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Geometry of moduli spaces of semistable rank 2 sheaves on projective space

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Introduction

Following the proof of existence of a projective moduli scheme parametrizing S-equivalence classes of semistable sheaves on a projective variety by Maruyama [12], the study of the geometry of such moduli spaces has been a central topic of research within algebraic geometry. Although a lot is known for curves and surfaces, general results for three dimensional varieties are still lacking. In fact, moduli spaces of sheaves on 3-folds turn out to be quite complicated spaces (as it is illustrated by Vakil's Murphy's law [7]), particularly with several irreducible components of various dimensions.

The goal of this thesis is to advance on the study of the moduli space of semistable rank 2 sheaves on $\mathbb{P}^3$ with fixed Chern classes $c_1 = 0, c_2 = k, c_3 = 0$, which we will denote by $\mathcal{M}(k)$. Questions on the geometry of such spaces, such as the number of irreducible components, seem to be less explored if compared to the study of the geometry of the Hilbert schemes of curves in the projective 3-space for instance.

The summary is organized as follows. In Section 1 we introduce different notions of stability of sheaves on projective smooth varieties and provide some properties of semistable sheaves. In Section 2 we remind the basic GIT construction of the moduli space of semistable sheaves. Sections 1 and 2 mainly follow the content of the book [8]. In Section 3 we define reflexive sheaves and discuss their properties. In Section 4 we describe all known irreducible components of the moduli schemes $\mathcal{M}(k),\ k \geq 3$. Finally, Section 5 contains the main results of the present thesis, namely, the description of new irreducible components of $\mathcal{M}(k),\ k \geq 3$.

1 Stability of sheaves

Historically, the notion of stability for coherent sheaves first appeared in the context of vector bundles on curves [14]: let X be a smooth projective curve over an algebraically closed field k, and let E be a locally free sheaf of rank r and degree $\deg(E)$. Define *the slope* of E as $\mu(E) = \frac{\deg(E)}{r}$. Then E is said to be (semi)stable, if for all subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$ one has $\mu(F)(\leq)\mu(E)$.

If we pass from sheaves on curves to higher dimensional varieties the notion of stability can be generalized as follows. Let $(X, \mathcal{O}(1))$ be a polarized smooth projective variety of dimension d over an algebraically closed field k. Recall that the Euler characteristic of a coherent sheaf E is $\chi(E) = \Sigma(-1)^i h^i(X,E)$, where $h^i(X,E) = \dim_k H^i(X,E)$. The Hilbert polynomial $P(E)$ is given by $m \mapsto \chi(E \otimes \mathcal{O}(m))$.

Definition 1 *The support of E is the closed set* $\mathrm{Supp}(E) = \{x \in X \mid E_x \neq 0\}$. *Its dimension is called the dimension of the sheaf E and is denoted by* $\dim(E)$.

Definition 2 *E is pure of dimension d if* $\dim(F) = d$ *for all non-trivial coherent subsheaves $F \subset E$.*

Definition 3 *For any sheaf E over X there exists an open dense subset $U \subset X$ such that $E|_U$ is locally free. Then the rank of the vector bundle $E|_U$ is called the rank* $\mathrm{rk}(E)$ *of E.*

Definition 4 *The reduced Hilbert polynomial $p(E)$ of a coherent sheaf E of dimension d is defined by*

$$p(E,m) := \frac{P(E,m)}{\mathrm{rk}(E)}.$$

Recall that there is a natural ordering of polynomials given by the lexicographic order of their coefficients. Explicitly, $f \leq g$ if and only if $f(m) \leq g(m)$ for $m \gg 0$. Analogously, $f < g$ if and only if $f(m) < g(m)$ for $m \gg 0$. We are now prepared for the definition of stability.

Definition 5 *A coherent sheaf E of dimension d is semistable if E is pure and for any proper subsheaf $F \subset E$ one has $p(F) \leq p(E)$. E is called stable if E is semistable and the inequality is strict, i.e. $p(F) < p(E)$ for any proper subsheaf $F \subset E$.*

Proposition 1 (see [8, Cor. 1. 2. 8]) *Any stable sheaf E over X is simple, i. e.* $\mathrm{End}(E) \simeq k$.

Another way to define stable sheaves on the higher dimensional varieties is the straightforward generalization of the notion of the slope. More precisely, let E be a coherent sheaf of dimension $d = \dim(X)$ and H be an ample divisor associated with $\mathcal{O}(1)$. The degree of E can be defined in the following way

$$\deg(E) := c_1(E).H^{n-1}$$

and its slope

$$\mu(E) := \frac{\deg(E)}{\mathrm{rk}(E)}.$$

Definition 6 *A coherent sheaf E of dimension $d = \dim(X)$ is μ-(semi)stable if E is pure and $\mu(F)(\leq)\mu(E)$ for all subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$.*

Proposition 2 (see [8, Lemma 1. 2. 13]) *If E is a pure coherent sheaf of dimension $d = \dim(X)$, then one has the following chain of implications*

$$E \text{ is } \mu\text{-stable} \Rightarrow E \text{ is stable} \Rightarrow E \text{ is semistable } \Rightarrow E \text{ is } \mu\text{-semistable.}$$

Further a (μ-)semistable sheaf which is not (μ-)stable we will call *properly (μ-)semistable sheaf* (the notation *strictly semistable sheaf* also appears in the literature).

Let E be a non-trivial pure sheaf of dimension d. *A Harder-Narasimhan filtration* for E is an increasing filtration

$$0 = \mathrm{NH}_0(E) \subset \mathrm{NH}_1(E) \subset ... \subset \mathrm{NH}_l(E) = E$$

such that the factors $\mathrm{gr}_i^{\mathrm{NH}} = \mathrm{NH}_i(E)/\mathrm{NH}_{i+1}(E)$ for $i = 1, ..., l$, are semistable sheaves of dimension d with reduced Hilbert polynomials $p_i = p(\mathrm{gr}_i^{\mathrm{NH}})$ satisfying

$$p_{\max}(E) := p_1 > ... > p_l =: p_{\min}(E).$$

Assume now that E is semistable. *A Jordan-Hölder filtration* of E is a filtration

$$0 = E_0 \subset E_1 \subset ... \subset E_l = E,$$

such that the factors $\mathrm{gr}_i(E) = E_i/E_{i-1}$ are stable with reduced Hilbert polynomial $p(E)$.

Proposition 3 (see [8, Th. 1.3.4]) *Every pure sheaf E has a unique Harder-Narasimhan filtration.*

Proposition 4 (see [8, Prop. 1.5.2]) *Every semistable sheaf E has a Jordan-Hölder filtration. The graded object $\mathrm{gr}(E) = \bigoplus_i \mathrm{gr}_i(E)$ does not depend on the choice of the Jordan-Hölder filtration.*

Definition 7 *Two semistable sheaves E_1 and E_2 with the same reduced Hilbert polynomial are called S-equivalent if $\mathrm{gr}(E_1) \simeq \mathrm{gr}(E_2)$.*

2 The construction of moduli scheme

For a fixed polynomial $P \in \mathbb{Q}[z]$ define a functor

$$\mathfrak{M}' : (Sch/k)^o \to (Sets)$$

as follows. If $S \in \mathrm{Ob}(Sch/k)$, let $\mathfrak{M}'(S)$ be the set of isomorphism classes of S-flat families of semistable sheaves on X with Hilbert polynomial P. And if $f : S' \to S$ is a morphism in (Sch/k), let $\mathfrak{M}'(f)$ be the map obtained by pulling-back sheaves via $f_X = f \times \mathrm{id}_X$:

$$\mathfrak{M}'(f) : \mathfrak{M}'(S) \longrightarrow \mathfrak{M}'(S'), \quad [E] \to [f_X^* E].$$

If $E \in \mathfrak{M}'(S)$ is an S-flat family of semistable sheaves on X, and if L is an arbitrary line bundle on S, then $E \otimes p^*(L)$ is also an S-flat family, where $p : X \times S \to S$ is the natural projection, and the fibres E_s and $(E \otimes p^* L)_s = E_s \otimes_{k(s)} L(s)$ are isomorphic for each point $s \in S$. It is therefore reasonable to consider the quotient functor $\mathfrak{M} = \mathfrak{M}'/\sim$, where $\sim$ is the equivalence relation:

$E \sim E'$ for $E,\ E' \in \mathfrak{M}'(S)$ if and only if $E \simeq E' \otimes p^*L$ for some $L \in \mathrm{Pic}(S)$.

If we take families of stable sheaves only, we get open subfunctors $(\mathfrak{M}')^s \subset \mathfrak{M}'$ and $\mathfrak{M}^s \subset \mathfrak{M}$. A scheme $\mathcal{M}$ is called *a moduli space of semistable sheaves* if it corepresents the functor $\mathfrak{M}$.

Proposition 5 (see [8, Lemma 4.1.2]) *Suppose $\mathcal{M}$ corepresents $\mathfrak{M}$. Then S-equivalent sheaves correspond to identical closed points in $\mathcal{M}$. In particular, if there is an indecomposable properly semistable sheaf E then $\mathfrak{M}$ cannot be represented.*

The family of semistable sheaves on X with Hilbert polynomial equal to P is bounded (see [8, Th. 3.3.7]). In particular, it means that there is an integer m such that $E(m)$ is globally generated and $h^0(E(m)) = P(m)$. Thus if we let $V := k^{\oplus P(m)}$ and $\mathcal{K} := V \otimes \mathcal{O}_X(-m)$, then there is a surjection $\rho : \mathcal{K} \twoheadrightarrow E$ obtained by composing the canonical evaluation map $\mathrm{H}^0(E(m)) \otimes \mathcal{O}_X(-m) \longrightarrow E$ with an isomorphism $V \longrightarrow \mathrm{H}^0(E(m))$. This defines a closed point $[\rho : \mathcal{K} \twoheadrightarrow E] \in \mathrm{Quot}(\mathcal{K}, P)$ of the corresponding Quot-scheme parameterising quotient sheaves of the sheaf $\mathcal{K}$ with the Hilbert polynomial P. In fact, this point is contained in the open subset $R \subset \mathrm{Quot}(\mathcal{K}, P)$ of all those quotients $[\mathcal{K} \twoheadrightarrow E]$, where E is semistable and the induced map

$$V = \mathrm{H}^0(\mathcal{K}(m)) \longrightarrow \mathrm{H}^0(E(m))$$

is an isomorphism. Let $R^s \subset R$ denote the open subscheme of those points which parametrize stable sheaves E.

Theorem 1 (see [8, Th. 4.3.4]) *There is a projective scheme $\mathcal{M}_{\mathcal{O}(1)}(P)$ that universally corepresents the functor $\mathfrak{M}_{\mathcal{O}(1)}(P)$. Closed points in $\mathcal{M}_{\mathcal{O}(1)}(P)$ are in bijection with S-equivalence classes of semistable sheaves with Hilbert* *polynomial P. Moreover, there is an open subset $\mathcal{M}^s_{\mathcal{O}(1)}(P)$ that universally corepresents the functor $\mathfrak{M}^s_{\mathcal{O}(1)}(P)$.*

Theorem 2 (see [8, Cor. 4.3.5]) *The morphism $\pi : R^s \longrightarrow \mathcal{M}^s$ is a principal $PGL(V)$-bundle.*

One also needs to consider relative moduli spaces, i.e. moduli spaces of semistable sheaves on the fibres of a projective morphism $X \to S$. Consider for a given polynomial P the functor $\mathfrak{M}_{X/S} : (Sch/S)^o \longrightarrow (Sets)$, which by definition associates to an S-scheme T of finite type the set of isomorphism classes of T-flat families of semistable sheaves on the fibres of the morphism $X_T := T \times_S X \to T$ with Hilbert polynomial P. If we take families of stable sheaves only, we get open subfunctor $\mathfrak{M}^s_{X/S}(P) \subset \mathcal{M}_{X/S}(P)$.

Theorem 3 (see [8, Th. 4.3.7]) *Let $f : X \to S$ be a projective morphism of k-schemes of finite type with geometrically connected fibres, and let $\mathcal{O}(1)$ be a line bundle on X very ample relative to S. There is a projective morphism $\mathcal{M}_{X/S}(P) \to S$ which universally corepresents the functor $\mathfrak{M}_{X/S}(P)$. In particular, for any closed point $s \in S$ one has $\mathcal{M}_{X/S}(P)_s \simeq \mathcal{M}_{X_s}(P)$. Moreover, there is an open subscheme $\mathcal{M}^s_{X/S}(P) \subset \mathcal{M}_{X/S}(P)$ that universally corepresents the subfunctor $\mathfrak{M}^s_{X/S}(P) \subset \mathfrak{M}_{X/S}(P)$.*

Further we will be concentrated on the Gieseker-Maruyama moduli scheme of semistable rank-2 sheaves with Chern classes $c_1 = 0,\ c_2 = k,\ c_3 = 2n$ on the projective space $\mathbb{P}^3$ which we will denote by $\mathcal{M}(0, k, 2n)$. Also denote $\mathcal{M}(k) = \mathcal{M}(0, k, 0)$. In addition, we define $\mathcal{B}(k)$ to be the open subset of $\mathcal{M}(k)$ consisting of stable locally free sheaves. For simplicity we will not make a distinction between a stable sheaf E and corresponding isomorphism class $[E]$ as a point of moduli scheme. Also by a general point of an irreducible scheme we understand a closed point belonging to some Zariski open dense subset of this scheme.

3 Reflexive sheaves

A sheaf F is called reflexive if the natural map $F \to F^{\vee\vee}$ is an isomorphism. The singularity set $\mathrm{Sing}(F)$ of a reflexive sheaf F on X is of codimension ≥ 3. In particular, a reflexive sheaf on $\mathbb{P}^3$ has zero-dimensional singularities. Moreover, any reflexive rank 1 sheaf is invertible.

Theorem 4 (see [4, Th. 4.1]) *Fix an integer c_1. Then there is a one-to-one correspondence between*

(i) pairs (F, s) where F is a rank 2 reflexive sheaf on $\mathbb{P}^3$ with $c_1(F) = c_1$, and $s \in \mathrm{H}^0(F)$ is a global section whose zero-set has codimension 2, and

(ii) pairs (Y, ξ), where Y is a Cohen-Macaulay curve in $\mathbb{P}^3$, generically locally complete intersection, and $\xi \in \mathrm{H}^0(\omega_Y(4 - c_1))$ is a global section which generates the sheaf $\omega_Y(4 - c_1)$ except at finitely many points.

Furthermore under this correspondence

$$c_2 = d, \quad c_3 = 2p_a - 2 + d(4 - c_1),$$

where c_2, c_3 are the Chern classes of F, and d, p_a are the degree and arithmetic genus of Y.

The moduli scheme $\mathcal{R}(0, m, 2n)$ parameterizing stable reflexive rank-2 sheaves on $\mathbb{P}^3$ with Chern classes $c_1 = 0$, $c_2 = m$, $c_3 = 2n$ can be considered as an open subset of the Gieseker-Maruyama moduli scheme $\mathcal{M}(0, m, 2n)$, so it is a quasi-projective scheme (see [4]). It is known that for $(m, n) = (2, 1)$, $(2, 2)$, $(3, 4)$ this scheme is smooth, irreducible and rational; for $(m, n) = (3, 2)$ it is irreducible and reduced at general point; for $(m, n) = (3, 1)$, $(3, 3)$ the corresponding reduced scheme is irreducible (see [3]). Moreover, the scheme $\mathcal{R}(0, m, m^2 - m + 2)$ is irreducible and smooth for each $m \geq 2$ (see [22]).

Theorem 5 (see [10, Th. 8]) *For each triple (a,b,c) of positive integers such that $3a+2b+c$ is nonzero and even, the rank 2 reflexive sheaves given by*

$$0 \longrightarrow G_{(a,b,c)} \xrightarrow{\alpha} (a+b+c+2)\cdot \mathcal{O}_{\mathbb{P}^3} \longrightarrow F(k) \longrightarrow 0,$$

$$G_{(a,b,c)} := a\cdot \mathcal{O}_{\mathbb{P}^3}(-3) \oplus b\cdot \mathcal{O}_{\mathbb{P}^3}(-2) \oplus c\cdot \mathcal{O}_{\mathbb{P}^3}(-1),$$

fill out an irreducible, nonsingular, component $\mathcal{S}(a,b,c)$ of $\mathcal{R}(0,m,2n)$ of expected dimension $8m-3$, where m and n are given by the expressions:

$$m = \frac{1}{4}(3a+2b+c)^2 + \frac{3}{2}(3a+2b+c) - (b+c),$$

$$2n = 27\binom{a+2}{3} + 8\binom{b+2}{3} + \binom{c+2}{3} + 3(3a+2b+5)ab +$$
$$+\frac{3}{2}(3a+c+4)ac + (2b+3c+3)bc + 6abc.$$

More precisely, let $\widetilde{\mathcal{S}}(a,b,c) \subset \mathrm{Hom}(G_{(a,b,c)}, (a+b+c+2)\cdot\mathcal{O}_{\mathbb{P}^3})$ be the open subset consisting of monomorphisms with 0-dimensional degeneracy loci; then

$$\mathcal{S}(a,b,c) = \widetilde{\mathcal{S}}(a,b,c)/\Big(\Big(\mathrm{Aut}(G_{(a,b,c)}) \times GL(a+b+c+2)\Big)/\mathbb{C}^*\Big).$$

Also we can construct a scheme $\mathcal{V}(0,m,2n)$ parameterizing some set of reflexive properly μ-semistable rank-2 sheaves with the corresponding Chern classes in the following way. Consider the Hilbert scheme $\mathrm{Hilb}_{m,g}(\mathbb{P}^3)$ of smooth space curves of degree m and genus g; let $n = g+2m-1$. Now denote by $\mathcal{Z} \hookrightarrow \mathrm{Hilb}_{m,g}(\mathbb{P}^3)\times\mathbb{P}^3$ the corresponding universal curve and $\mathrm{pr}: \mathrm{Hilb}_{m,g}(\mathbb{P}^3)\times\mathbb{P}^3 \longrightarrow \mathrm{Hilb}_{m,g}(\mathbb{P}^3)$ the projection onto the first factor. We define the scheme $\mathcal{V}(0,m,2n)$ as an open subset of $\mathbf{P}((\mathrm{pr}_*\omega_{\mathcal{Z}}(4))^\vee)$ the points $(Y,\ \mathbb{P}\xi) \in \mathbf{P}((\mathrm{pr}_*\omega_{\mathcal{Z}}(4))^\vee)$ of which satisfy the following property

$$\xi \in \mathrm{H}^0(\omega_Y(4)) \text{ generates } \omega_Y(4) \text{ except at finitely many points.}$$

By the construction we have the formula for the dimension of this scheme

$$\dim\ \mathcal{V}(0,m,2n) = \dim\ \mathrm{Hilb}_{m,g}(\mathbb{P}^3) + \dim\ \mathbb{P}(\mathrm{H}^0(\omega_Y(4))) = \tag{1}$$

$$= h^0(N_{Y/\mathbb{P}^3}) + h^0(\omega_Y(4)) - 1,$$

where Y is an arbitrary curve from $\mathrm{Hilb}_{m,g}(\mathbb{P}^3)$. Next, note that due to the isomorphism $\mathrm{H}^0(\omega_Y(4)) \simeq \mathrm{Ext}^1(I_Y, \mathcal{O}_{\mathbb{P}^3})$ any point $(Y, \mathbb{P}\xi) \in \mathcal{V}(0,m,2n)$ uniquely defines the sheaf F which fits in the exact triple

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow F \longrightarrow I_Y \longrightarrow 0. \tag{2}$$

One can show that F is a reflexive properly μ-semistable rank-2 sheaf with Chern classes $c_1 = 0,\ c_2 = m,\ c_3 = 2n$. Therefore, there exists one-to-one correspondence between points of $\mathcal{V}(0,m,2n)$ and some family of reflexive properly μ-semistable rank-2 sheaves with Chern classes $c_1 = 0,\ c_2 = m$, $c_3 = 2n$ (for more details, see [4, Thm. 4.1, Prop. 4.2]).

If curve Y from the previous construction is rational then we have $n = 2m - 1$. Denote the corresponding parameter space $\mathcal{V}(0,m,4m-2)$ by just $\mathcal{V}_m$.

4 Irreducible decomposition of $\mathcal{M}(k),\ k \geq 1$

It is not difficult to check that $\mathcal{M}(1)$ is irreducible. The key point is to show that every semistable rank 2 sheaf E on $\mathbb{P}^3$ with $c_1(E) = 0,\ c_2(E) = 1$ and $c_3(E) = 0$ is a nullcorrelation sheaf in the sense of [1], that is, given by an exact sequence of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow \Omega^1_{\mathbb{P}^3}(1) \longrightarrow E \longrightarrow 0.$$

It follows that E is uniquely determined by the section $\sigma \in \mathrm{H}^0(\Omega^1_{\mathbb{P}^3}(2))$ up to scalar multiples, so that $\mathcal{M}(1) \simeq \mathbb{P}\mathrm{H}^0(\Omega^1_{\mathbb{P}^3}(2)) \simeq \mathbb{P}^5$.

For $k \geq 2$ the moduli schemes $\mathcal{M}(k)$ become reducible. However, some infinite series of components of $\mathcal{M}(k)$ have been constructed. We will describe them in this section.

First of all, for any $k \geq 1$ there exists so called the moduli space of instanton bundles $\mathcal{I}(k)$ which can be considered as a subscheme of the moduli scheme of stable bundles $\mathcal{B}(k) \subset \mathcal{M}(k)$. An instanton bundle of charge k is a rank-2 bundle E satisfying the following properties

$$c_1(E) = 0, \quad c_2(E) = k, \quad h^0(E(-1)) = h^1(E(-2)) = 0.$$

It is known that the moduli space $\mathcal{I}(k)$ is irreducible (see [19, 20]), non-singular (see [21]) and affine (see [18]). The closure $\overline{\mathcal{I}(k)}$ within $\mathcal{M}(k)$ is the irreducible component of $\mathcal{M}(k)$ of the dimension $8k-3$. In particular, $\mathcal{M}(1) \simeq \overline{\mathcal{I}(1)}$, $\mathcal{M}(1) \setminus \mathcal{I}(1) \simeq \mathrm{Gr}(2,4) \subset \mathbb{P}^3$, and $\mathcal{I}(1)$ parameterizes nullcorrelation bundles.

For $k = 1, 2$ we have that $\mathcal{B}(k) = \mathcal{I}(k)$. However, for $k \geq 3$ this is no longer the case and the scheme $\mathcal{B}(k)$ also becomes reducible. Further we describe the series of components of $\mathcal{B}(k)$ which do not coincide with $\mathcal{I}(k)$. More precisely, for any three integers $c > b \geq a \geq 0$ consider the monad

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-c) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^3}(-b) \oplus \mathcal{O}_{\mathbb{P}^3}(-a) \oplus \mathcal{O}_{\mathbb{P}^3}(a) \oplus \mathcal{O}_{\mathbb{P}^3}(b) \xrightarrow{\beta} \mathcal{O}_{\mathbb{P}^3}(c) \longrightarrow 0, \tag{3}$$

with morphisms given by

$$\alpha = \begin{pmatrix} \sigma_4 \\ \sigma_3 \\ -\sigma_2 \\ -\sigma_1 \end{pmatrix}$$

and $\beta = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where

$$\sigma_1 \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(c+b)), \quad \sigma_2 \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(c+a)),$$

$$\sigma_3 \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(c-a)), \quad \sigma_4 \in \mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(c-b))$$

do not vanish simultaneously.

Theorem 6 (see [2, Prop. 1.2(a)]) *For $c > a + b$ there exists an irreducible component $\mathcal{N}(a, b, c)$ of $\mathcal{B}(c^2 - b^2 - a^2)$ whose points correspond to locally free sheaves given as the cohomologies of monads as in (3). The closure $\overline{\mathcal{N}(a, b, c)}$ is an irreducible component of the moduli scheme $\mathcal{M}(c^2 - b^2 - a^2)$. These components are called Ein components.*

The next theorem describes the series of components of $\mathcal{M}(k)$ whose general sheaves have 0-dimensional singularities. These components are constructed by using components of the moduli schemes parameterizing stable reflexive sheaves. For example, we can use the series $\mathcal{S}(a, b, c)$ from Theorem 5.

Theorem 7 (see [10, Th. 7]) *For every nonsingular irreducible component $\mathcal{F}$ of $\mathcal{R}(0, k, 2s)$ of expected dimension $8k-3$, there exists an irreducible component $\overline{\mathcal{T}(k, s)}$ of dimension $8k - 3 + 4s$ in $\mathcal{M}(k)$ whose generic point $[E]$ fits into the exact triple*

$$0 \longrightarrow E \longrightarrow F \longrightarrow Q \longrightarrow 0,$$

where $[F] \in \mathcal{F}$ and Q is a zero-dimensional sheaf of the length s.

In the following theorem the series of components of $\mathcal{M}(k)$ parameterising semistable sheaves with 1-dimensional singularities is presented.

Theorem 8 (see [10, Th. 17]) *For any positive integers $0 < d_1 \leq d_2$ and non-negative integer $c \geq 0$ there exists the irreducible component $\overline{\mathcal{C}(d_1, d_2, c)}$ of $\mathcal{M}(d_1 d_2 + c)$ whose general sheaf $[E]$ fits in the following exact triple*

$$0 \longrightarrow E \longrightarrow F \longrightarrow L(2) \longrightarrow 0,$$

where $[F] \in \mathcal{I}(c)$ and L is a line bundle over smooth complete intersection curve C of bidegree (d_1, d_2) and genus $g = 1 + \frac{1}{2} d_1 d_2 (d_1 + d_2 - 4)$ such that

$$\deg L = g - 1, \quad h^0(L) = h^1(L) = 0, \quad L^{\otimes 2} \not\simeq \omega_C.$$

Next, we recall the description of $\mathcal{M}(2)$ given by Hartshorne [5] and Le Potier [6]. Firstly, the moduli scheme $\mathcal{M}(2)$ contains the instanton component $\overline{\mathcal{I}(2)}$. Moreover, all locally free sheaves from $\mathcal{M}(2)$ are instanton bundles. Next, according to [6, Thm. 7.12], $\mathcal{M}(2)$ contains two additional irreducible components, which are given by the closures of the subschemes

$$\mathcal{P}(2)_s = \{[E] \in \mathcal{M}(2) \mid \dim \mathrm{Ext}^2(E, \mathcal{O}_{\mathbb{P}^3}) = s\},\ \ s = 1, 2$$

within $\mathcal{M}(2)$; furthermore, $\dim \overline{\mathcal{P}(2)_s} = 13 + 4s$. Le Potier calls these the Trautmann components.

Note that these actually coincides with the components $\overline{\mathcal{T}(2,s)}$ described above. Indeed, note that if $[E] \in \mathcal{T}(2,s)$, then

$$\dim \mathrm{Ext}^2(E, \mathcal{O}_{\mathbb{P}^3}) = h^0(\mathcal{E}xt^2(E, \mathcal{O}_{\mathbb{P}^3})) = h^0(\mathcal{E}xt^3(Q_E, \mathcal{O}_{\mathbb{P}^3})) = h^0(Q_E).$$

However, the length of Q_E is half of $c_3(E^{\vee\vee})$, which means that $[E] \in \mathcal{P}(2)_s$, thus $\mathcal{T}(2,s) \subset \mathcal{P}(2)_s$.

From the previous section we know that, for each $s = 1, 2$, $\mathcal{R}(0, 2, 2s)$ is irreducible, nonsingular of dimension 13. It follows from Theorem 7 that, for each $s = 1, 2$, $\overline{\mathcal{T}(2,s)}$ is an irreducible component of $\mathcal{M}(2)$ of dimension $13 + 4s$; therefore, we must have that $\overline{\mathcal{T}(2,s)} = \overline{\mathcal{P}(2)_s}$.

Consequently, Le Potier's result can be restated in the following form:

$$\mathcal{M}(2) = \overline{\mathcal{I}(2)} \cup \overline{\mathcal{T}(2,1)} \cup \overline{\mathcal{T}(2,2)}.$$

Ellingsrud and Stromme showed in [13] that $\mathcal{B}(3)$ has precisely two irreducible components, both nonsingular, rational and of the expected dimension 21; these can be described as follows:

- the instanton component $\mathcal{I}(3)$, whose points are the cohomology of monads of the form

$$0 \longrightarrow 3\mathcal{O}_{\mathbb{P}^3}(-1) \longrightarrow 8\mathcal{O}_{\mathbb{P}^3} \longrightarrow 3\mathcal{O}_{\mathbb{P}^3} \longrightarrow 0;$$

- the Ein component $\mathcal{N}(0,1,2)$ whose points are the cohomology of monads of the form

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus 2\mathcal{O}_{\mathbb{P}^3} \oplus \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2) \longrightarrow 0.$$

As it was mentioned in the previous section, $\mathcal{R}(0,3,2s)$ is irreducible and of expected dimension 21 for each $s = 1, ..., 4$. Therefore, we can apply Theorem 7 to show that there are four irreducible components $\overline{\mathcal{T}(3,s)}$ of dimensions $21+4s$ for each $s = 1, ..., 4$ within $\mathcal{M}(3)$.

Furthermore, Theorem 8 provides one additional irreducible component whose generic point corresponds to sheaves with 1-dimensional singularities, labeled $\overline{\mathcal{C}(1,3,0)}$.

We therefore conclude that $\mathcal{M}(3)$ has at least seven irreducible components, divided into 3 types, as below:

1. $\overline{\mathcal{I}(3)}$ and $\overline{\mathcal{N}(0,1,2)}$, both of dimension 21, and whose generic points correspond to locally free sheaves;

2. $\overline{\mathcal{C}(1,3,0)}$, of dimension 21; whose generic point corresponds to a sheaf which is singular along smooth plane cubic;

3. $\overline{\mathcal{T}(3,s)}$ for $s = 1,2,3,4$, of dimension $21+4s$; whose generic point corresponds to a sheaf which is singular along $3s$ distinct points.

5 New components of $\mathcal{M}(k), k \geq 3$

In the series of papers [15, 16, 17] there were described new irreducible components of the moduli schemes $\mathcal{M}(k)$, $k \geq 3$ starting with construction of one component of $\mathcal{M}(3)$ and then sequentially generalizing this construction to description of series of components. New feature of these new components is that their general sheaves have singularities of mixed dimension, namely, union of a curve and collection of points in $\mathbb{P}^3$.

The first component of this series was described in [15]. A general sheaf of this component has singularities along the union of projective line and two points. Similarly to the construction of the series $\overline{\mathcal{T}(k,s)}$ and $\overline{\mathcal{C}(d_1,d_2,c)}$ from Theorems 7 and 8 the construction of this new component is based on the technique of so called elementary transformations. More precisely, we consider all sheaves E fitting in the following exact triple

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_l(2) \longrightarrow 0, \tag{4}$$

where F is a reflexive sheaf from $\mathcal{R}(0,2,2)$ and l is a projective line such that $l \cap \mathrm{Sing}(F) = \emptyset$. One can show that E is a stable sheaf with Chern classes $c_1 = 0,\ c_2 = 3,\ c_3 = 0$. It happens that the dimension of the tangent space $T_{[E]}\mathcal{M}(3)$ of the moduli scheme $\mathcal{M}(3)$ at the point $[E]$ is equal to the dimension of the family of sheaves obtained by the exact triple of the form (4). We denote this family by $\mathcal{X}(1,0)$. Therefore, we have the following theorem

Theorem 9 (see [15, Th.]) *The closure of the family $\mathcal{X}(1,0)$ within $\mathcal{M}(3)$ is a new irreducible component of $\mathcal{M}(3)$ of dimension 22.*

One can see that a general sheaf of this component has singularities along $l \sqcup \mathrm{Sing}(F)$. In fact, it was the first example of a component whose general sheaf has singularities of mixed dimension.

Further this result was generalized in [16]. There were constructed two more components of the moduli scheme $\mathcal{M}(3)$. The general sheaves E of these components fit into the exact triple

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_l(r) \oplus \mathcal{O}_W \longrightarrow 0, \tag{5}$$

where $[F] \in \mathcal{R}(0,2,2n),\ n = 1,2,$ l is a projective line and $W = \{q_1, ..., q_s\} \in \mathrm{Sym}^s(\mathbb{P}^3)^*$ is a set of points in $\mathbb{P}^3$ such that $l \cap W = \emptyset,\ (l \sqcup W) \cap \mathrm{Sing}(F) = \emptyset$. The condition $c_1(E) = 0$ imposes the following restriction $r = n + 1 - s$, so we have the seven families of sheaves $\mathcal{X}(n,s)$, where

$$(n,s) = (1,0),\ (1,1),\ (1,2),\ (2,0),\ (2,1),\ (2,2),\ (2,3).$$

Four of them lie inside the known components $\overline{\mathcal{T}(3,s)}$.

Theorem 10 (see [16, Th.4]) *We have the following proper inclusions:*

$$\mathcal{X}(1,1) \subsetneq \overline{\mathcal{T}(3,1)}, \quad \mathcal{X}(2,2) \subsetneq \overline{\mathcal{T}(3,2)},$$

$$\mathcal{X}(1,2) \subsetneq \overline{\mathcal{T}(3,2)}, \quad \mathcal{X}(2,3) \subsetneq \overline{\mathcal{T}(3,3)}.$$

So their closures do not give new components. The family $\mathcal{X}(1,0)$ was already discussed previously. However, dimensions of the families $\mathcal{X}(2,0)$ and $\mathcal{X}(2,1)$ coincide with the corresponding dimensions of the tangent spaces of $\mathcal{M}(3)$, so their closures constitute irreducible components $\overline{\mathcal{X}(2,0)}$ and $\overline{\mathcal{X}(2,1)}$. Due to the fact that the singularity sets of general sheaves of these two components, namely, $l \sqcup \mathrm{Sing}(F) \sqcup W$, where $|\mathrm{Sing}(F)| = 4$, $|W| = 0,1$, do not coincide with the singularity sets of general sheaves of other components of $\mathcal{M}(3)$, the components $\overline{\mathcal{X}(2,0)}$ and $\overline{\mathcal{X}(2,1)}$ are new.

Theorem 11 (see [16, Th. 3]) *The closures of the families $\mathcal{X}(2,0)$ and $\mathcal{X}(2,1)$ are new irreducible components of $\mathcal{M}(3)$ of dimensions 24 and 26, respectively.*

The construction from [15, 16] was generalized in the paper [11]. More precisely, in this paper new irreducible components of the moduli schemes $\mathcal{M}(e,n,m)$, $e = -1, 0$ whose general sheaves have singularities along the disjoint union of a projective line and a collection of points in $\mathbb{P}^3$ were constructed.

These results were further generalized in [17]. Namely, an infinite series of components of $\mathcal{M}(k)$, $k \geq 3$ whose general sheaves have singularities of mixed dimension was constructed. The construction generalizes computations from [10, 15, 16, 11]. More precisely, let us consider two series of components of the Hilbert schemes Hilb_d, $d \geq 1$, and $\mathrm{Hilb}_{(d_1,d_2)}$, $1 \leq d_1 \leq d_2$, where Hilb_d parameterizes smooth rational curves of degree d and $\mathrm{Hilb}_{(d_1,d_2)}$ parameterizes smooth complete intersection curves of bidegree (d_1, d_2) in $\mathbb{P}^3$. For the Hilbert schemes $\mathrm{Hilb}_{(d_1,d_2)}$ we will assume that $1 \leq d_1 \leq d_2$ and $(d_1, d_2) \neq (1, 1), (1, 2)$. Denote by $\mathcal{H}$ some component from the collection $\{\mathrm{Hilb}_d \mid d \geq 1\} \sqcup \{\mathrm{Hilb}_{(d_1,d_2)} \mid 1 \leq d_1 \leq d_2, (d_1, d_2) \neq (1, 1), (1, 2)\}$. Let C be a curve from $\mathcal{H}$ with degree d and genus g. Next, consider the subset $W \subset \mathbb{P}^3$ of s disjoint points in $\mathbb{P}^3$ satisfying the condition $C \cap W = \emptyset$.

Now consider the series of components $\mathcal{S}(a, b, c)$ of moduli spaces of stable reflexive sheaves from Theorem 5 and also consider the series of moduli spaces $\mathcal{V}_m$ for properly μ-semistable reflexive sheaves from Section 3. Denote by $\mathcal{R}$ some component from the collection $\{\mathcal{S}(a, b, c)\} \sqcup \{\mathcal{V}_m\}$ such that the following restrictions are satisfied

$$\begin{cases} s < n, \text{ if } \mathcal{H} = \mathrm{Hilb}_d \text{ for some } d, \\ s \leq n, \text{ if } \mathcal{H} = \mathrm{Hilb}_{(d_1,d_2)} \text{ for some } (d_1, d_2), \\ m \leq d, \text{ if } \mathcal{R} = \mathcal{V}_m \text{ for some } m, \end{cases} \tag{6}$$

where m is c_2 and n is $\frac{1}{2}c_3$ of sheaves from $\mathcal{R}$. Suppose $F \in \mathcal{R}$ and consider a line bundle L over C of degree $g - 1 + 2d + n - s$ satisfying the property

$$\mathrm{Hom}_e(F, L \oplus \mathcal{O}_W) \neq 0, \quad h^1(\mathcal{H}om(F, L)) = 0, \quad h^0(\omega_C(4) \otimes L^{-2}) = 0. \tag{7}$$

Next, similarly to the exact triple (5) we can construct a sheaf E as follows

$$0 \longrightarrow E \longrightarrow F \longrightarrow L \oplus \mathcal{O}_W \longrightarrow 0. \tag{8}$$

One can show that the sheaf E is stable and belongs to $\mathcal{M}(m+d)$. Moreover, we have the following theorem.

Theorem 12 (see [17, Th.]) *The closure of the family $\mathcal{C}(\mathcal{R}, \mathcal{H}, s)$ of sheaves E obtained by the exact triple (8) is the irreducible component of $\mathcal{M}(m + d)$.*

Therefore, varying the moduli space $\mathcal{R} \in \{\mathcal{S}(a, b, c)\} \sqcup \{\mathcal{V}_m\}$, the Hilbert scheme $\mathcal{H} \in \{\mathrm{Hilb}_d \mid d \geq 1\} \sqcup \{\mathrm{Hilb}_{(d_1,d_2)} \mid 1 \leq d_1 \leq d_2, (d_1, d_2) \neq (1, 1), (1, 2)\}$

and the number s satisfying (6) we obtain a new infinite series of irreducible components $\{\overline{\mathcal{C}(\mathcal{R}, \mathcal{H}, s)}\}$ of $\mathcal{M}(k)$, $k \geq 3$. General sheaves of these components have singularities of mixed dimension. In particular, it means that they do not coincide with known components.

The smallest k for which $\mathcal{M}(k)$ contains a new component from this series is equal to 3. More precisely, we have the corollary

Corollary 1 *The closure of the family* $\mathcal{C}(\mathcal{V}_1, \mathrm{Hilb}_2, 0)$ *is the irreducible component of the moduli scheme* $\mathcal{M}(3)$ *of dimension 21. Therefore, the moduli scheme* $\mathcal{M}(3)$ *has at least 11 irreducible components.*

A general sheaf E of this component satisfies the following triple

$$0 \longrightarrow E \longrightarrow F \longrightarrow \mathcal{O}_C(2) \longrightarrow 0,$$

where $[F] \in \mathcal{V}_1$ and C is a smooth conic. The properly μ-semistable reflexive sheaf F fits into the exact triple

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow F \longrightarrow I_l \longrightarrow 0, \quad l \in \mathrm{Gr}(2,4).$$

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1. A.N. Ivanov, A.S. Tikhomirov, The moduli component of the space of semistable rank-2 sheaves on $\mathbb{P}^3$ with singularities of mixed dimension, Doklady Mathematics, 2017, Vol. 96, No. 2, pp. 506–509.

2. A. N. Ivanov, A. S. Tikhomirov, Semistable rank 2 sheaves with singularities of mixed dimension on $\mathbb{P}^3$, Journal of Geometry and Physics, Vol. 129, 2018, pp. 90-98.

3. A. N. Ivanov, A new series of moduli components of rank-2 semistable sheaves on $\mathbb{P}^3$ with singularities of mixed dimension, Sbornik: Mathematics, 211:7 (2020), pp. 967-986.

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