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Milnor K-groups and Differential Forms

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Let R be a commutative associative ring with a unit. Denote by R^* its multiplicative group. Then its Milnor K-group $K_n^M(R)$ of degree n can be defined as the n-th graded component of the quotient of the tensor ring $(R^*)^{\otimes\bullet}$ by a two-sided ideal, generated by elements of the type $r\otimes(1-r)$, where both r and $1-r$ are invertible. Such elements are called Steinberg relations.

Milnor K-groups are important algebraic invariants that play a fundamental role in various domains of algebra and arithmetics, such as class field theory. Unfortunately, they are usually quite hard to commute, since their definition involves a delicate interplay between the additive and multiplicative structures in the ring R.

At the same time, the R-module of (absolute) differential forms Ω_R^n of degree n is relatively easy to calculate explicitly. There is a functorial group homomorphism $d\log\colon K_n^M(R)\to\Omega_R^n$, however in the general case it is far from being an isomorphism.

Let $I\subset R$ be a nilpotent ideal of degree N such that there exists a section of the quotient map $R\to R/I$ that is also a ring homomorphism (in this case we call the pair (R,I) a split nilpotent extension of the ring R/I). By definition, the corresponding Milnor K-group $K_n^M(R,I)$ is the kernel of the natural homomorphism $K_n^M(R)\to K_n^M(R/I)$. In 1975 Bloch [4, § 1] constructed canonical integral of the relative map $d\log$, that is, the functorial group homomorphism

$$\mathrm{B}\;:\;K_n^M(R,I)\longrightarrow\Omega_{R,I}^{n-1}/d\,\Omega_{R,I}^{n-2}\,,\qquad n\geqslant 1\,,$$

such that there is an equality

$$d\circ\mathrm{B}=d\log\;:\;K_n^M(R,I)\longrightarrow\Omega_{R,I}^{n}\,.$$

It was done under the assumption that all the natural numbers from 1 to N are invertible in R.

Later the combination of results, obtained by Bloch [4, theorem 0.1], Maazen and Stienstra [22, § 3.12], van der Kallen [18, corollary 8.5] and Dribus [14] showed that under the additional assumption of R being 5-fold stable the map B is an isomorphism. This result might be interpreted as a variant of famous Goodwillie Theorem [8] with Milnor K-groups replacing algebraic K-groups.

Some time later Gorchinskiy and Osipov [9, Theor. 2.9] proved that the map B is an isomorphism in the case $R=S[\varepsilon]$, $I=(\varepsilon)$, where is a formal variable such that $\varepsilon^2=0$ and S is a weakly 5-fold stable ring such that

2 is invertible in it. They applied this result to the study of the higher-dimensional Contou-Carrère symbol. The approach in [9] was based on the explicit analysis of elements in Milnor K-groups.

First major result of this paper (which we also call the isomorphism theorem for Bloch map) states that in order for the map B to be an isomorphism it is enough for R to be weakly 5-fold stable. This result was published in paper [10], written together with S.O. Gorchinsky (see [10, Theorem 2.12]). Note that the condition of R being weakly 5-fold stable is substantially more general that the condition of R being just 5-fold stable (a good example is a ring of Laurent series with a suitable ring of coefficients). In addition, the proof of this theorem was carried in a much more simpler way, than the proof described in the articles mentioned above. In particular, the proof is reduced to the case of [9, Theor. 2.9] by using the fact that relative Milnor K-groups and modules of differential forms commute with a certain class of non-filtered colimits and also applying several new tricks to deal with elements in Milnor K-groups.

Now let us fix some prime p bigger than two. Note that in case of p-adically complete ring R with all natural numbers exept the ones divisible by p being invertible in it, the integration of $d\log$ is not possible in general. However, it turns out that one can define a p-adic equivalent of the Bloch map B. Moreover, one might actually not regress to the relative case for some nilpotent ideal. However, in order to do that one must consider the (derived) p-adic completions of the corresponding modules of differential forms.

Originally, Katou [19, § I.3] defined such a p-adic equivalent of the Bloch map for the case of smooth schemes over the ring of Witt vectors of some perfect field with characteristic $p > 2$, equipped with a lifting of the Frobenius homomorphism. (In fact, Katou defined this map for a more general case of syntomic schemes over the ring of Witt vectors, without any chosen lifting of the Frobenius homomorphism; in this case the image of this map lies in syntomic cohomologies). The main non-trivial fact here is that the constructed map satisfies the Steinberg property, that is, it sends all the Steinberg relations to zero (see [19, proposition I.3.2]). The proof of the Steinberg property provided by Katou is based on two statements. Firstly, one shows that p-adic Bloch map in a right way does not depend on the choice of a lifting of the Frobenius homomorphism (see [19, p. 212]). For this purpose one has to reduce the syntomic cohomologies to the crystallic ones. Secondly, one considers the separate case of the ring $\mathbb{Z}_p[x, x^{-1}, (1-x)^{-1}]$, equipped with a lifting of the Frobenius homomorphism that maps x to x^p [19, p. 217]. For this purpose the proof is reduced to the case of the ring $\mathbb{Z}_p((x))$ of Laurent series. However, we think that the last reduction in [19] is not entirely clear.

Note that (affine) smooth schemes over the ring of Witt vectors, considered by Katou, can be viewed as a special case of a δ-ring. The notion of a δ-ring was firstly introduced by Joyal [17] and was later studied by Buium [7], who called them rings, equipped with p-derivations. The article of Bhatt–Scholze [2, § 2] can also serve as a great source. Briefly, by a δ-structure on the ring R one means a map $\delta\colon R \to R$ that satisfies the set of certain properties, from which, in particular, it follows that the map $\varphi\colon r \mapsto r^p + p\delta(r)$ is a correctly defined endomorphism of the ring R and thus is also a lifting to the Frobenius homomorphism. Notably, if the ring R has trivial p-torsion, then the notions of a δ-structure and a lifting of the Frobenius homomorphism are equivalent.

It is easy to show that a δ-structure on a R allows to define a group homomorphism $\frac{\varphi}{p^n}$ on the module Ω^n_R, that coincides with the natural action of φ on Ω^n_R after being multiplied by p^n and commutes with the differential map.

Then there exists a canonical integral of the map $\left(1 - \frac{\varphi}{p}\right) d\log$. In other words for any p-adically complete δ-ring (R, δ) there exists a functorial group homomorphism

$$\mathrm{B}_\delta \;:\; (R^*)^{\otimes n} \longrightarrow {}^D\widehat{\Omega}^{n-1}_R / d\, {}^D\widehat{\Omega}^{n-2}_R\,, \qquad n \geqslant 1$$

that satisfies the equality

$$d \circ \mathrm{B}_\delta = \left(1 - \frac{\varphi}{p^n}\right) d\log \;:\; (R^*)^{\otimes n} \longrightarrow {}^D\widehat{\Omega}^{n}_R\,.$$

Here, by ${}^D\widehat{\Omega}^n_R$ we denote the derived p-adic completion of the group ${}^D\widehat{\Omega}^n_R$.

Second major result of this paper (which we call the exictence theorem for Bloch–Artin–Hasse map) states that the map B_δ quotients through Steinberg relations. Thus, there is a group homomorphism

$$\mathrm{B}_\delta \;:\; K^M_n(R) \longrightarrow {}^D\widehat{\Omega}^{n-1}_R / d\, {}^D\widehat{\Omega}^{n-2}_R\,.$$

The proof of this theorem is explicit and does not use divided powers theory or chrystallic cohomologies. We call the homomorphism B_δ the Bloch–Artin–Hasse map, because in case $n = 1$ the corresponding group homomorphism from R^* to R might be considered as a generalization of the classic Artin–Hasse logarithm, which is an isomorphism the groups $1+t\mathbb{Z}_p[[t]] \xrightarrow{\sim} t\mathbb{Z}_p[[t]]$, mapping element $1+t$ to $\sum_{p\nmid i}(-1)^{i-1}\frac{t^i}{i}$ (see [34, § 1]).

Now let $R = S \oplus I$ be a split nilpotent extension of S such that both rings R and S are p-adically complete and have trivial p-torsion and $I^N = 0$ for some $N \in \mathbb{N}$. Suppose that there is a δ-structure on R such that $\delta(S) \subset S$ and $\delta(I) \subset I$. It is easy to see that the restriction Bloch–Artin–Hasse map B_δ defines the homomorphism

$$\mathrm{B}_\delta : {}^d\widehat{K}^M_{n+1}(R, I) \to {}^D\widehat{\Omega}^n_{R,I}/d\,{}^D\widehat{\Omega}^{n-1}_{R,I}\,.$$

Analogously to isomorphism theorem for the Bloch map, there is a reason to believe that under some additional assumptions this map is an isomorphism. For instance, it is easy to show that if a δ-ring R has trivial p-torsion and there is also an inclusion $\delta(I) \subset I^2$ then the corresponding Bloch–Artin–Hasse map $\mathrm{B}_\delta \colon 1+I \xrightarrow{\sim} I$ is an isomorphism (compare this to the results of [13], and also compare the particular case of the ring $\mathbb{Z}_p[[t]]$ with [34, Proposition 1]).

Our third major result (that we call the isomorphism theorem for the Bloch–Artin–Hasse map) states that if S is a p-adically complete weakly 5-fold stable δ-ring with trivial p-torsion, then for any $N \in \mathbb{N}$ and for any extension of the δ-structure, such that $\delta(I_N) \subset I_N^2$ the homomorphism $B_\delta :$ ${}^D\widehat{K}^M_2(R_N, I_N) \to {}^D\widehat{\Omega}^1_{R_N,I_N}/dI_N$ is an isomorphism. Here by R_N we denote the ring $S[t]/(t^N)$ and by I_N — its nilpotent ideal $(\bar{t})$. The proof is carried by induction on N and actively uses the machinery, developed in paper [10]. We would also like to note that, while the existence theorem for Bloch–Artin–Hasse map stays true for the case of classic p-adic completion, in order to achieve this particular result we had to turn to derived p-adic completion, since classic p-adic completion does not satisfy some necessary conditions that are required for the proof (for example, the cokernel of a map of p-adically complete modules can fail to be p-adically complete).

In summary, there is a list of our main results:

(i) Let $I \subset R$ be a nilpotent ideal and $N \geqslant 1$ be a natural number such that $I^N = 0$. Suppose that the quotient map $R \to R/I$ admits a splitting by a ring homomorphism $R/I \to R$, that $N!$ is invertible in R, and that R is weakly 5-fold stable. Then for any natural number $n \geqslant 0$, the Bloch map is an isomorphism

$$\mathrm{B}\ :\ K^M_{n+1}(R, I) \xrightarrow{\sim} \Omega^n_{R,I}/d\,\Omega^{n-1}_{R,I}\,.$$

(ii) There exists a group homomorphism

$$\mathrm{B}_\delta \;:\; K_n^M(R) \longrightarrow {}^D\widehat{\Omega}_R^{n-1}/d\,{}^D\widehat{\Omega}_R^{n-2}$$

that is functorial on the category of p-adically complete δ-rings and satisfies the equality

$$d \circ \mathrm{B}_\delta = \Big(1 - \frac{\varphi}{p^n}\Big) d\log\;.$$

(iii) If S is a p-adically complete weakly 5-fold stable δ-ring with trivial p-torsion, then for any $N \in \mathbb{N}$ and for any extension of the δ-structure, such that $\delta(I_N) \subset I_N^2$ the homomorphism

$$B_\delta : {}^d\widehat{K}_2^M(R_N, I_N) \to {}^D\widehat{\Omega}^1_{R_N,I_N}/dI_N$$

is an isomorphism.

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- D.Tyurin, "Generalization of Artin–Hasse logarithm for milnor K-groups of δ-rings". The paper is accepted for publication by *Sbornik: Mathematics.*

References

[1] D. Arapura, S.-J. Kang, "Kähler–de Rham cohomology and Chern classes", *Comm. Algebra*, **39**:4, 1153–1167, (2011)

[2] B. Bhatt, P. Scholze, "Prisms and prismatic cohomology", preprint (2019) arXiv:1905.08229.

[3] S. Bloch "On the tangent space to Quillen K-theory", *Lecture Notes in Math.*, **341**, 205–210, (1973)

[4] S. Bloch, "K_2 of Artinian $\mathbb{Q}$-algebras, with application to algebraic cycles", *Comm. Algebra*, **3** (1975), 405–428.

[5] S. Bloch, "Higher regulators, algebraic K-theory, and zeta functions of elliptic curves", *CRM Monograph Series*, **11**, American Mathematical Society (2000).

[6] S. Bloch, "Applications of the dilogarithm function in algebraic K-theory and algebraic geometry", *Proceedings of the International Symposium on Algebraic Geometry (Kyoto, 1977)*, (M. Nagata, ed.), Kinokuniya, Tokyo (1978), 103–114.

[7] A. Buium, "Arithmetic analogues of derivations", *Journal of Algebra*, **198** (1997), 290–299.

[8] T. G. Goodwillie, "Relative algebraic K-theory and cyclic homology", *Ann. of Math. (2)*, **124**:2, 347–402, (1986)

[9] S. O. Gorchinskiy, D. V. Osipov, "Tangent space to Milnor K-groups of rings", *Proceedings of the Steklov Institute of Mathematics*, **290**, 26–34 (2015)

[10] S. O. Gorchinskiy, D. N. Tyurin, "Relative Milnor K-groups and differential forms of split nilpotent extensions", *Izv. Math.*, **82**:5 (2018), 880–913.

[11] A. B. Goncharov, "Geometry of configurations, polylogarithms, and motivic cohomology", *Adv. Math.*, **114**:2, 197–318, (1995)

[12] H. Grauert, H. Kerner, "Deformationen von Singularitäten komplexer Räume", *ath. Ann.*, **153**, 236–260, (1964)

[13] T. tom Dieck, "The Artin–Hasse logarithm for λ-rings", *Algebraic topology (Arcata, CA, 1986), Lecture Notes in Math.*, **1370** (1989), 409–415.

[14] B. Dribus, "A Goodwillie-type theorem for Milnor K-Theory", preprint arXiv:1402.2222.

[15] H.-J. Reiffen, "Das Lemma von Poincaré für holomorphe Differentialformen auf komplexen Räumen", *Math. Z.*, **101**, 269–284, (1967)

[16] M. Rossi, L. Terracini, "Maple subroutines for computing Milnor and Tyurina numbers of hypersurface singularities with application to Arnol'd adjacencies", *Rend. Semin. Mat. Univ. Politec. Torino*, **73**:3-4, 269–316, (2015)

[17] A. Joyal, "δ-anneaux et vecteurs de Witt", *C. R. Math. Rep. Acad. Sci. Canada*, **7**:3 (1985), 177–182.

[18] W. van der Kallen, "The K_2 of rings with many units", *Ann. Sci. École Norm. Sup. (4)*, **10**:4 (1977), 473–515.

[19] K. Kato, "On p-adic vanishing cycles", *Adv. Stud. Pure Math.*, **10** (1987), 207–251.

[20] K. Kato, "The explicit reciprocity law and the cohomology of Fontaine–Messing", *Bulletin de la S.M.F.*, **119**:4 (1991), 397–441.

[21] M. Kerz, "The Gersten conjecture for Milnor K-theory", *Invent. Math.*, **175**:1 (2009), 1–33.

[22] H. Maazen, J. Stienstra, "A presentation for K_2 of split radical pairs", *J. Pure Appl. Algebra*, **10**:3 (1977), 271–294.

[23] B. Malgrange, "Intégrales asymptotiques et monodromie",*Ann. Sci. École Norm. Sup. (4)*, **7**, 405–430, (1974),

[24] H. Matsumura,"Commutative ring theory",*Cambridge Studies in Advanced Mathematics***8** (1989)

[25] J. Milnor, "Singular points of complex hypersurfaces", *Annals of Mathematics Studies, No. 61*, Princeton University Press, University of Tokyo Press, (1968)

[26] D. Mond, "From the index of a differential operator to the Milnor number of a singularity", *Differential algebra, complex analysis and orthogonal polynomials, Contemp. Math.*, **509**, 129–141, (2010),

[27] M. Morrow, "K_2 of localisations of local rings", *J. Algebra*, **399** (2014), 190–204.

[28] Yu. P. Nesterenko, A. A. Suslin, "Homology of the general linear group over a local ring, and Milnor's K-theory", *Math. USSR-Izv.*, **34**:1 (1990), 121–146.

[29] V. P. Palamodov, "Multiplicity of holomorphic mappings", *Funct. Anal. Its Appl.*, **1**:3, 218–226, (1967)

[30] G. N. Tjurina, "Locally semi-universal flat deformations of isolated singularities of complex spaces", *Izv. Akad. Nauk SSSR Ser. Mat.*, **33**:5, 967–999 (1969)

[31] Lê Dũng Tráng, "Calculation of Milnor number of isolated singularity of complete intersection", *Funct. Anal. Its Appl.*, **8**:2, 127–131, (1974)

[32] G.-M. Greuel, "Der Gauss–Manin–Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten", *Math. Ann.*, **214**, 235–266, (1975)

[33] K. Saito, "Quasihomogene isolierte Singularitäten von Hyperflächen" *Invent. Math.*, **14**, 123–142, (1971)

[34] S. V. Vostokov, "An explicit form of the reciprocity law", *Math. USSR-Izv.*, **13**:3 (1979), 557–588.