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**The enumeration of the coverings of compact 3 dimensional Euclidean manifolds**

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- G. Chelnokov, M. Deryagina and A. Mednykh, On the coverings of Euclidean manifolds $\mathbb{B}_1$ and $\mathbb{B}_2$, *Comm. Algebra* 45 (2017), no. 4, 1558–1576.
- G. Chelnokov and A. Mednykh, On the coverings of Euclidean manifolds $\mathbb{G}_2$ and $\mathbb{G}_4$, Comm. Algebra 48 (2020), no.7, 2725–2739, doi:10.1080/00927872.2019.1705468.
- G. Chelnokov and A. Mednykh, On the coverings of Euclidean manifolds $\mathbb{G}_3$ and $\mathbb{G}_5$, *J. Algebra* 560 (2020), 48–66, https://doi.org/10.1016/j.jalgebra.2020.05.010.
- G. Chelnokov and A. Mednykh On the coverings of Hantzsche-Wendt manifold (expected in Tohoku Math. J. in October 2021) (preprint https://arxiv.org/abs/2009.06691)

# Contents

# Introduction

The classification of of compact three-dimensional Euclidean manifolds (that is locally isometric to the Euclidean space $\mathbb{E}^3$) up to homeomorphism was obtained by W. Nowacki [20] and W. Hantzsche and H. Wendt [3]. This classification is based on the Bieberbach theorem (1911), which claims that each such manifold can be represented as a quotient $\mathbb{R}^3/\Gamma$, where $\Gamma$ is a Bieberbach group. Recall that a subgroup of isometry group of $\mathbb{R}^3$ is called *Bieberbach group* if it is discrete, cocompact and torsion free. In this case, $\Gamma$ is isomorphic to the fundamental group of the manifold, that is $\Gamma \cong \pi_1(\mathbb{R}^3/\Gamma)$. In virtue of the above classification there are only 10 Euclidean forms: six are orientable $\mathcal{G}_1, \ldots, \mathcal{G}_6$ and four are non-orientable $\mathcal{B}_1, \ldots, \mathcal{B}_4$. The fundamental groups of this manifolds are explicitly known, see, for example, monograph [21]. Here $\mathcal{G}_1$ is the three-dimensional torus, for which the following questions are trivial, so it will not be an object of our consideration.

The purpose of the articles composing this thesis is to classify the homeomorphism types of manifolds, which can be a finite-sheeted covering space for one of the manifolds $\mathcal{G}_2, \ldots, \mathcal{G}_6$, $\mathcal{B}_1$, $\mathcal{B}_2$; and also to enumerate the equivalence classes of $n$-fold coverings of the above manifolds for each possible homeomorphism type of the coverage. Recall that two coverings

$$p_1 : \mathcal{M}_1 \to \mathcal{M} \text{ and } p_2 : \mathcal{M}_2 \to \mathcal{M}$$

are said to be equivalent if there exists a homeomorphism $h : \mathcal{M}_1 \to \mathcal{M}_2$ such that $p_1 = p_2 \circ h$. The studies of coverings up to equivalence have a long history. The problem of enumeration for nonequivalent coverings over a Riemann surface with given branch type goes back to Hurwitz. In his paper (1891, [6]) the number of coverings over the Riemann sphere with a given number of simple (of order two) branching points was determined. Later, in [7], it has been proved that this number can be expressed in terms of irreducible characters of symmetric groups. The Hurwitz problem was studied by many authors. For closed Riemann surfaces, this problem was completely solved in [14].

However, of most interest is the case of unramified coverings. We will use the following notations: let $s_G(n)$ denote the number of subgroups of index $n$ in the group $G$, and let $c_G(n)$ be the number of conjugacy classes of such subgroups. Similarly, by $s_{H,G}(n)$ denote the number of subgroups of index $n$ in the group $G$, which are isomorphic to $H$, and by $c_{H,G}(n)$ the number of conjugacy classes of such subgroups. According to what was said above, $c_G(n)$ coincides with the number of nonequivalent $n$-fold coverings over a manifold $\mathcal{M}$ with fundamental group $G$, and $c_{H,G}(n)$ coincides with the number of nonequivalent $n$-fold coverings $p : \mathcal{N} \to \mathcal{M}$, where $\pi_1(\mathcal{N}) \cong H$ and $\pi_1(\mathcal{M}) \cong M$. If $\mathcal{M}$ is a compact surface with nonempty boundary of Euler characteristic $\chi(\mathcal{M}) = 1 - r$, where $r \geq 0$ (e.g., a disk with $r$ holes), then its fundamental group $\Gamma = F_r$ is the free group of rank $r$. For this case, M. Hall (1949, [5]) calculated the number $s_\Gamma(n)$ and V. A. Liskovets (1971, [9]) found the number $c_\Gamma(n)$. A different approach for enumeration of the conjugacy classes of subgroups in the free group was given by J. H. Kwak and Y. Lee (1996, [8]). The numbers $s_G(n)$ and $c_G(n)$ for the fundamental group $G$ of a closed surface (orientable or not) were found by A. D. Mednykh (1978 [12], 1979 [13], 1986 [15]). In the paper (2008) [16], a general method for calculating the number $c_G(n)$ of conjugacy classes of subgroups in an arbitrary finitely generated group $G$ was given. Asymptotic formulas for $s_G(n)$ in many important cases were obtained by T. W. Müller and his collaborators (2000 [17], 2002 [18], 2002 [19]).

In the three-dimensional case, for a large class of Seifert fibrations, the value of $s_G(n)$ was calculated by V. A. Liskovets and A. D. Mednykh in (2000, [10]) and (2000, [11]).

According to the general theory of covering spaces, any $n$-fold covering is uniquely determined by a subgroup of index $n$ in the fundamental group of the covered manifold $\mathcal{M}$. The equivalence classes of $n$-fold coverings of $\mathcal{M}$ are in one-to-one correspondence with the conjugacy classes of subgroups of index $n$ in the fundamental group $\pi_1(\mathcal{M})$. See, for example, ([4], p. 67). In such a way the following natural problems arise: to describe the isomorphism classes of subgroups of finite index in the fundamental group of a given manifold and to enumerate the finite index subgroups and their conjugacy classes with respect to isomorphism type. In the articles composing the present thesis the above problems are solved for the groups $\pi_1(\mathcal{G}_2)$, $\pi_1(\mathcal{G}_3)$, $\pi_1(\mathcal{G}_4)$, $\pi_1(\mathcal{G}_5)$, $\pi_1(\mathcal{G}_6)$, $\pi_1(\mathcal{B}_1)$, $\pi_1(\mathcal{G}_2)$. Also, we provide the Dirichlet generating functions for all the above sequences.

# 1 Notations

Let $G$ and $H$ be some groups. By $s_{H,G}(n)$ we denote the number of subgroups of index $n$ in the group $G$ isomorphic to the group $H$; by $c_{H,G}(n)$ the number of conjugacy classes of subgroups of index $n$ in the group $G$ isomorphic to the group $H$.

Also we will need the following number-theoretic functions. Given a fixed $n$ we widely use summation over all representations of $n$ as a product of two or three positive integer factors $\sum_{ab=n}$ and $\sum_{abc=n}$. The order of factors in the product is important. We assume this sum vanishes if $n$ is not integer.

To start with, this is the natural language to express the function $\sigma_0(n)$ – the number of representations of number $n$ as a product of two factors $\sigma_0(n) = \sum_{ab=n} 1$. We will also need the following number-theoretic functions $\sigma_0$:

$$\sigma_1(n) = \sum_{ab=n} a, \qquad \sigma_2(n) = \sum_{ab=n} \sigma_1(a) = \sum_{abc=n} a,$$

$$d_3(n) = \sum_{ab=n} \sigma_0(a) = \sum_{abc=n} 1, \qquad \omega(n) = \sum_{ab=n} a\sigma_1(a) = \sum_{abc=n} a^2 b,$$

$$\chi(n) = \sum_{ab=n} a\sigma_1(b) = \sum_{ab=n} a\sigma_0(a) = \sum_{abc=n} ab;$$

$$\tau(n) = |\{(s,t)|s,t\in\mathbb{Z}, s>0, t\geq 0, s^2+t^2=n\}| = \sum_{\substack{ab=n\\ a\equiv 1 \bmod 4}} 1 - \sum_{\substack{ab=n\\ a\equiv 3 \bmod 4}} 1 = \sum_{k|n} \sin\frac{\pi k}{2},$$

$$\theta(n) = |\{(p,q)\in\mathbb{Z}^2|p>0, q\geq 0, p^2-pq+q^2=n\}| = \sum_{\substack{ab=n\\ a\equiv 1 \bmod 3}} 1 - \sum_{\substack{ab=n\\ a\equiv 2 \bmod 3}} 1 = \sum_{k|n} \frac{2}{\sqrt{3}}\sin\frac{2\pi k}{3}.$$

Given a sequence $\{f(n)\}_{n=1}^{\infty}$, a formal power series

$$\widehat{f}(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

is called the Dirichlet generating function for $\{f(n)\}_{n=1}^{\infty}$, see ([1], Ch. 12). To reconstruct the sequence $f(n)$ from $\widehat{f}(s)$ one can use Perron's formula ([1], Th. 11.17).

By $\zeta(s)$ we denote the Riemann zeta function $\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$. Following [1] note

$$\widehat{\sigma}_0(s) = \zeta^2(s), \quad \widehat{\sigma}_1(s) = \zeta(s)\zeta(s-1), \quad \widehat{\sigma}_2(s) = \zeta^2(s)\zeta(s-1), \quad \widehat{\omega}(s) = \zeta(s)\zeta(s-1)\zeta(s-2).$$

Define sequence $\{\phi(n)\}_{n=1}^{\infty}$ by $\phi(n) = \begin{cases} 1 \text{ if } n\equiv 1 \mod 4 \\ 0 \qquad\qquad \text{if } 2 \mid n. \\ -1 \text{ if } n\equiv 3 \mod 4 \end{cases}$

Put $\eta(s) = \widehat{\phi}(s)$. Note that $\eta(s)$ is the Dirichlet L-series for the multiplicative character $\phi(n)$. Then $\widehat{\tau}(s) = \zeta(s)\eta(s)$. In more algebraic terms,

$$\widehat{\tau}(s) = \frac{1}{1-2^{-s}} \prod_{p\equiv 1 \mod 4} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 3 \mod 4} \frac{1}{1-p^{-2s}}.$$

For details see ([2] p. 4).

Similarly, define sequence $\{\psi(n)\}_{n=1}^{\infty}$ by $\psi(n) = \frac{2}{\sqrt{3}}\sin\frac{2\pi n}{3}$ or equivalently

$$\psi(n) = \begin{cases} 1 \text{ if } n\equiv 1 \mod 3 \\ -1 \text{ if } n\equiv 2 \mod 3 \\ 0 \text{ if } n\equiv 0 \mod 3. \end{cases}$$

Denote $\vartheta(s) = \widehat{\psi}(s)$. Note that $\vartheta(s)$ is the Dirichlet L-series for the multiplicative character $\psi(n)$. Then $\widehat{\theta}(s) = \zeta(s)\vartheta(s)$. In more algebraic terms,

$$\widehat{\theta}(s) = \frac{1}{1-3^{-s}} \prod_{p\equiv 1 \bmod 3} \frac{1}{(1-p^{-s})^2} \prod_{p\equiv 2 \bmod 3} \frac{1}{1-p^{-2s}}.$$

## 2 Manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$

The paper in Appendix B is dedicated to the coverings of manifolds $\mathcal{G}_2$ and $\mathcal{G}_4$. The main interest is the following statements.

**Theorem 1.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{G}_2)$ is isomorphic to either $\pi_1(\mathcal{G}_2)$ or $\mathbb{Z}^3$. The respective numbers of subgroups are*

$$(i) \qquad s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \omega(n) - \omega(\frac{n}{2});$$

$$(ii) \qquad s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n) = \omega(\frac{n}{2}).$$

**Theorem 2.** *Let $\mathcal{N} \to \mathcal{G}_2$ be an $n$-fold covering over $\mathcal{G}_2$. If $n$ is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_2$. If $n$ is even then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_2$ or $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_2)}(n) = \sigma_2(n) + 2\sigma_2(\frac{n}{2}) - 3\sigma_2(\frac{n}{4}); \tag{i}$$

$$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_2)}(n) = \frac{1}{2}\Big(\omega(\frac{n}{2}) + \sigma_2(\frac{n}{2}) + 3\sigma_2(\frac{n}{4})\Big). \tag{ii}$$

**Theorem 3.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{G}_4)$ is isomorphic to either $\pi_1(\mathcal{G}_4)$, or $\pi_1(\mathcal{G}_2)$, or $\mathbb{Z}^3$. The respective numbers of subgroups are*

$$(i) \qquad s_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n) = \sum_{a|n} a\tau(a) - \sum_{a|\frac{n}{2}} a\tau(a);$$

$$(ii) \qquad s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)}(n) = \omega(\frac{n}{2}) - \omega(\frac{n}{4});$$

$$(iii) \qquad s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(n) = \omega(\frac{n}{4}).$$

**Theorem 4.** *Let $\mathcal{N} \to \mathcal{G}_4$ be an $n$-fold covering over $\mathcal{G}_4$. If $n$ is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_4$. If $n$ is even but not divisible by 4 then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_4$ or $\mathcal{G}_2$. Finely, if $n$ is divisible by 4 then $\mathcal{N}$ is homeomorphic to one of $\mathcal{G}_4$, $\mathcal{G}_2$ and $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

$$(i) \qquad c_{\pi_1(\mathcal{G}_4),\pi_1(\mathcal{G}_4)}(n) = \sum_{a|n} \tau(a) - \sum_{a|\frac{n}{4}} \tau(a);$$

$$(ii) \qquad c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_4)}(n) = \frac{1}{2}\Big(\sigma_2(\frac{n}{2}) + 2\sigma_2(\frac{n}{4}) - 3\sigma_2(\frac{n}{8}) + \sum_{a|\frac{n}{2}} \tau(a) - \sum_{a|\frac{n}{8}} \tau(a)\Big);$$

$$(iii) \qquad c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_4)}(n) = \frac{1}{4}\Big(\omega(\frac{n}{4}) + \sigma_2(\frac{n}{4}) + 3\sigma_2(\frac{n}{8}) + 2\sum_{a|\frac{n}{4}} \tau(a) + 2\sum_{a|\frac{n}{8}} \tau(a)\Big).$$

Theorems 1 and 2 classify isomorphism types of finite-index subgroups of $\pi_1(\mathcal{G}_2)$, also enumerate such subgroups and their conjugacy classes with respect to isomorphism class. Theorems 3 and 4 do the same for the subgroups of $\pi_1(\mathcal{G}_4)$. Theorems 1–4 correspond to Theorems 1–4 in the article.

Of independent interest are the following propositions, which provide an explicit systems of invariants, enumerating subgroups of given finite index and isomorphism class in the groups $\pi_1(\mathcal{G}_2)$ and $\pi_1(\mathcal{G}_4)$, and the conjugacy classes of such subgroups.

**Definition 1.** *A 3-plet $(a, H, \nu)$ is called* $n$-essential for $\pi_1(\mathcal{G}_2)$ *if the following conditions holds:*

*(i) $a$ is a positive divisor of $n$,*

*(ii) $H$ is a subgroup of index $n/a$ in $\mathbb{Z}^2$,*

*(iii) $\nu$ is an element of $\mathbb{Z}^2/H$.*

**Proposition 1.** *There is a bijection between the set of $n$-essential for $\pi_1(\mathcal{G}_2)$ 3-plets $(a, H, \nu)$ and the set of subgroups $\Delta$ of index $n$ in $\pi_1(\mathcal{G}_2)$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $a(\Delta)$ is even and $\Delta \cong \pi_1(\mathcal{G}_2)$ if $a(\Delta)$ is odd.*

**Proposition 2.** *The conjugacy classes of subgroups $\Delta \cong \mathbb{Z}^3$ of index $n$ in $\pi_1(\mathcal{G}_2)$ are enumerated by the triplets $(a, H, \bar{\nu})$, where $a, H$ have the same meaning as in the proposition 1 and $\bar{\nu}$ is a set of the form $\bar{\nu} = \{\nu, -\nu\}$, where $\nu$ as above.*

*The conjugacy classes of subgroups $\Delta \cong \pi_1(\mathcal{G}_2)$ of index $n$ in $\pi_1(\mathcal{G}_2)$ are enumerated by the triplets $(a, H, \tilde{\nu})$, where $a, H$ have the same meaning as in the proposition 1 and $\tilde{\nu}$ is an element of $\mathbb{Z}^2/\langle H, (2,0), (0,2)\rangle$.*

**Definition 2.** *A 3-plet $(a, H, \nu)$ is called* $n$-essential for $\pi_1(\mathcal{G}_4)$ *if the following conditions holds:*

*(i) $a$ is a positive divisor of $n$,*

*(ii) $H$ is a subgroup of index $n/a$ in $\mathbb{Z}^2$ also if $a$ is odd then $H$ is preserved by the automorphism $\ell : (x, y) \to (-y, x)$,*

*(iii) $\nu$ is an element of $\mathbb{Z}^2/H$.*

**Proposition 3.** *There is a bijection between the set of $n$-essential for $\pi_1(\mathcal{G}_4)$ 3-plets $(a, H, \nu)$ and the set of subgroups of index $n$ in $\pi_1(\mathcal{G}_4)$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $a(\Delta) \equiv 0 \mod 4$, $\Delta \cong \pi_1(\mathcal{G}_2)$ if $a(\Delta) \equiv 2 \mod 4$ and $\Delta \cong \pi_1(\mathcal{G}_4)$ if $a(\Delta) \equiv 1 \mod 2$.*

Propositions 1 and 3 correspond to Propositions 3 and 5 in the article, Proposition 3, throw not explicitly formulated in the article, proven inside the proof of Theorem 2.

Dirichlet generating series for the above sequences are given in the following table.

| H \ G | | $\mathcal{G}_2$ | $\mathcal{G}_4$ |
|---|---|---|---|
| $\mathbb{Z}^3$ | $\widehat{s}_{H,G}$ | $2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ | $4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| | $\widehat{c}_{H,G}$ | $2^{-s-1}\zeta(s)\zeta(s-1)\Big(\zeta(s-2)+(1+3\cdot 2^{-s})\zeta(s)\Big)$ | $2^{-2s-2}\zeta(s)\Big(\zeta(s-1)\zeta(s-2)+(1+3\cdot 2^{-s})\zeta(s)\zeta(s-1)+2(1+2^{-s})\zeta(s)\eta(s)\Big)$ |
| $\mathcal{G}_2$ | $\widehat{s}_{H,G}$ | $\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)\zeta(s-2)$ | $2^{-s}\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| | $\widehat{c}_{H,G}$ | $\big(1-2^{-s}\big)\big(1+3\cdot 2^{-s}\big)\zeta(s)^2\zeta(s-1)$ | $2^{-s-1}(1-2^{-s})\zeta(s)^2\Big((1+3\cdot 2^{-s})\zeta(s-1)+(1+2^{-s})\eta(s)\Big)$ |
| $\mathcal{G}_4$ | $\widehat{s}_{H,G}$ | 0 | $\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)\eta(s-1)$ |
| | $\widehat{c}_{H,G}$ | 0 | $\big(1-2^{-s}\big)\big(1+2^{-s}\big)\zeta(s)^2\eta(s)$ |

# 3 Manifolds $\mathcal{G}_3$ and $\mathcal{G}_5$

The paper in Appendix C is dedicated to the coverings of manifolds $\mathcal{G}_3$ and $\mathcal{G}_5$. The main interest is the following statements.

**Theorem 5.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{G}_3)$ is isomorphic to either $\pi_1(\mathcal{G}_3)$ or $\pi_1(\mathcal{G}_1) \cong \mathbb{Z}^3$. The respective numbers of subgroups are*

(*i*)
$$s_{\pi_1(\mathcal{G}_3),\pi_1(\mathcal{G}_3)}(n) = \sum_{k|n} k\theta(k) - \sum_{k|\frac{n}{3}} k\theta(k);$$

(*ii*)
$$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_3)}(n) = \omega(\frac{n}{3}).$$

**Theorem 6.** *Let $\mathcal{N} \to \mathcal{G}_3$ be an $n$-fold covering over $\mathcal{G}_3$. If $n$ is not divisible by 3 then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_3$. If $n$ is divisible by 3 then $\mathcal{N}$ is homeomorphic to either $\mathcal{G}_3$ or $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

(*i*)
$$c_{\pi_1(\mathcal{G}_3),\pi_1(\mathcal{G}_3)}(n) = \sum_{k|n} \theta(k) + \sum_{k|\frac{n}{3}} \theta(k) - 2\sum_{k|\frac{n}{9}} \theta(k);$$

(*ii*)
$$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_3)}(n) = \frac{1}{3}\Big(\omega(\frac{n}{3}) + 2\sum_{k|\frac{n}{3}} \theta(k) + 4\sum_{k|\frac{n}{9}} \theta(k)\Big).$$

The next two theorems are analogues of Theorem 5 and Theorem 6 respectively for the manifold $\mathcal{G}_5$.

**Theorem 7.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{G}_5)$ is isomorphic to either $\pi_1(\mathcal{G}_5)$ or $\pi_1(\mathcal{G}_3)$ or $\pi_1(\mathcal{G}_2)$ or $\pi_1(\mathcal{G}_1) \cong \mathbb{Z}^3$. The respective numbers of subgroups are*

(*i*)
$$s_{\pi_1(\mathcal{G}_5),\pi_1(\mathcal{G}_5)}(n) = \sum_{k|n,\,(\frac{n}{k},6)=1} k\theta(k);$$

(*ii*)
$$s_{\pi_1(\mathcal{G}_3),\pi_1(\mathcal{G}_5)}(n) = \sum_{k|\frac{n}{2}} k\theta(k) - \sum_{k|\frac{n}{6}} k\theta(k);$$

(*iii*)
$$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_5)}(n) = \omega(\frac{n}{3}) - \omega(\frac{n}{6});$$

(*iv*)
$$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_5)}(n) = \omega(\frac{n}{6}).$$

**Theorem 8.** *The numbers of equivalence classes of $n$-fold coverings over $\mathcal{G}_5$ is given by the following formulas:*

(i)
$$c_{\pi_1(\mathcal{G}_5),\pi_1(\mathcal{G}_5)}(n) = \sum_{k|n,\,(\frac{n}{k},6)=1} \theta(k);$$

(ii)
$$c_{\pi_1(\mathcal{G}_3),\pi_1(\mathcal{G}_5)}(n) = \sum_{k|\frac{n}{2}} \theta(k) - \sum_{k|\frac{n}{18}} \theta(k);$$

(iii)
$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_5)}(n) = \frac{1}{3}\Big(\sigma_2(\frac{n}{3}) + 2\sigma_2(\frac{n}{6}) - 3\sigma_2(\frac{n}{12}) + 2\sum_{k|\frac{n}{3}} \theta(k) - 2\sum_{k|\frac{n}{6}} \theta(k)\Big);$$

(iv)
$$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_5)}(n) = \frac{1}{6}\Big(\omega(\frac{n}{6}) + \sigma_2(\frac{n}{6}) + 3\sigma_2(\frac{n}{12}) + 4\sum_{k|\frac{n}{6}} \theta(k) + 4\sum_{k|\frac{n}{18}} \theta(k)\Big).$$

Theorems 5–8 correspond to Theorems 1–4 in the article.

Of independent interest are the following propositions, which provide an explicit systems of invariants, enumerating subgroups of given finite index and isomorphism class in the groups $\pi_1(\mathcal{G}_3)$ and $\pi_1(\mathcal{G}_5)$.

**Definition 3.** *A triple $(a, H, \nu)$ is called $n$-*essential for $\pi_1(\mathcal{G}_3)$ or $\pi_1(\mathcal{G}_5)$ *if the following conditions hold:*

*(i) $a$ is a positive divisor of $n$,*

*(ii) $H$ is a subgroup of index $n/a$ in $\mathbb{Z}^2$ also if $3 \nmid a$, then $H$ is preserved by the automorphism $\ell : (x, y) \to (-y, x - y)$.*

*(iii) $\nu$ is an element of $\mathbb{Z}^2/H$.*

**Remark** As a family of objects $n$-essential for $\pi_1(\mathcal{G}_3)$ triples and $n$-essential for $\pi_1(\mathcal{G}_5)$ triples coincide, we just use slightly different procedure to recover a subgroup from it's triple is cases of $\pi_1(\mathcal{G}_3)$ and $\pi_1(\mathcal{G}_5)$. So, to avoid confusion we call a triple $n$-essential for $\pi_1(\mathcal{G}_3)$ if we are going to reconstruct a subgroup in $\pi_1(\mathcal{G}_3)$ and vice versa.

**Proposition 4.** *There is a bijection between the set of $n$-essential for $\pi_1(\mathcal{G}_3)$ triple $(a, H, \nu)$ and the set of subgroups $\Delta$ of index $n$ in $\pi_1(\mathcal{G}_3)$ Moreover, $\Delta \cong \mathbb{Z}^3$ if $3 \mid a(\Delta)$ and $\Delta \cong \pi_1(\mathcal{G}_3)$ otherwise.*

**Proposition 5.** *There is a bijection between the set of $n$-essential triple $(a, H, \nu)$ and the set of subgroups $\Delta$ of index $n$ in $\pi_1(\mathcal{G}_5)$. Moreover, $\Delta \cong \mathbb{Z}^3$ if $(a, 6) = 6$, $\Delta \cong \pi_1(\mathcal{G}_2)$ if $(a, 6) = 3$, $\Delta \cong \pi_1(\mathcal{G}_3)$ if $(a, 6) = 2$ and $\Delta \cong \pi_1(\mathcal{G}_5)$ if $(a, 6) = 1$.*

Propositions 4 and 5 correspond to Propositions 3 and 5 in the article.
Dirichlet generating series for the above sequences are given in the following table.

| H \ G | | $\pi_1(\mathcal{G}_3)$ | $\pi_1(\mathcal{G}_5)$ |
|---|---|---|---|
| $\pi_1(\mathcal{G}_1)$ | $\widehat{s}_{H,G}$ | $3^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ | $6^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| | $\widehat{c}_{H,G}$ | $3^{-s-1}\zeta(s)\Big(\zeta(s-1)\zeta(s-2)+2(1+2\cdot 3^{-s})\zeta(s)\vartheta(s)\Big)$ | $6^{-s-1}\zeta(s)\Big(\zeta(s-1)\zeta(s-2)+(1+3\cdot 2^{-s})\zeta(s)\zeta(s-1)+4(1+3^{-s})\zeta(s)\vartheta(s)\Big)$ |
| $\pi_1(\mathcal{G}_2)$ | $\widehat{s}_{H,G}$ | 0 | $3^{-s}\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| | $\widehat{c}_{H,G}$ | 0 | $3^{-s-1}\big(1-2^{-s}\big)\zeta(s)^2\Big((1+3\cdot 2^{-s})\zeta(s-1)+2\vartheta(s)\Big)$ |
| $\pi_1(\mathcal{G}_3)$ | $\widehat{s}_{H,G}$ | $\big(1-3^{-s}\big)\zeta(s)\zeta(s-1)\vartheta(s-1)$ | $2^{-s}\big(1-3^{-s}\big)\zeta(s)\zeta(s-1)\vartheta(s-1)$ |
| | $\widehat{c}_{H,G}$ | $\big(1-3^{-s}\big)\big(1+2\cdot 3^{-s}\big)\zeta(s)^2\vartheta(s)$ | $2^{-s}\big(1-3^{-s}\big)\big(1+3^{-s}\big)\zeta(s)^2\vartheta(s)$ |
| $\pi_1(\mathcal{G}_5)$ | $\widehat{s}_{H,G}$ | 0 | $\big(1-2^{-s}\big)\big(1-3^{-s}\big)\zeta(s)\zeta(s-1)\vartheta(s-1)$ |
| | $\widehat{c}_{H,G}$ | 0 | $\big(1-2^{-s}\big)\big(1-3^{-s}\big)\zeta(s)^2\vartheta(s)$ |

# 4 Hantzsche-Wendt Manifold

The paper in Appendix D is dedicated to the coverings of Hantzsche-Wendt manifold. The main theorems are:

**Theorem 9.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{G}_6)$ is isomorphic to either $\pi_1(\mathcal{G}_6)$, or $\pi_1(\mathcal{G}_2)$, or $\mathbb{Z}^3$. The respective numbers of subgroups are*

*(i)*
$$s_{\mathbb{Z}^3,\pi_1(\mathcal{G}_6)}(n) = \omega\left(\frac{n}{4}\right);$$

*(ii)*
$$s_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_6)}(n) = 3\omega\left(\frac{n}{2}\right) - 3\omega\left(\frac{n}{4}\right);$$

*(iii)*
$$s_{\pi_1(\mathcal{G}_6),\pi_1(\mathcal{G}_6)}(n) = n\left(d_3(n) - 3d_3\left(\frac{n}{2}\right) + 3d_3\left(\frac{n}{4}\right) - d_3\left(\frac{n}{8}\right)\right).$$

**Theorem 10.** *Let $\mathcal{N} \to \mathcal{G}_6$ be an $n$-fold covering over $\mathcal{G}_6$. Then $\mathcal{N}$ is homeomorphic to one of $\mathcal{G}_6$, $\mathcal{G}_2$ or $\mathcal{G}_1$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

*(i)*
$$c_{\mathbb{Z}^3,\pi_1(\mathcal{G}_6)}(n) = \frac{1}{4}\omega\left(\frac{n}{4}\right) + \frac{3}{4}\sigma_2\left(\frac{n}{4}\right) + \frac{9}{4}\sigma_2\left(\frac{n}{8}\right);$$

*(ii)*
$$c_{\pi_1(\mathcal{G}_2),\pi_1(\mathcal{G}_6)}(n) = \frac{3}{2}\left(\sigma_2\Big(\frac{n}{2}\Big) + 2\sigma_2\Big(\frac{n}{4}\Big) - 3\sigma_2\Big(\frac{n}{8}\Big) + d_3\Big(\frac{n}{2}\Big) - d_3\Big(\frac{n}{4}\Big) - 3d_3\Big(\frac{n}{8}\Big) + 5d_3\Big(\frac{n}{16}\Big) - 2d_3\Big(\frac{n}{32}\Big)\right);$$

*(iii)*
$$c_{\pi_1(\mathcal{G}_6),\pi_1(\mathcal{G}_6)}(n) = d_3(n) - 3d_3\left(\frac{n}{2}\right) + 3d_3\left(\frac{n}{4}\right) - d_3\left(\frac{n}{8}\right).$$

**Remark.** If $n$ is odd then $\mathcal{N} \cong \mathcal{G}_6$. If $n \equiv 2 \mod 4$ then $\mathcal{N} \cong \mathcal{G}_2$. Finally, if $4 \mid n$ then $\mathcal{N} \cong \mathcal{G}_2$ or $\mathcal{N} \cong \mathcal{G}_1$.

Dirichlet generating series for the above sequences are given in the following table.

Table 2. Dirichlet generating functions for the sequences $s_{H,\mathcal{G}_6}(n)$ and $c_{H,\mathcal{G}_6}(n)$.

| $H$ | $s_{H,\mathcal{G}_6}$ | $c_{H,\mathcal{G}_6}$ |
|---|---|---|
| $\pi_1(\mathcal{G}_1)$ | $4^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ | $4^{-s-1}\zeta(s)\zeta(s-1)\big(\zeta(s-2)+3(1+3\cdot 2^{-s})\zeta(s)\big)$ |
| $\pi_1(\mathcal{G}_2)$ | $2^{-s}\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)\zeta(s-2)$ | $3\cdot 2^{-s-1}(1-2^{-s})\zeta^2(s)\big((1+3\cdot 2^{-s})\zeta(s-1)+$ $(1-2^{-s})^2(1+2^{-s+1})\zeta(s)\big)$ |
| $\pi_1(\mathcal{G}_1)$ | $\big(1-2^{-s+1}\big)^3\zeta^3(s-1)$ | $\big(1-2^{-s}\big)^3\zeta^3(s)$ |

## 5 Manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$

The paper in Appendix A is dedicated to the coverings of manifolds $\mathcal{B}_1$ and $\mathcal{B}_2$. The main interest is the following statements.

**Theorem 11.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{B}_1)$ is isomorphic to either $\mathbb{Z}^3$, or $\pi_1(\mathcal{B}_1)$, or $\pi_1(\mathcal{B}_2)$, and*

*(i)* $$s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) = \omega(\frac{n}{2});$$

*(ii)* $$s_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n) = \chi(n) - \chi(\frac{n}{2});$$

*(iii)* $$s_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n) = 2\chi(\frac{n}{2}) - 2\chi(\frac{n}{4}).$$

**Theorem 12.** *Let $\mathcal{N} \to \mathcal{B}_1$ be an $n$-fold covering over $\mathcal{B}_1$. If $n$ is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{B}_1$. If $n$ is even then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_1$ or $\mathcal{B}_1$ or $\mathcal{B}_2$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

*(i)* $$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_1)}(n) = \frac{1}{2}\Big(\omega(\frac{n}{2}) + \sigma_2(\frac{n}{2}) + 3\sigma_2(\frac{n}{4})\Big);$$

*(ii)* $$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_1)}(n) = \sigma_2(n) - \sigma_2(\frac{n}{4});$$

*(iii)* $$c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_1)}(n) = 2\sigma_2(\frac{n}{2}) - 2\sigma_2(\frac{n}{4}).$$

**Theorem 13.** *Every subgroup $\Delta$ of finite index $n$ in $\pi_1(\mathcal{B}_2)$ is isomorphic to either $\mathbb{Z}^3$, or $\pi_1(\mathcal{B}_2)$, or $\pi_1(\mathcal{B}_1)$, and*

*(i)* $$s_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(n) = \omega(\frac{n}{2});$$

*(ii)* $$s_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_2)}(n) = \chi(n) - 5\chi(\frac{n}{2}) + 12\chi(\frac{n}{4}) - 8\chi(\frac{n}{8});$$

*(iii)* $$s_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_2)}(n) = 2\chi(\frac{n}{2}) - 2\chi(\frac{n}{4}).$$

**Theorem 14.** *Let $\mathcal{N} \to \mathcal{B}_2$ be an $n$-fold covering over $\mathcal{B}_2$. If $n$ is odd then $\mathcal{N}$ is homeomorphic to $\mathcal{B}_2$. If $n$ is even then $\mathcal{N}$ is homeomorphic to $\mathcal{G}_1$ or $\mathcal{B}_1$ or $\mathcal{B}_2$. The corresponding numbers of nonequivalent coverings are given by the following formulas:*

*(i)* $$c_{\mathbb{Z}^3,\pi_1(\mathcal{B}_2)}(n) = \frac{1}{2}\Big(\omega(\frac{n}{2}) + \sigma_2(\frac{n}{2}) - \sigma_2(\frac{n}{4}) + 4\sigma_2(\frac{n}{8})\Big);$$

*(ii)* $$c_{\pi_1(\mathcal{B}_2),\pi_1(\mathcal{B}_2)}(n) = \sigma_2(n) - 4\sigma_2(\frac{n}{2}) + 7\sigma_2(\frac{n}{4}) - 4\sigma_2(\frac{n}{8});$$

*(iii)* $$c_{\pi_1(\mathcal{B}_1),\pi_1(\mathcal{B}_2)}(n) = 2\sigma_2(\frac{n}{2}) - 2\sigma_2(\frac{n}{4}).$$

Theorems 11, 12, 13 and 14 coincide with theorems 1,3,4,6 of the original article respectively. Note that here we present the this theorems in uniform notation, invented later.

The sequences obtained above have a remarkably smooth expression in terms of the Dirichlet generating series, which we present in the following table.

| H \ G | | $\mathcal{B}_1$ | $\mathcal{B}_2$ |
|---|---|---|---|
| $\mathbb{Z}^3$ | $\widehat{s}_{H,G}$ | $2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ | $2^{-s}\zeta(s)\zeta(s-1)\zeta(s-2)$ |
| | $\widehat{c}_{H,G}$ | $2^{-s-1}\zeta(s)\zeta(s-1)\Big(\zeta(s-2)+(1+3\cdot 2^{-s})\zeta(s)\Big)$ | $2^{-s-1}\zeta(s)\zeta(s-1)\Big(\zeta(s-2)+(1-2^{-s}+4\cdot 2^{-2s})\zeta(s)\Big)$ |
| $\mathcal{B}_1$ | $\widehat{s}_{H,G}$ | $\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)^2$ | $2^{-s+1}\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)^2$ |
| | $\widehat{c}_{H,G}$ | $\big(1-2^{-s}\big)\big(1+2^{-s}\big)\zeta(s)^2\zeta(s-1)$ | $2^{-s+1}\big(1-2^{-s}\big)\zeta(s)^2\zeta(s-1)$ |
| $\mathcal{B}_2$ | $\widehat{s}_{H,G}$ | $2^{-s+1}\big(1-2^{-s}\big)\zeta(s)\zeta(s-1)^2$ | $\big(1-2^{-s}\big)\big(1-4\cdot 2^{-s}+8\cdot 2^{-2s}\big)\zeta(s)\zeta(s-1)^2$ |
| | $\widehat{c}_{H,G}$ | $2^{-s+1}\big(1-2^{-s}\big)\zeta(s)^2\zeta(s-1)$ | $\big(1-2^{-s}\big)\big(1-3\cdot 2^{-s}+4\cdot 2^{-2s}\big)\zeta(s)^2\zeta(s-1)$ |

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