e. with the study of functions that are invariant or almost invariant under the action of the corresponding discrete group. In the case of finite reflection groups, there is the classical Chevalley theorem, which states that if such a group acts on a n-dimensional real space then the invariant polynomials of this group form a free graded algebra with n generators.

If we add an additional reflection with respect to some hyperplane, then we arrive to the notion of affine Weyl groups. Their fundamental domains are simplexes. In the case of affine Weyl groups, there is a result similar to Chevalley's theorem: the algebras of invariants are freely generated by n generators, but they are already trigonometric polynomials, because of the additional invariance with respect to translations. Here, again, n is the dimension of the real space on which the group acts.

The action of both the Coxeter groups and the affine Weyl groups can be extended on the complexification of the real spaces on which the groups act. In this case, the fundamental domains of the affine Weyl groups are not compact anymore. If we require the compactness condition, then it is necessary to consider complex crystallographic Coxeter groups, namely, the reflection groups of a complex space with a compact fundamental domain such that the matrices corresponding to the elements of the groups are real in some basis. These groups were studied and described by I.N. Bernstein and O.V. Schwartzman in [5], where it was shown that the complex crystallographic Coxeter groups are labeled by Dynkin diagrams and a parameter τ belonging to the complex upper half-plane. E. Loyenga noticed that such reflection groups are compatible with the action of the group $SL_2(\mathbb{Z})$, and therefore one can consider their semidirect product and obtain a group, which is called the Jacobi group.

The passage to the complex crystallographic Coxeter groups was studied in [5], [16], [17] and [15]. In this case the search of invariant functions makes no sense, since the quotient space is a complex torus and any holomorphic function is constant on it. Instead, one should study the action of a complex crystallographic Coxeter group on some suitable bundle and look for invariant sections. In this case, we get a free algebra generated by $n+1$ theta functions.

For the first time the theory of invariants for Jacobi forms was taken up by M. Eichler and D. Zagier in the book [9], where the case of the root system

A_1 was studied. A complete analogue of Chevalley's theorem for Jacobi forms was obtained by K. Wirtmüller in [23]. In his work, he studied the structures of algebras of weak Jacobi form associated with all root systems, except E_8. The root system E_8 in this context was considered in the recent paper by H. Wang [22] and, as it turned out, in this case the algebra of Jacobi forms is not polynomial. K. Wirtmüller's proof does not contain a direct construction of all generators of the corresponding algebras, but their explicit form can be extremely useful in applications. One such application is the construction of flat coordinates on suitable Frobenius manifolds (see [18], [19], [8, §4], [21], [3] and [4]). For example, in the papers by M. Bertola [3] and [4], the cases of root systems A_n, B_n and G_2 were analyzed independently, and I. Satake in [21] and K. Sakai in [20] considered the cases of the root systems E_6 and E_7.

Until now, the root systems C_n, D_n and F_4, to which this dissertation is devoted, remained as open cases. Namely, we prove Wirtmüller theorem for these root systems and give an explicit construction of the generators of the corresponding algebras of weak Jacobi forms. It turns out that all these three types of root systems are closely related. The main difficulty is the construction of three main generators of index 1 and weights -4, -2 and 0 for root systems of type D_n. Also, in addition, besides the new proof of Wirtmüller theorem, we give examples of interesting differential equations for the mentioned generators.

Note also that weak Jacobi forms invariant under the action of the full orthogonal group for D_n with $2 \leqslant n \leqslant 8$ corresponds to the D_8-tower of strongly reflective modular forms on the orthogonal groups $O^+(2U \oplus D_n(-1))$ (see [11]). The strongly reflective modular forms of this tower of the orthogonal groups $\widetilde{O}^+(2U \oplus D_n(-1))$ $(3 \leq n \leq 8)$ determine the Lorentzian Kac–Moody algebras corresponding to the BCOV (Bershadsky-Cecotti-Ooguri-Vafa)-analytic torsions (see [11], [24]).

Also there exists the D_8-tower of the reflective automorphic discriminants starting with the Borcherds–Enriques modular form

$$\Phi_4 \in M_4(O^+(U \oplus U(2) \oplus E_8(-2)), \chi_2) = M_4(O^+(U \oplus U \oplus D_8(-1)), \chi_2),$$

which is the discriminant of the moduli space of Enriques surfaces (see [6, 14] and [11, §5] or [12]).

The main methods in our work are the construction of generators using the Jacobi theta functions, modular forms, and a modular differential operator. We require a stronger condition on the generators than in the Wirtmüller theorem. Namely, we construct them in such a way that they satisfy the so-called "tower condition". That is, if we restrict all generators to a lattice of lower rank, we must also obtain all generators for the corresponding sublattice (in this case, some generators vanish). By use of generators constructed with this condition, we can then prove the algebraic independence and completeness of the set of obtained weak Jacobi forms by induction based on the case of the lattice A_1, which was studied in [9]. But we also give our short proof of this fact, based on the study of the divisors of the supposed generators.

Further directions of research, except applications in the theory of Frobenius manifolds, may be related with the study of t-deformations of automorphic discriminants and their application in geometry and physics. One of the even Siegel theta functions of genus 2 is a discriminant of the moduli space of abelian surfaces, and it could be represented as Borcherds product [13]. Its t-deformation could be constructed by the procedure, which is dual to the procedure described in [10], and it would be a new result. Then it could be generalized to singular forms for $U \oplus U \oplus D_8$ (here, U is the standard hyperbolic lattice of signature $(1, 1)$). After that it is interesting to study similar constructions for special Borcherds products, automorphic discriminants of spaces of versal deformations of Arnold exceptional singularities.

Another direction is studying of modular differential operators with respect to the orthogonal group $O(2, n)$. The main goals here are the following. The first one is to construct cusp forms of small weights for studying geometrical types of moduli spaces. In particular, for moduli spaces of hyper-Kähler manifolds. The second one is to find differential equations on generators of polynomial rings of modular forms. For example, modular forms for $U \oplus U \oplus D_8$.

1 Main results

In [23], K. Wirtmüller proved the following theorem.

Theorem 1.1. *For all root systems of rank n, except E_8, the corresponding algebras of weak Jacobi forms are freely generated over the ring of modular forms by $n+1$ generators φ_j of weight $-k(j)$ and index $m(j)$. Here $m(0) = 1$, and other $m(j)$ are the coefficients of the dual vector to the highest root of the dual root system in basis of the initial root system. As for weights of this generators, $k(0) = 0$, and other $k(j)$ correspond to the degrees of polynomials invariant under the action of Weyl group. In other words, exponents increased by 1.*

For each root system we have the following lattices, Weyl groups and weights and indices.

R	L	W	$(k(j), m(j))$
A_n	A_n	$W(A_n)$	$(0,1), (j,1) : 2 \leqslant j \leqslant n+1$
B_n	nA_1	$W(nA_1)$	$(2j,1) : 0 \leqslant j \leqslant n$
C_n	D_n	$W(C_n)$	$(0,1), (2,1), (4,1), (2j,2) : 3 \leqslant j \leqslant n$
D_n	D_n	$W(D_n)$	$(0,1), (2,1), (4,1), (n,1), (2j,2) : 3 \leqslant j \leqslant n-1$
E_6	E_6	$W(E_6)$	$(0,1), (2,1), (5,1), (6,2), (8,2), (9,2), (12,3)$
E_7	E_7	$W(E_7)$	$(0,1), (2,1), (6,2), (8,2), (10,2), (12,3), (14,3), (18,4)$
G_2	A_2	$O(A_2)$	$(0,1), (2,1), (6,2)$
F_4	D_4	$O(D_4)$	$(0,1), (2,1), (6,2), (8,2), (12,3)$

Remark 1.2. *The Weyl group for the root system C_n is a full orthogonal group for the root system D_n only if $n \neq 4$. Therefore, we write $W(C_n)$ to avoid overloaded notations.*

The thesis is devoted mainly to root systems of the C_n, D_n and F_4 type. Let us formulate the results to be proved for these lattices.

Theorem 1.3. *The set of all W-invariant weak Jacobi forms for the root system D_n has the structure of the free algebra over the ring of modular forms with the following generators*

$$J^W_{*,*}(D_2) = M_*[\varphi^{D_2}_{0,1}, \varphi^{D_2}_{-2,1}, \varphi^{D_2}_{-4,1}],$$

$$J^W_{*,*}(D_3) = M_*[\varphi^{D_3}_{0,1}, \varphi^{D_3}_{-2,1}, \varphi^{D_3}_{-4,1}, \omega^{D_3}_{-3,1}],$$

$$J^W_{*,*}(D_4) = M_*[\varphi^{D_4}_{0,1}, \varphi^{D_4}_{-2,1}, \varphi^{D_4}_{-4,1}, \varphi^{D_4}_{-6,2}, \omega^{D_4}_{-4,1}],$$

and

$$J^W_{*,*}(D_n) = M_*[\varphi^{D_n}_{0,1}, \varphi^{D_n}_{-2,1}, \varphi^{D_n}_{-4,1}, \varphi^{D_n}_{-6,2}, \ldots, \varphi^{D_n}_{-2n+2,2}, \omega^{D_n}_{-n,1}]$$

for $5 \leqslant n$. Moreover, for all $n \neq 4$ all generators, except $\omega^{D_n}_{-n,1}$, are invariant under the action of the full integral orthogonal group (and the group $O'(D_4)$ in the case $n = 4$), and there is the following natural tower with respect to restrictions from D_n to D_{n-1} by setting $z_n = 0$ up to multiplication by *constants*

$$
\begin{array}{cccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi^{D_n}_{0,1}, & \varphi^{D_n}_{-2,1}, & \varphi^{D_n}_{-4,1}, & \varphi^{D_n}_{-6,2}, & \cdots & \cdots & \varphi^{D_n}_{-2n+2,2}, & (\omega^{D_n}_{-n,1})^2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\varphi^{D_{n-1}}_{0,1}, & \varphi^{D_{n-1}}_{-2,1}, & \varphi^{D_{n-1}}_{-4,1}, & \varphi^{D_{n-1}}_{-6,2}, & \cdots & \varphi^{D_{n-1}}_{-2n+4,2}, & (\omega^{D_{n-1}}_{-n,1})^2 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & \\
\varphi^{D_4}_{0,1}, & \varphi^{D_4}_{-2,1}, & \varphi^{D_4}_{-4,1}, & \varphi^{D_4}_{-6,2}, & (\omega^{D_4}_{-4,1})^2, & 0 & & \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & & & \\
\varphi^{D_3}_{0,1}, & \varphi^{D_3}_{-2,1}, & \varphi^{D_3}_{-4,1}, & (\omega^{D_4}_{-3,1})^2, & 0 & & & \\
\downarrow & \downarrow & \downarrow & \downarrow & & & & \\
\varphi^{D_2}_{0,1}, & \varphi^{D_2}_{-2,1}, & \varphi^{D_2}_{-4,1}, & 0, & & & &
\end{array}
$$

Remark 1.4. *As it follows from the hypothesis of the theorem, we require the generators to be consistent with the operation of restricting to the sublattice. Further in this work, this requirement will be called the "tower condition", thereby emphasizing that the corresponding Jacobi forms form a tower, like the lattices themselves:*

$$
\begin{array}{c}
\cdots \\
\downarrow \\
D_n \\
\downarrow \\
D_{n-1} \\
\downarrow \\
\cdots \\
\downarrow \\
D_4 \\
\downarrow \\
D_3 \simeq A_3 \\
\downarrow \\
D_2 \simeq A_1 \oplus A_1
\end{array}
$$

Remark 1.5. *For* $n = 4$ *we have two generators of weight* -4*, they are* $\varphi_{-4,1}$*, which exists for all* n*, and* $\omega_{-4,1}$*. But trere is no contradiction here.*

We get an immediate corollary from this theorem.

Corollary 1.6. *For* $n \geqslant 3$ *the set of all* W*-invariant weak Jacobi forms for root system* C_n *has the structure of the free algebra over the ring of modular*

forms with the following generators
for $n \geqslant 5$:

$$J^{W}_{*,*}(C_n) = M_*[\varphi^{D_n}_{0,1}, \varphi^{D_n}_{-2,1}, \varphi^{D_n}_{-4,1}, \varphi^{D_n}_{-6,2}, \ldots, \varphi^{D_n}_{-2n+2,2}, (\omega^{D_n}_{-n,1})^2;$$

for $n = 4$:

$$J^{W}_{*,*}(C_4) = M_*[\varphi^{D_4}_{0,1}, \varphi^{D_4}_{-2,1}, \varphi^{D_4}_{-4,1}, \varphi^{D_4}_{-6,2}, (\omega^{D_4}_{-4,1})^2];$$

for $n = 3$:

$$J^{W}_{*,*}(C_3) = M_*[\varphi^{D_3}_{0,1}, \varphi^{D_3}_{-2,1}, \varphi^{D_3}_{-4,1}, (\omega^{D_3}_{-3,1})^2].$$

Finally, for the root system F_4 we prove the following theorem.

Theorem 1.7. *The ring of weak Jacobi forms of even index for the root system F_4 and invariant under the action of the Weyl group $W(F_4)$ has the structure of the free algebra over the ring of modular forms. More precisely,*

$$J^{w,W}_{*,2*}(F_4) \simeq J^{w,W}_{*,*}(F_4(2)) = M_*[\varphi^{F_4}_{0,1}, \varphi^{F_4}_{-2,1}, \varphi^{F_4}_{-6,2}, \varphi^{F_4}_{-8,2}, \varphi^{F_4}_{-12,3}] \simeq J^{w,O}_{*,*}(D_4).$$

Remark 1.8. *A first idea of the proof is to construct the generators of the algebra of weak $W(F_4)$-invariant Jacobi forms as the averages of the generators of $J^{w,W}_{*,*}(D_4)$ over the group $W(F_4)/W(D_4) \simeq S_3$. However, a straightforward computation proves that the average of the form $\varphi_{-4,1}$ is identically equal to zero, and this is a stumbling block in the construction, because this form is central in the case of the lattice D_4. Nevertheless, we construct generators of $J^{w,W}_{*,*}(F_4)$ by use of generators of $J^{w,W}_{*,*}(D_4)$, but this construction is a little bit tricky than averaging over S_3.*

2 Differential equations for the root system D_n

The modular differential operator H_k

$$H_k := H + \left(\frac{\operatorname{rk} L}{2} - k\right) E_2 \times,$$

where H is the heat operator

$$H := \sum_{n=0}^{\infty} \sum_{l \in L^\vee} \left(12n - \frac{6}{m}(l,l)\right) a(n,l) q^n \zeta^l,$$

plays an important role in constructing the generators in the thesis.

In the thesis we noticed that for the root system D_4

$$H_{-4}(\varphi_{-4,1}^{D_4}) = 0$$

or

$$H(\varphi_{-4,1}^{D_4}) = 6E_2\varphi_{-4,1}^{D_4}.$$

An interesting question arises, do there exist similar differential equations or systems of differential equations for generators of index 1 for lattices of a different rank? These equations can be with respect to both the modular differential operator and the heat operator, for example, in the spirit of the Ramanujan system of differential equations for the differential operator for modular forms. It is not difficult to check that

$$H_{-n}(\omega_{-n,1}^{D_n}) = 0.$$

Thus, we will not consider this form in our equations.

It turns out, that $\varphi_{-4,1}^{D_n}$ and $\varphi_{0,1}^{D_n}$ satysfy the following differential equations of order 3 in terms of the modular differential equation:

$$4H_0(H_{-2}(H_{-4}(\varphi_{-4,1}))) - (3n^2 - 12n + 32)E_4H_{-4}(\varphi_{-4,1}) +$$

$$+ (n-4)^2(n+8)E_6\varphi_{-4,1} = 0,$$

and

$$4H_4(H_2(H_0(\varphi_{0,1}))) - (3n^2 + 12n + 32)E_4H_0(\varphi_{0,1}) + n^2(n+12)E_6\varphi_{0,1} = 0.$$

In case of $\varphi_{-2,1}^{D_n}$, the minimal possible order is equal to 4:

$$4H_4(H_2(H_0(H_{-2}(\varphi_{-2,1}^{D_n})))) - (3n^2 - 12n + 224)E_4H_0(H_{-2}(\varphi_{-2,1}^{D_n})) +$$

$$+ (n^3 + 24n^2 - 144n + 384)E_6H_{-2}(\varphi_{-2,1}^{D_n}) - 12(n-8)(n-2)(n+4)E_4^2\varphi_{-2,1}^{D_n} = 0.$$

There is also a system of differential equations for all three generators.

$$\begin{cases} 4H_{-2}(\varphi_{-2,1}^{D_n}) - 3n\varphi_{0,1}^{D_n} - (n-8)E_4\varphi_{-4,1}^{D_n} = 0 \\ 3H_0(\varphi_{0,1}^{D_n}) - 2nE_4\varphi_{-2,1}^{D_n} - nE_6\varphi_{-4,1}^{D_n} = 0 \\ H_{-4}(\varphi_{-4,1}^{D_n}) - (n-4)\varphi_{-2,1}^{D_n} = 0 \end{cases}$$

In terms of the heat operator it could be written as

$$\begin{cases} 2H(\varphi_{-4,1}^{D_n}) = 2(n-4)\varphi_{-2,1}^{D_n} - (n+8)E_2\varphi_{-4,1}^{D_n} \\ 4H(\varphi_{-2,1}^{D_n}) = 3n\varphi_{0,1}^{D_n} - 2(n+4)E_2\varphi_{-2,1}^{D_n} + (n-8)E_4\varphi_{-4,1}^{D_n} \\ 6H(\varphi_{0,1}^{D_n}) = -3nE_2\varphi_{0,1}^{D_n} + 4nE_4\varphi_{-2,1}^{D_n} - 2nE_6\varphi_{-4,1}^{D_n} \end{cases}$$

The results of the thesis are published in two papers.

• D. Adler, *The structure of the algebra of weak Jacobi forms for the root system F_4.* Funct. Anal. Appl., **54**:3 (2020), 155–168.

• D. Adler, V. Gritsenko, *The D_8-tower of weak Jacobi forms and applications.* J. Geom. Phys., 150, Article ID 103616 (2020), 12 p.

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