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Applications of automorphic forms in algebraic geometry

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Introduction

The name "*K3 surfaces*" was coined by A. Weil in 1957 when he formulated a research programme for these surfaces and their moduli. For many years the last open question of Weil's programme was that of the geometric type of the moduli spaces of polarised K3 surfaces. In [D4] this problem was solved 50 years after its formulation. This is one of the main results of the dissertation, summarised in §5. (See also C. Voisin's talk at the Bourbaki seminar [Vo2] and our large survey [GHS1] in the first volume of the monograph "Handbook of Moduli".) In a very short form, the result can be formulated as follows:

The moduli space $\mathcal{F}_{2d}$ of polarized K3 *surfaces of degree 2d is of general type for all* $d > 61$.

To prove this statement, a fairly general and efficient method was developed, which is successfully applied to various modular manifolds of orthogonal type. The proof of the general type of modular varieties uses algebraic geometry, modular forms with respect to the indefinite orthogonal group $\mathrm{O}(2, n)$, automorphic Borcherds products, combinatorics of root systems, and arithmetic theory of quadratic forms. This general method is the main result of the dissertation.

The need to study modular forms on orthogonal groups $\mathrm{O}(2, n)$ was first noted by A. Weil in the late 1950s in his program for the study of K3 surfaces: *"One interesting feature here is the occurrence, in a problem of moduli, of the automorphic functions belonging to the group of unites of a quadratic form of signature* $(n, 2)$ *(with* $n = 19$ *in the present case)."* (See "Final report on research contract AF 18(603)-57", [We, p. 390–395].)

Below we present the author's results in the theory of automorphic forms on orthogonal groups, which allowed us to solve some classical problems in the theory of moduli spaces of polarized Abelian surfaces and their corresponding Kummer surfaces, polarized K3 surfaces and polarized hyper-Kähler manifolds. The texts of the papers [D1]–[D10] are collected in Appendices A–J of the dissertation. Their exact bibliographic description is given below on page 3. Let us briefly formulate our main results.

1) We prove irrationality (more exactly, non-negativity of Kodaira dimension) of the moduli spaces of $(1, t)$-polarized abelian surfaces (it was a question of Siegel) for all t except twenty polarizations.

2) The general type of moduli spaces of polarized K3 surfaces of degree $2d$ for $d > 61$ is proved. This was the last open problem of A. Weil's program on K3 surfaces.

3) In the mid-1980s, multidimensional analogs of K3 surfaces were discovered. They are *irreducible holomorphic symplectic varieties* or *hyperkähler manifolds*. We proved the general type of moduli spaces of polarized hyper-Kähler manifolds of type $\mathrm{K3}^{[2]}$ (moduli of dimension 20) and moduli spaces of polarized 10-dimensional O'Grady manifolds (moduli of dimension 21).

4) It is proved the irrationality of the moduli spaces of Kummer surfaces constructed from polarized Abelian surfaces. This question has been opened since 1996.

The author's results in the field of modular forms with respect to orthogonal groups are key for solving these algebraic-geometric problems. We list the main automorphic results.

5) A method for lifting Jacobi forms to modular forms on paramodular groups and on orthogonal groups of signature $(2, n)$. This method allows one to construct canonical differential forms on modular varieties.

6) Two automorphic criteria are proved: *"Low weight cusp form trick"* for general type and *an automorphic criterion for unirouledness* of modular varieties. Both criteria are technically related to reflective modular forms.

7) It is found a new representation of automorphic Borcherds products in one-dimensional cusps of the modular group in terms of Jacobi forms. This approach allows one to construct automorphic forms for the both automorphic criteria from point 6). This is the main transcendental part of the proof of the algebraic-geometric results 1)– 4). In addition, this approach allows one to find the Cartan matrix and the multiplicity of all positive roots of Lorentzian Kac–Moody algebras.

8) Automorphic products in terms of Jacobi forms in one variable are interpreted in physics as the secondary quantised elliptic genus of Calabi–Yau manifolds. We have studied the automorphic properties of the elliptic genus and its secondary quantisation.

List of publications of the dissertation.

[D1] V. Gritsenko, *Irrationality of the moduli spaces of polarized Abelian surfaces.* International Mathematics Research Notices **6** (1994), 235—243.

[D2] V. Gritsenko *Reflective modular forms and applications.* Russian Math. Surveys **73**:5 (2018), 797–864.

[D3] V. Gritsenko, V.V. Nikulin, *Lorentzian Kac-Moody algebras with Weyl groups of 2-reflections.* Proceedings London Math. Soc. **116**:3 (2018), 485–533.

[D4] V. Gritsenko, K. Hulek, G. Sankaran, *The Kodaira dimension of the moduli of K3 surfaces.* Inventiones Mathematicae **169** (2007), 519—567.

[D5] V. Gritsenko, K. Hulek, G. Sankaran, *Moduli spaces of irreducible symplectic manifolds.* Compositio Mathematica **146** (2010), 404—434.

[D6] V. Gritsenko, K. Hulek, G. Sankaran, *Moduli spaces of polarised symplectic O'Grady varieties and Borcherds products.* J. of Differential Geometry **88** (2011), 61—85.

[D7] V. Gritsenko, K. Hulek, *Uniruledness of orthogonal modular varieties.* J. Algebraic Geometry **23** (2014), 711–725.

[D8] V. Gritsenko, *Elliptic genus of Calabi–Yau manifolds and Jacobi and Siegel modular forms.* St. Petersburg Math. Journal **11**:5 (2000), 781—804 (with Appendix of F. Hirzebruch, *On the Euler characteristic of manifolds with $c_1 = 0$. A letter to V. Gritsenko.* 805—807).

[D9] V. Gritsenko, C. Poor, D. S. Yuen, *Antisymmetric Paramodular Forms of Weights 2 and 3.* International Mathmatical Research Notices, Issue **20** (2020), 6926—6946.

[D10] V. Gritsenko, H. Wang *Antisymmetric paramodular forms of weight 3.*
Sbornik: Mathematics, **210**:12 (2019), 1702–1723.

1 Modular varieties and modular forms

First, we define a class of varieties, the modular varieties of orthogonal type, which is important in algebraic geometry.

Let L be a integral quadratic lattice, more exactly, a free $\mathbb{Z}$-module of finite rank with a non-degenerate symmetric bilinear intergal form $(\cdot\,,\cdot)\colon L \times L \to \mathbb{Z}$ of signature $(2, n)$. It means that $(l,l) \in 2\mathbb{Z}$ for any $l \in L$ and $\mathrm{sign}(L \otimes \mathbb{R}) = (2,n)$. We define the associated with L n-dimensional classical Hermitian domain of type IV

$$\mathcal{D}(L) = \{[Z] \in \mathbb{P}(L \otimes \mathbb{C}) \mid \ (Z,Z) = 0,\ (Z,\overline{Z}) > 0\}^+,$$

where the superscript $^+$ denotes a choice of one of the two connected components. We denote by $\mathrm{O}^+(L)$ the index 2 subgroup of the integral orthogonal group $\mathrm{O}(L)$ preserving $\mathcal{D}(L)$.

We let Γ be a subgroup of finite index in $\mathrm{O}^+(L)$. Any such Γ acts properly discontinuously on $\mathcal{D}(L)$ as a discrete group of automorphisms. We define the factor space

$$\mathcal{M}_L(\Gamma) = \Gamma \backslash \mathcal{D}(L),$$

which is called *modular variety of orthogonal type.*

For geometric applications, the most important arithmetic groups are *stable orthogonal group* and its special subgroup

$$\widetilde{\mathrm{O}}^+(L) = \{g \in \mathrm{O}^+(L) \mid g|_{L^\vee/L} = \mathrm{id}\}, \quad \widetilde{\mathrm{SO}}^+(L) = \mathrm{SO}(L) \cap \widetilde{\mathrm{O}}^+(L), \tag{1}$$

where $L^\vee$ is the dual lattice and $L^\vee/L$ is finite discriminant group of order $|\det(L)|$.

Many classical modular spaces are orthogonal modular varieties. Below are some important examples of such varieties that we study in the dissertation.

Moduli spaces.

(1) The moduli space $\mathcal{A}_t$ of $(1,t)$-polarized abelian or Kummer surfaces. They are the following Siegel modular varieties of dimension 3

$$L_t = U \oplus U \oplus \langle -2t \rangle, \qquad \mathrm{sign}(L_t) = (2,3), \tag{2}$$

$$\mathcal{A}_t = \widetilde{\mathrm{SO}}^+(L_t) \setminus \mathcal{D}(L_t), \quad \mathcal{K}_t = \mathrm{O}^+(L_t) \setminus \mathcal{D}(L_t), \quad \dim \mathcal{A}_t = \dim \mathcal{K}_t = 3, \tag{3}$$

where $U \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the hyperbolic plane, $\langle -2t \rangle$ is the lattice of rang 1 with Gramm matrix $(-2t)$.

(2) The moduli space of polarized K3-surfaces of degree $2d$

$$L_{2d} = 2U \oplus 2E_8(-1) \oplus \langle -2d \rangle, \qquad \mathrm{sign}(L_t) = (2,19), \tag{4}$$

$$\mathcal{M}_{2d} = \widetilde{\mathrm{O}}^+(L_{2d}) \setminus \mathcal{D}(L_{2d}), \tag{5}$$

where $2E_8(-1)$ denotes two orthogonal copies of $E_8(-1)$.

(3) The moduli space of *polarized irreducible holomorphic symplectic varieties of type* $\mathrm{K3}^{[2]}$ with split polarization of Beauville–Bogomolov degrees $2d$. See [Be], [Bo], [D5], [GHS1].

$$L_{2,2d} = 2U \oplus 2E_8(-1) \oplus \langle -2 \rangle \oplus \langle -2d \rangle, \qquad \mathrm{sign}(L_{2,2d}) = (2,20), \tag{6}$$

$$\mathcal{M}^{split}_{\mathrm{K3}^{[2]},2d} = \widetilde{\mathrm{O}}^+(L_{2,2d}) \setminus \mathcal{D}(L_{2,2d}). \tag{7}$$

(4) The moduli space of *polarized* 10-*dimensional O'Grady varieties* with a split polarization of the Bogomolov–Beauville degree $2d$

$$L_{A_2,2d} = 2U \oplus 2E_8(-1) \oplus \langle A_2(-1) \rangle \oplus \langle -2d \rangle, \qquad \mathrm{sign}(L_{A_2,2d}) = (2,21), \tag{8}$$

$$\mathcal{M}^{split}_{O'G_{10},2d} = \widetilde{\mathrm{O}}^+(L_{A_2,2d}) \setminus \mathcal{D}(L_{2,2d}). \tag{9}$$

By its construction modular variety of orthogonal type is a complex analytic space. Its compactification of Satake's type, more exactly the Baily–Borel compactification, was constructed in [BB]. The boundary of the compactification $\mathcal{D}^*$ decomposes as a disjoint union of components F_P, which are themselves symmetric spaces associated with certain rational parabolic subgroups of the orthogonal group of signature $(2,n)$, which are the the stabiliser of totally isotropic subspaces in $L\otimes\mathbb{Q}$. Since $\operatorname{sign}(L)=(2,n)$, the isotropic subspaces may have dimension 1 or 2. The following result is true (see [BB]): *the Baily–Borel compactification $\mathcal{M}_L(\Gamma)^*$ is an irreducible normal projective variety over $\mathbb{C}$ and it is decomposed as a disjoint union of the components*

$$\mathcal{M}_L(\Gamma)^* = \mathcal{M}_L(\Gamma) \amalg \coprod_{\mathcal{P}} X_{\mathcal{P}} \amalg \coprod_{\ell} Q_\ell, \tag{10}$$

where ℓ and $\mathcal{P}$ runs through representatives of Γ-orbits of isotropic lines and planes in $L\otimes\mathbb{Q}$. The components $X_{\mathcal{P}}$ and Q_ℓ are usually called 1- and 0-dimensional boundary components of the modular variety or its one-dimensional and zero-dimensional cusps.

The theory of modular forms is one of the main tools to study the geometry of modular manifolds of orthogonal type. For example, the Baily—Borel compactification $\mathcal{M}_L(\Gamma)^*$ can be defined as $\operatorname{Proj}\Big(\bigoplus_k M_k(\Gamma)\Big)$, where $M_k(\Gamma)$ denotes the finite dimensional spaces of modular forms of weight k with trivial character.

Definition 1.1. *Consider a lattice L has signature $(2,n)$ with $n\geqslant 3$ and the affine cone*

$$\mathcal{D}(L)^\bullet = \{y\in L\otimes\mathbb{C} \mid x=\mathbb{C}^*y\in\mathcal{D}(L)\}$$

over $\mathcal{D}(L)$. Let $k\in\mathbb{Z}$ and let $\chi\colon\Gamma\to\mathbb{C}^*$ be a character of a subgroup $\Gamma<\mathrm{O}^+(L)$ of finite index. A holomorphic function $F\colon\mathcal{D}(L)^\bullet\to\mathbb{C}$ is called modular form of weight k and character χ for the group Γ, if

$$F(tZ)=t^{-k}F(Z) \qquad \forall\, t\in\mathbb{C}^*, \qquad F(gZ)=\chi(g)F(Z) \qquad \forall\, g\in\Gamma.$$

A modular form is called parabolic or cusp form if if it vanishes at every cusp, i. e. at every boundary components of the Baily–Borel compactification of the modular variety $\Gamma\backslash\mathcal{D}(L)$.

By $M_k(\Gamma,\chi)$ (respectively, by $S_k(\Gamma,\chi)$) we denote the linear spaces of modular (respectively, cusp) forms of weight k and character χ. These spaces are finite dimensional.

Differential forms on $\mathcal{M}_L(\Gamma)$ can be interpreted as modular forms with respect to the group Γ. We select a holomorphic volume element dZ on $\mathcal{D}(L)$. Then, if F is a modular form of weight kn and character $\det^k$ for group Γ, then $F\,(dZ)^k$ is a Γ-invariant section of the pluricanonical bundle $\Omega(\mathcal{D}(L))^{\otimes k}$. Therefore the arithmetic information on modular forms can be used in order to obtain a geometric information of the modular variety of orthogonal type $\mathcal{M}_L(\Gamma)$.

The weight $k=n$ is called *canonical* because by a lemma of Freitag (see [Fr, Proposition 2.1 in Ch. 3]):

$$S_n(\Gamma,\det)\cong H^0\big(\widetilde{\mathcal{M}}_L(\Gamma),\Omega(\widetilde{\mathcal{M}}_L(\Gamma))\big),$$

where $\widetilde{\mathcal{M}}_L(\Gamma)$ is a smooth compact model of the modular variety $\mathcal{M}_L(\Gamma)$ and $\Omega(\widetilde{\mathcal{M}}_L(\Gamma))$ is the sheaf of canonical differential forms. Therefore we have the following important formula for the geometric genus of the modular variety of orthogonal type:

$$p_g(\widetilde{\mathcal{M}}_L(\Gamma)) = h^{n,0}(\widetilde{\mathcal{M}}_L(\Gamma)) = \dim S_n(\Gamma,\det). \tag{11}$$

Main problem: *How to construct at least one parabolic form of canonical weight for modular groups from the moduli theory algebraic varieties?*

For the manifold $\mathcal{A}_t$ this problem was solved by the author in [D1], and for the variety $\mathcal{K}_p$ with a prime p the first non- trivial results obtained in [D9]–[D10].

2 Arithmetic lifting of Jacobi forms and module spaces Abelian surfaces (Siegel problem)

In this dissertation, we consider lattices containing two hyperbolic planes U,

$$L = U \oplus L_1 = U \oplus (U_1 \oplus L_0(-1)), \quad U \cong U_1 \cong \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right), \tag{12}$$

where L_0 is an even integer positive definite lattice of rank $n_0 > 0$, L_1 is of signature $(1, n_0+1)$, and L of signature $(2, n_0+2)$. The first decomposition of the lattice L in (12) gives a cylindrical model (the so-called *tube of future*) of the homogeneous domain:

$$\mathcal{H}(L_1) \cong \mathcal{H}(L_0) = \{Z = \omega e_1 + \mathfrak{z} + \tau f_1 \in L_1 \otimes \mathbb{C} \mid \\ \tau,\, \omega \in \mathbb{H}_1,\ \mathfrak{z} \in L_0 \otimes \mathbb{C},\ 2\operatorname{Im}\tau \cdot \operatorname{Im}\omega - (\operatorname{Im}\mathfrak{z}, \operatorname{Im}\mathfrak{z})_{L_0} > 0\}. \tag{13}$$

Any modular form $F \in M_k(\widetilde{\mathrm{SO}}^+(L))$ is periodic, i.e. $F(Z + l) = F(Z)$ for any $l \in L_1$. This defines the Fourier expansion in the variable $Z \in \mathcal{H}(L_1)$ at the zero-dimensional cusp

$$F(Z) = \sum_{l \in L_1^\vee,\, (l,l) \geq 0} f(l) \exp{(2\pi i\, (l, Z))}. \tag{14}$$

Condition on the hyperbolic norm of indices of nonzero Fourier coefficients $(l, l)_{L_1} \geq 0$ follows from the holomorphism of the modular form. (See the description of the Fourier expansion in an arbitrary cusp in [GN2, §2.3] and [GHS1, §8.2–8.3].) The Fourier–Jacobi decomposition is a one-dimensional cusp decomposition. More precisely, this is the Fourier expansion in the variable ω from (13)

$$F(\tau, \mathfrak{z}, \omega) = \sum_{m \geq 0} \varphi_m(\tau, \mathfrak{z}) \exp{(2\pi i\, m\omega)}. \tag{15}$$

The coefficients $\varphi_m(\tau, \mathfrak{z})$ are called *Fourier–Jacobi coefficients* at 1-dimensional cusp.

We define Jacobi forms of weight k and index m with respect to the lattice L_0 as automorphic forms of the type $\varphi(\tau, \mathfrak{z}) \exp{(2\pi i\, m\omega)}$ relative a parabolic subgroup preserving a given one-dimensional cusp. This can be expressed with functional equations of two types given below.

Definition 2.1. (See [D2, Definition 2.2], [G2] and [CG2].) *Holomorphic function* $\varphi : \mathbb{H} \times (L_0 \otimes \mathbb{C}) \to \mathbb{C}$ *is called* **nearly holomorphic Jacobi form of weight $k \in \mathbb{Z}$, index $t \in \mathbb{N}$ for lattice L_0**, *if it satisfies the functional equations*

$$\varphi\left(\frac{a\tau + b}{c\tau + d}, \frac{\mathfrak{z}}{c\tau + d}\right) = (c\tau + d)^k\, e^{i\pi t \frac{c(\mathfrak{z},\mathfrak{z})}{c\tau + d}}\, \varphi(\tau, \mathfrak{z}), \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}),$$
$$\varphi(\tau, \mathfrak{z} + x\tau + y) = e^{-i\pi t((x,x)\tau + 2(x,\mathfrak{z}))}\, \varphi(\tau, \mathfrak{z}), \quad \forall\, x, y \in L_0,$$

ant it has the following Fourier expansion

$$\varphi(\tau, \mathfrak{z}) = \sum_{n \geq c_0} \sum_{\ell \in L_0^\vee} f(n, \ell) q^n \zeta^\ell, \tag{16}$$

where $c_0 \in \mathbb{Z}$, $q = e^{2\pi i\tau}$ *and* $\zeta^\ell = e^{2\pi i(\ell,\mathfrak{z})}$. *If* φ *satisfies*

$$f(n,\ell) \neq 0 \Longrightarrow 2n - (\ell,\ell) \geq 0,$$

then φ *is called* **holomorphic Jacobi form**. *If* φ *satisfies a stronger condition* $(f(n,\ell) \neq 0 \Longrightarrow 2n - (\ell,\ell) > 0)$ *then* φ *is called* **parabolic**. *We denote by* $J^!_{K,L_0,t}$ *(respectively,* $J_{k,L_0,t}$, *and* $J^{cusp}_{k,L_0,t}$*) the vector space of weakly holomorphic (respectively, holomorphic or parabolic) Jacobi forms of weight* k *and index* t *with respect to the lattice* L.

Theorem 2.2. (See [D2, Theorem 3.1] and [G2].) *The lifting of a Jacobi form* $\varphi_k(\tau,\mathfrak{z}) \in J_{k,1}(L_0)$ *(with* $f(0,0) = 0$*) of weight* k *was defined as an action of a formal Hecke L-function of* $SL_2(\mathbb{Z})$ *by the formula*

$$\begin{aligned}\mathrm{Lift}(\varphi_k)(\tau,\mathfrak{z},\omega) &= \sum_{m\geq 1} m^{k-1}\big(\varphi_k(\tau,\mathfrak{z})e^{2\pi i m\omega}\big) \mid_k T_-(m)\\ &= \sum_{m\geq 1} m^{-1} \sum_{\substack{ad=m\\ b \mod d}} a^k \varphi_k\big(\frac{a\tau+b}{d},\ a\mathfrak{z}\big)\, e^{2\pi i m\omega}.\end{aligned} \tag{17}$$

On can write its Fourier expansion

$$\mathrm{Lift}(\varphi_k)(Z) = \sum_{\substack{n,m>0,\,\ell\in L_0^\vee\\ 2nm-(\ell,\ell)\geq 0}} \sum_{d|(n,\ell,m)} d^{k-1} f\big(\frac{nm}{d^2},\frac{\ell}{d}\big)\, e(n\tau + (\ell,\mathfrak{z}) + m\omega)$$

where $d|(n,\ell,m)$ *denotes a positive integral divisor of the vector in* $U \oplus L_0^\vee(-1)$. *The lifting is a modular form with respect to stable special orthogonal group* $\widetilde{\mathrm{SO}}^+(L)$ *(see* (1)*?)*

$$\mathrm{Lift}(\varphi_k) \in M_k(\widetilde{\mathrm{SO}}^+(L)).$$

$\mathrm{Lift}(\varphi_k)$ *is a cusp form if* φ_k *is a Jacobi cusp form.*

Jacobi modular forms of weight k and index 1 corresponding to a lattice in $\langle 2t\rangle$ are the usual modular Jacobi forms of weight k and index t in the sense of Eichler–Zagier. For any parabolic Jacobi form of weight 3 and index t, the lifting construction gives a nonzero parabolic form of weight 3 with respect to the paramodular group Γ_t. According Freitag's criterion, we obtain a nonzero canonical differential form on any compactification of the modular manifold $\mathcal{A}_t$. This gives the following theorem.

Theorem 2.3. (See [D1, Theorem 1.1].) *The moduli space of* $(1,t)$*-polarized abelian surfaces* $\mathcal{A}_t$ *(see (2)–(3)) has positive geometric genus for all* t *except twenty exceptional polarizations* $t = 1, 2, \ldots, 12, 14, 15, 16, 18, 20, 24, 30, 36$. *In particular,* $H^3(\Gamma_t, \mathbb{C})$ *is not trivial for all non-exclusive polarizations.*

The rationality or unirationality of the corresponding moduli space is known only for exceptional $t \leq 20$ (see [GP]).

3 Reflective automorphic forms: two automorphic criteria in geometry of modular varieties

Differential forms on $\mathcal{F}_L(\Gamma)$ may be interpreted as modular forms for Γ: see Section **??** for more details. Therefore arithmetic information (modular forms) may be used to obtain geometric information about $\mathcal{F}_L(\Gamma)$. In particular we can use modular forms to decide whether $\mathcal{F}_L(\Gamma)$ is of general type, or more generally to try to determine its Kodaira dimension.

If Y is a connected smooth projective variety of dimension n, the *Kodaira dimension* $\kappa(Y)$ of Y is defined by

$$\kappa(Y) = \mathrm{Tr.deg}\Big(\bigoplus_{k\geq 0} H^0(Y, kK_Y)\Big) - 1,$$

or $-\infty$ if $H^0(Y, kK_Y) = 0$ for all $k > 0$. Thus $h^0(Y, kK_Y) \sim k^{\kappa(Y)}$ for k sufficiently divisible. The possible values of $\kappa(Y)$ are $-\infty, 0, 1, \ldots, n = \dim Y$, and Y is said to be of *general type* if $\kappa(Y)$. The Kodaira dimension is a bimeromorphic invariant so it makes sense to extend the definition to arbitrary irreducible quasi-projective varieties X by putting $\kappa(X) = \kappa(\widetilde{X})$ for $\widetilde{X}$ a desingularisation of a compactification of X.

The branch divisor of the modular projection $\pi_\Gamma : \mathcal{D}(L) \to \Gamma\backslash\mathcal{D}L)$ which is one major obstacle to continue pluricanonic differential forms with an open subdomain $\mathcal{F}_L(\Gamma)^o$ on a smooth compactification of this quasi-projective variety.

Theorem 3.1. (See [D4, Corollary 2.13].) *The branch divisor* $\mathrm{R.div}(\pi_\Gamma)$ *of the modular projection* $\pi_\Gamma : \mathcal{D}(L) \to \Gamma\backslash\mathcal{D}(L)$ *is induced by all* $g \in \Gamma$ *such that* g *or* $-g$ *is a reflection with respect to a vector in* L

$$\mathrm{R.div}(\pi_\Gamma) = \bigcup_{\substack{r\in L/\pm 1\\ r\ \text{primitive}\\ \sigma_r\in\Gamma\ \text{or}\ -\sigma_r\in\Gamma}} \mathcal{D}_r(L). \tag{18}$$

Definition 3.2. *A modular form* $F \in M_k(\Gamma, \chi)$ *is called* **reflective** *if*

$$\mathrm{supp}(\mathrm{div}\, F) \subset \mathrm{R.div}(\pi_\Gamma),$$

where $\mathrm{R.div}(\pi_\Gamma)$ *is the divisor of modular projection from Theorem 3.1.* F *is called* **strongly reflective if** *, if the multiplicity of all of irreducible components of* $\mathrm{div}\, F$ *is equal to* 1.

Modular forms with a small or large divisor. According to the definition given above, the modular form $F \in M_k(\Gamma, \chi)$ is strictly reflective if and only if $\mathrm{div}\, F \leq \mathrm{R.div}(\pi_\Gamma)$, i.e. the divisor of a strictly reflective form if *small.* We say that the divisor of the modular form $F \in M_k(\Gamma, \chi)$ is *large* if $\mathrm{div}\, F \geq \mathrm{R.div}(\pi_\Gamma)$.

Modular forms of canonical weight. Small and big weights. Let $\mathrm{sign}(L) = (2, n)$. Arbitrary modular divisor a modular form of canonical weight $F \in M_n(\Gamma, \det)$ always contains ramification divisor $\mathrm{R.div}(\pi_\Gamma)$.

Canonical weight is borderline in geometric applications. We say that the weight k of the modular form $F \in M_k(\Gamma, \chi)$ is *small* if $k < n$, and *large* if $k > n$. Below we give **the first automorphic criterion**.

Theorem 3.3. (**Low weight cusp form trick**, see [D4,Theorem 1.1].) *Let be* $\mathrm{sign}(L) = (2, n)$ *and* $n \geqslant 9$. *The modular varity* $\mathcal{M}_L(\Gamma)$ *is of general type if there exists a cusp form* $F \in S_k(\Gamma, \det^\varepsilon)$ $(\varepsilon = 0, 1)$ *of low weight* $k < n$ *such that* $\mathrm{div}(F) \geqslant \mathrm{R.div}(\pi_\Gamma)$.

The second criterion is in some sense the opposite of the first. We will prove that the Kodaira dimension of the modular variety $\mathcal{M}_L(\Gamma)$ is equal to $-\infty$ if there exists a modular form of large weight with small divisor.

Theorem 3.4. (See [D2, Theorem 5.8].) *Let* $\operatorname{sign}(L) = (2, n)$*, and let* $n \geqslant 3$*. Let* $F_k \in M_k(\Gamma, \chi)$ *be a strongly reflective modular form of weight* k *and character* χ*, where* $\Gamma < \mathrm{O}^+(L)$ *is of finite index. Denote by* $\kappa(X)$ *the Kodaira dimension of* X*.*

1) If $k > n$*, then* $\kappa(\Gamma\backslash\mathcal{D}(L)) = -\infty$*.*

2) Let $k = n$*. Assume that* Γ *has at least one cusp, i. e.* $\Gamma\backslash\mathcal{D}(L)$ *is not compact. If the form* F *is not cusp, then* $\kappa(\Gamma\backslash\mathcal{D}(L)) = -\infty$*. If* F *is a cusp form (with multiplicity of zeroes at least* 1 *along the boundary), then for the subgroup* $\Gamma_\chi = \ker(\chi \cdot \det) < \Gamma$ *we obtain* $\kappa(\Gamma_\chi\backslash\mathcal{D}(L)) = 0$*.*

Note that there is the following its algebraic-geometric refinement. Remind that the manifold X is called uniruled if there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$, where Y is a manifold with $\dim Y = \dim X - 1$. If X is uniruled then $\kappa(X) = -\infty$. It is hypothetically assumed that the reciprocal is also true, however, this is proved only for $\dim X = 3$.

Theorem 3.5. (See [D7, Theorem 2.1].) *Let* $k > n$ *like in the conditions of Theorem 3.4. Then the modular variety* $\Gamma\backslash\mathcal{D}(L)$ *is at least uniruled.*

4 Automorphic Borcherds products in terms of Jacobi modular forms of weight 0, Lorentzian Kac–Moody algebras, elliptic genus of Calabi–Yau varieties

In the previous section, we described the role of automorphic forms with special divisors. The main task is to construct automorphic forms that satisfy the conditions of the first or second automorphic criteria.

4.1. Automorphic product at a one-dimensional cusp.

Consider the *η-Dedekind function*

$$\eta(\tau) = q^{1/24} \prod_{n \geq 1} (1 - q^n) \in S_{1/2}(\mathrm{SL}_2(\mathbb{Z}), v_\eta), \tag{19}$$

which is a cusp form of weight $1/2$ with a system of multipliers $v_\eta : SL_2(\mathbb{Z}) \to U_{24}$ of order 24. The basic object in our construction is **an odd Jacobi theta function** ($\vartheta(\tau, -z) = -\vartheta(\tau, z)$)

$$\vartheta(\tau, z) = q^{\frac{1}{8}} (\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}) \prod_{n \geq 1} (1 - q^n \zeta)(1 - q^n \zeta^{-1})(1 - q^n). \tag{20}$$

In [GN3] was noted that $\vartheta(\tau, z) \in J_{1/2,1/2}(v_\eta^3 \times v_H)$ happens to be a holomorphic Jacobi form of weight $1/2$ and index $1/2$ in terms of definitions from §4.

Let $L = 2U \oplus L_0(-1)$ as above. $\widetilde{\mathrm{O}}^+(L)$ is a stable orthogonal group. For any $v \in L \otimes \mathbb{Q}$ satisfying $(v, v) < 0$, define the corresponding rational quadratic divisor For any $v \in L \otimes \mathbb{Q}$ satisfying $(v, v) < 0$, define the corresponding rational quadratic divisor $\mathcal{D}_v = \{[Z] \in \mathcal{D}(M) : (Z, v) = 0\}$. We fix the affine cylindrical realization $\mathcal{H}(L_0)$ of the homogeneous domain $\mathcal{D}(L)$ (see (13)).

Theorem 4.1. (See [D2, Theorem 4.2].) *Suppose that for the Jacobi form $\varphi \in J^{\mathrm{nh}}_{0,L_0;1}$ the condition $f(n,\ell) = f(Q,\mu) \in \mathbb{Z}$ is satisfied, if $Q = 2n - \ell^2 \leqslant 0$. Then we define the meromorphic modular form $\mathcal{B}_\varphi(Z)$ of weight $k = f(0,0)/2$ with respect to $\widetilde{\mathrm{O}}^+(L)$, where $L = 2U \oplus L_0(-1)$, with character χ*

$$\mathcal{B}_\varphi(Z) = q^A r^{\vec{B}} s^C \prod_{\substack{n,m\in\mathbb{Z},\ \ell\in L_0^\vee \\ (n,\ell,m)>0}} (1 - q^n r^\ell s^m)^{f(nm,\ell)},$$

where

$$Z = (\tau, \mathfrak{z}, \omega) \in \mathcal{H}(L_0), \quad q = \exp(2\pi i\tau),$$
$$r^\ell = \exp(2\pi i(\ell, \mathfrak{z})), \quad s = \exp(2\pi i\omega),$$

and $(n,\ell,m) > 0$ means that either $m > 0$, or $m = 0$ and $n > 0$, or $m = n = 0$, and $\ell < 0$, and

$$A = \frac{1}{24}\sum_{\ell\in L_0^\vee} f(0,\ell), \quad \vec{B} = \frac{1}{2}\sum_{\ell>0} f(0,\ell)\ell \in \frac{1}{2}L_0^\vee, \quad C = \frac{1}{2\,\mathrm{rank}\,L_0}\sum_{\ell\in L_0^\vee} f(0,\ell)(\ell,\ell).$$

The poles and zeros of the meromorphic form $\mathcal{B}_\varphi$ coinside with the Heegner divisors $H_Q(\mu)$ of the modular variety and the divisor multiplicity is equal to

$$\mathrm{mult}\, H_Q(\mu) = \sum_{d\geqslant 1} f(d^2Q, d\mu).$$

where $v \equiv \ell \mod 2U \oplus L_0(-1)$, $\ell \in L_0^\vee$, $n \in \mathbb{Z}$ such that $(v,v) = 2n - (\ell,\ell)$. In particular, $\mathcal{B}_\varphi$ is holomorphic, if all nonzero Fourier coefficients with indices of negative hyperbolic norms are positive.

Remark. Our version of the Borcherds product is it an exponential variant of the arithmetic lifting. The Infinite Product of Theorem 4.1 in our version it is written in a different way. Let φ be the original Jacobi form of weight 0 and

$$\widetilde{\varphi}(Z) = \widetilde{\varphi}(\tau, \mathfrak{z}, \omega) = \varphi(\tau, \mathfrak{z}) \exp(2pii\omega).$$

Then

$$\mathrm{Borch}(\varphi)(Z) = \widetilde{\psi}_{L;C}(Z) \exp\big(-\sum_{m\geq 1} m^{-1}\widetilde{\varphi}|T_-(m)(Z)\big), \tag{21}$$

where the sum under exponent is *arithmetic lifting of the Jacobi form of weight* 0 from Theorem 2.2. The first factor, i.e. the first nonzero Fourier–Jacobi coefficient with index C of the form $\mathrm{Borch}(\varphi)$ in a given one- dimensional cusp is a generalized theta block

$$\psi_{L,C}(\tau, \mathfrak{z}) = \eta(\tau)^{f(0,0)} \prod_{\ell>0} \left(\frac{\vartheta(\tau, (\ell, \mathfrak{z}))}{\eta(\tau)}\right)^{f(0,\ell)}. \tag{22}$$

A special case of Theorem 4.1 for the signature of signature lattices $(2,3)$ was suggested in the article by Gritsenko and Nikulin [GN1]–[GN3]. Below, in Sections 4.2–4.5, we give several applications of Theorem 4.1.

4.2. Twenty-three new representations of the Borcherds modular form Φ_{12}.

We consider an even unimodular lattice $II_{2,26}$ of signature $(2,26)$. The boundary of the Bailey–Borel compactification $\mathrm{O}^+(II_{2,26}) \setminus \mathcal{D}(II_{2,26})$ (this is the so-called moduli space of bosonic string) consists of one zero-dimensional cusp and 24 one-dimensional cusps corresponding to the classes of twenty-four even unimodular lattices (see [D4, Lemma 4.4]).

Theorem 4.2. (See [D2, Introduction].) *The Borchers form $\Phi_{12} \in M_{12}(\mathrm{O}^+(II_{2,26}), \det)$ vanishes with multiplicity 1 in the zero-dimensional cusp of the group $\mathrm{O}^+(II_{2,26})$. On the one-dimensional cusp corresponding to the Leach lattice, the value of Φ_{12} is equal to the Ramanujan form $\Delta_{12}(\tau)$. In one-dimensional cusps corresponding to Niemeyer lattices $N(R)$ with a nontrivial root system $R(N)$, Φ_{12} vanishes with multiplicity $h(R)$, where $h(R)$ the Coxeter number of irreducible components of the root system R. The first Fourier–Jacobi coefficient Φ_{12} in a neighborhood of this one-dimensional component coincides, up to a sign, with the Kac–Weil denominator function of the corresponding affine algebra $\hat{\mathfrak{g}}(R)$*

$$\Phi_{12}(\tau, \mathfrak{z}, \omega) = \pm\eta(\tau)^{24} \prod_{v \in R_+} \frac{\vartheta(\tau, (v, \mathfrak{z}))}{\eta(\tau)}\, e^{2\pi i h(R)\omega} + \ldots,$$

where $\vartheta(\tau, z)$ is the odd Jacobi theta-series (see (20)), $\eta(\tau)$ is the Dedekind η-function (see (19)). The product is taken over all positive roots finite root system R. The first theta block in the formula for $\Phi_{12}(\tau, \mathfrak{z}, \omega)$ is the same as the denominator Kac–Weyl function of the affine Lie algebra $\hat{\mathfrak{g}}(R(N))$, where $R(N)$ is a non-empty root system of the Niemeier lattice N (see [KP]).

4.3. A Frenkel–Feingold's long-standing question about the simplest hyperbolic Kac–Moody algebra

In 1983, in the work [FF] of I. Frenkel and A. Feingold [FF] it was posed a question of possible relationships between affine Lie algebras, the simplest hyperbolic Kac–Moody algebra and Siegel modular forms of genus 2. Note that the odd Jacobi theta function $\vartheta(\tau, z)$ is a the Kac–Weil denominator function of the simplest affine algebra $\hat{\mathfrak{g}}(A_1)$ of the root system $A_1 = \langle 2 \rangle$.

The result of Theorem 4.2 gives an automorphic answer to the question Frenkel–Feingold in the case of the Borcherds algebra $\mathfrak{G}_{FM}$ of hyperbolic rank 26. This is the so-called **Fake Monster Lie Algebra**. The last theorem shows that this algebra continues (in the automorphic sense) 23 affine Lie algebras $\hat{\mathfrak{g}}(R)$ with the root systems of the Niemeyer lattices: $3E_8,\ E_8 \oplus D_{16},\ D_{24},\ 2D_{12},\ 3D_8,\ 4D_6,\ 6D_4, A_{24},\ 2A_{12},\ 3A_8,\ 4A_6,\ 6A_4,\ 8A_3,\ 12A_2,\ 24A_1,$ $E_7 \oplus A_{17},\ 2E_7 \oplus D_{10},\ 4E_6,\ E_6 \oplus D_7 \oplus A_{11}, A_{15} \oplus D_9,\ 2A_9 \oplus D_6, 2A_7 \oplus D_5,\ 4A_5 \oplus D_4$.

Note that Borcherds gave in [Bor1]–[Bor2] the construction of automorphic products in the zero-dimensional cusp. More precisely, he found the Fourier decomposition of the form Φ_{12} in a single zero-dimensional cusp group $\mathrm{O}^+(II_{2,26})$ in terms of the Leech lattice, which does not contain roots. That is why in Borcherds' formula for Φ_{12} did not appear systems of roots of affine Lie algebras.

4.4. Lorentzian Kac–Moody algebras and uniruled modular manifolds.

The construction of reflective modular forms is an important applied problem in algebraic geometry and the theory of Kac–Moody algebras. In [D3], a complete classification of Lorentzian Kac–Moody algebras with the hyperbolic Weyl group generated by all 2-reflections of the root lattice is carried out. We will not give definitions of these generalized hyperbolic (super) Kac–Moody algebras (see details in [D3]), since we concentrated in this dissertation on applications to algebraic geometry. The following theorem (we give below its abbreviated version of the original result) is an application of Theorem 4.2. Application of the construction of a quasi pull-back (see Theorem 5.3 in §5) gives the following result.

Theorem 4.3. (See [D3, Theorem 3.1] and [D2, Theorem 6.10].) *We put $L = 2U \oplus L_0(-1)$, where L_0 is one of the* 33 *root lattices indicated below. Quasi pull-back (see Theorem 5.3 below) of the Borcherds modular form*

$$\Phi_{12}|_{L_0} \in M_k(\widetilde{\mathrm{O}}^+(L), \det)$$

is a strictly (-2)-reflective modular form with complete (-2)-divisor of the weight k indicated in the bottom line,

$$A_1,\ 2A_1,\ 3A_1,\ 4A_1;\ \ A_2,\ 2A_2,\ 3A_2;\ \ A_3,\ 2A_3;\ \ A_4,\ A_5,\ A_6,\ A_7;$$

$$k = 35,\ \ 34,\ 33,\ \ 32;\ \ \ 45,\ \ 42,\ \ 39;\ \ 54,\ \ 48;\ \ \ 62,\ 69,\ 75,\ 80,$$

$$D_4,\ 2D_4,\ D_5,\ D_6,\ D_7,\ \ D_8;\ E_6,\ \ E_7,\ \ E_8,\ \ 2E_8;\ N_8;$$

$$k =\ 72,\ \ 60,\ \ \ 88,\ 102,\ 114,\ 124,\ 120,\ 165,\ 252,\ 132,\ \ 28;$$

or of weight $k = 12$ for

$$\langle 4\rangle,\ \langle 6\rangle,\ \langle 8\rangle,\ D_2(2) = \langle 4\rangle \oplus \langle 4\rangle,\ A_2(2),\ A_2(3),\ A_3(2),\ D_4(2),\ E_8(2).$$

Note that the first 24 *forms of weight $k > 12$ are cusp forms. All these reflective modular forms define the automorphic Lorentzian Kac–Moody algebras of the hyperbolic lattice $U \oplus L_0$.*

Theorem 4.4. (See [D2, §6.5].) *For all* 33 *lattices L from Theorem 4.3, the modular variety $\widetilde{\mathrm{O}}^+(L) \setminus \mathcal{D}(L)$ is at least uniruled.*

4.5. Elliptic genus in two variables of a manifold with $c_1 = 0$. Secondary quantization of the elliptic genus of Calabi–Yau varieties.

Let M be an (almost) complex compact manifold M of (complex) dimension d, T_M is the holomorphic tangent bundle of the manifold M, T_M^* is its dual. We put $q = \exp(2\pi i\tau)$ and $y = \exp(2\pi iz)$ ($\tau \in \mathbb{H}_1$, $z \in \mathbb{C}$). We define a formal power series $\mathbf{E}_{q,y} \in K(M)[[q, y^{\pm 1}]]$

$$\mathbf{E}_{q,y} = \bigotimes_{n=0}^{\infty} \bigwedge\nolimits_{-y^{-1}q^n} T_M^* \otimes \bigotimes_{n=1}^{\infty} \bigwedge\nolimits_{-yq^n} T_M \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_M^* \otimes \bigotimes_{n=1}^{\infty} S_{q^n} T_M,$$

where $\bigwedge_x E = \sum_{k\ge 0} (\wedge^k E)x^k$ and $S_x E = \sum_{k\ge 0}(S^k E)x^k$. Suppose the first Chern class $c_1(T_M) = 0$. Holomorphic Euler characteristic $\mathbf{E}_{q,y}$

$$EG(M_d; \tau, z) = y^{d/2} \int_M \mathrm{ch}(\mathbf{E}_{q,y})\mathrm{td}(T_M) = y^{d/2} \sum_{p=0}^{d} (-1)^p y^{-p} \chi^p(M) + q(\dots),$$

is called the **elliptic genus** of the variety M. Note that in the q^0-coefficient of elliptic genus coincides with Hirzebruch χ_y-genus of M where $\chi^p(M) = \chi(M, \wedge^p T_M^*) = \sum_{q=0}^{d} (-1)^q h^{p,q}(M)$.

Theorem 4.5. (See [D8].) *Elliptic genus $EG(M_d; \tau, z)$ of a complex variety M of dimension d with $c_1(M) = 0$ is a weak Jacobi form of weight 0 and index $d/2$ of Eichler–Zagier type with integral Fourier coefficients*

$$EG(M_d; \tau, z) \in J_{0,d/2}^{weak,\mathbb{Z}}.$$

The elliptic genus is uniquely determined by the $\chi_y(M)$ if the dimension of M is less than 12 or equal to 13.

The notion of elliptic genus for $N = 2$ supersymmetric theories was introduced in string theory by Witten, Eguchi and others (see [D8] for details). In physics, the elliptic genus of a Calabi-Yau manifold M_d is defined as the genus one partition function of the supersymmetric sigma model, whose target space is M_d. In [D8] we gave a mathematical proof that the elliptic genus of a Calabi-Yau variety of dimension d is a Jacobi form of weight 0 and index $d/2$. This Jacobi form can be used to construct an automorphic product by virtue of Theorem 4.1. The function $\mathrm{Borch}^{-1}(EG(M_d), Z)$ defines the second quantized elliptic genus of the Calabi–Yau manifold M_d by results of R. Dijkgraaf, G. Moore, E. Verlinde and H. Verlinde (see [SQ] and [D8]). In [D8], we study the connection between the second quantized elliptic genus and Lorentzian Kac–Moody algebras of signature $(1, 2)$, constructed by Gritsenko and Nikulin in [GN2].

Theorem 4.6. (See [D8, §3]). *Let M_d be a Calabi–Yau of dimension $d = 2$, 4, or 6. Then the second quantized elliptic genus of the manifold M_d is expressed as the product of the denominator functions of Lorentzian Kac–Moody algebras of rank 3 from the Gritsenko - Nikulin list in [GN1] – [GN4].*

5 Basic results on moduli spaces.

In the abstract, we keep the numbering of theorems from the main text of the dissertation.

5.1. Siegel's question on the geometric type of the moduli space of polarized Abelian surfaces. The answer to this question is given in Theorem 2.3

5.2. Moduli spaces of polarized K3 surfaces. As noted in the Introduction by the last open question of the Weyl program there was a question about the geometric type of moduli spaces of polarized K3 surfaces. A solution was suggested in [D4]. (See the report by C. Voisin at the Bourbaki seminar [Vo2].) Let us formulate one of the main results of the dissertation.

Theorem 5.1. ([D4, Theorem 1]) *The moduli space $\mathcal{F}_{2d}$ (see (4)–(5)) of polarized* K3 *surfaces of degree $2d$ is of general type for $d = 46, 50, 54, 57, 58, 60$ and for all $d > 61$. Kodaira dimension of $\mathcal{F}_{2d}$ is non-negative if $d \geq 40$ and $d \neq 41, 44, 45, 47$.*

Note that the question remains open for polarizations in the range $20 \leq d < 40$, since the studies of Mukai (1988–2010) give the following result.

Proposition. (See [Mu1]–[Mu5].) *The moduli space $\mathcal{F}_{2d}$ of polarized* K3 *surfaces of degree $2d$ is unirational for $1 \leq d \leq 12$ and $d = 15, 16, 17, 19$.*

To prove Theorem 5.1, we carried out in [D4] a detailed study of modular varieties. The following general result is very important.

Theorem 5.2. (See [D4, Theorem 2.1].) *Let L be a lattice of signature $(2, n)$ with $n \geq 9$, and let $\Gamma < \mathrm{O}^+(L)$ be a subgroup of finite index. Then there exists a projective toroidal* *compactification $\overline{\mathcal{F}}_L(\Gamma)$ of $\mathcal{F}_L(\Gamma) = \Gamma\backslash\mathcal{D}_L$ such that $\overline{\mathcal{F}}_L(\Gamma)$ has canonical singularities and there are no branch divisors in the boundary. The branch divisors in $\mathcal{F}_L(\Gamma)$ arise from the fixed divisors of $\pm$reflections. The ramification divisors in $\mathcal{F}_L(\Gamma)$ arise from the fixed divisors of $\pm$-reflections in the group Γ.*

The proof of this fundamental theorem consists of three parts We studied elliptic singularities of modular varieties, its ramification divisor, and the boundary of its compactification.

5.3. General type of moduli spaces of polarized hyper-Kähler manifolds. The above general method gives the following results for moduli of polarized hyper-Kähler manifolds.

Theorem 6.2 (See [D5, Theorem 4.1].) *Module space $\mathcal{M}^{split}(\mathrm{K3}^{[2]}, 2d)$ (see (6)–(7)) of polarized manifolds of type $\mathrm{K3}^{[2]}$ with a split polarization of the Beauville–Bogomolov degree $2d$ is of general type if $d \geq 12$. For $d = 9$ and $d = 11$ its Kodaira dimension is non-negative.*

For 10-dimensional O'Grady manifolds [OG1], there are split and non-split polarizations. They are fully described in [D7]. Below is the main result from [D7].

Theorem 6.3 ([D6, Theorem 4.1].) *Let d be a natural number not equal to 2^n for $n \geq 0$. The moduli Spaces of polarized ten-dimensional O'Grady varieties $\mathcal{M}^{split}_{O'G_{10},2d}$ (see (8) - (9)) with split polarization h of the Beauville–Bogomolov degree $h^2 = 2d \neq 2^{n+1}$ is of general type.*

5.4. Quasi pull-back of the Borcherds form. An important automorphic part of our method is the following theorem, which allows one to find automorphic forms used in the first and second automorphic criteria.

Theorem 5.3. (See [D4, Theorem 6.2] and a more general variant in [D2, §6].) *Let $L \hookrightarrow II_{2,26}$ be a primitive nondegenerate sublattice of signature $(2, n)$, $n \geq 3$, and let $\mathcal{D}_L \hookrightarrow \mathcal{D}_{II_{2,26}}$ be the corresponding embedding of the homogeneous domains. The set of (-2)-roots*

$$R_{-2}(L^\perp) = \{r \in II_{2,26} \mid r^2 = -2,\ (r, L) = 0\}$$

in the orthogonal complement is finite. We put $N(L^\perp) = \#R_{-2}(L^\perp)/2$. Then the function

$$\Phi|_L = \left.\frac{\Phi_{12}(Z)}{\prod_{r\in R_{-2}(L^\perp)/\pm 1}(Z, r)}\right|_{\mathcal{D}_L} \in M_{12+N(L^\perp)}(\widetilde{\mathrm{O}}(L), \det), \tag{23}$$

where in the product over r we fix a finite system of representatives in $R_{-2}(L^\perp)/\pm 1$. The modular form $\Phi|_L$ vanishes only on rational quadratic divisors of type $\mathcal{D}_v(L)$ where $v \in L^\vee$ is the orthogonal projection of a (-2)-root $r \in II_{2,26}$ on $L^\vee$.

We say that the modular form $\Phi|_L$ is a *quasi pull-back* of Φ_{12} if the set of roots $R_{-2}(L^\perp)$ is non-empty. We call the modular form $\Phi|_L$ a *quasi pull-back* of Φ_{12}, if the set of roots $R_{-2}(L^\perp)$ is not empty.

Theorem 5.4. *Let $L \hookrightarrow II_{2,26}$ be a nondegenerate sublattice of signature $(2, n)$, $n \geq 1$. We assume that the set $R_{-2}(L^\perp)$ of (-2)-roots in $L^\perp$ is non-empty. Then the quasi pull-back $\Phi|_L \in S_{12+N(L^\perp)}(\widetilde{\mathrm{O}}(L), \det)$ of the Borcherds form Φ_{12} is a cusp form.*

5.5. Vector of polarizations: the arithmetic of root lattices. To construct an automorphic form of a small weight with a large divisor, it is necessary to answer the following purely arithmetic question.

Key question for K3**:** *For which* $2d > 0$ *does the vector* $l \in E_8$ *exist such that*

$$l \in E_8, {}^2 = 2d,\ l \text{ is orthogonal to at least two and not more than 12 roots?} \tag{24}$$

If such a vector exists, then the quasi pull-back of Φ_{12} to subdomain $\mathcal{D}(2U \oplus 2E_8(-1) \oplus \langle l \rangle)$ in $\mathcal{D}(II_{2,26}) = \mathcal{D}(2U \oplus 3E_8(-1))$ will give us a cusp form of a small weight with a large divisor. According to *"the low weight cusp form trick"* (Theorem 4.3?), the module space $\mathcal{F}_{2d}$ will have the maximal Kodaira dimension.

Theorem 5.5. *(See [D4] and [D5].) A vector* l *satisfying* (24) *does exist if the inequality*

$$4N_{E_7}(2d) > 28N_{E_6}(2d) + 63N_{D_6}(2d) \tag{25}$$

is valid, where $N_L(2d)$ *denotes the number of representations of* $2d$ *by the lattice* L*.*

5.6. Moduli spaces of polarized Kummer surfaces (see (2)–(3)). In fact, we are talking about factors of finite order of the moduli space polarized Abelian surfaces. The paramodular group Γ_t, i.e. the modular group of module space $(1,t)$-polarized Abelian surfaces, and its maximal discrete extension in $\mathrm{Sp}_2(\mathbb{R})$ have a realization in the form of integral orthogonal groups signatures $(2,3)$. This implementation clearly describes the nature of normal extensions. Γ_t^+ and Γ_t^* (see [GH1]).

Let $L_t = 2U \oplus \langle -2t \rangle$ be an even integer lattice signatures $(2,3)$. According to [G2] and [GH1, Proposition 1.2 and Corollary 1.3]) we have the following isomorphisms

$$\Gamma_t^+/\{\pm E_4\} \cong \widetilde{\mathrm{O}}^+(L_t)/\{\pm E_5\}, \qquad \Gamma_t^*/\{\pm E_4\} \cong \mathrm{O}^+(L_t)/\{\pm E_5\}.$$

Coverings $\Gamma_t \setminus \mathbb{H}_2 \to \Gamma_t^+ \setminus \mathbb{H}_2$ and $\Gamma_t \setminus \mathbb{H}_2 \to \Gamma_t^* \setminus \mathbb{H}_2$ are Galois coverings with a finite abelian Galois group. According to [GH1, Proposition 1.5], the modular variety $\mathcal{A}_t^+ = \Gamma_t^+ \setminus \mathbb{H}_2$ (t is squarefree) is isomorphic to the *moduli space of polarized* $K3$ *surfaces with polarization of the type* $\langle 2t \rangle \oplus 2E_8(-1)$

According to [GH1, Theorem 1.5], the modular variety $\mathcal{K}_t = \Gamma_t^* \setminus \mathbb{H}_2$ isomorphic to it the moduli space of Kummer surfaces corresponding to Abelian surfaces with $(1,t)$-polarization. In (3), an orthogonal interpretation of this modular manifold was given.

Theorem 7.2 (See [D9, Corollary 8.1] and [D10, Theorem 6.2].) *The moduli space* $\mathcal{K}_p = \Gamma_p^* \setminus \mathbb{H}_2$ *of Kummer surfaces defined by* $(1,p)$*-polarized abelian surfaces has positive geometric genus for for* $p =$ 167*,* 173*,* 223*,* 227*,* 251*,* 257*,* 269*,* 271*,* 283*,* 293*. In addition, the following inequalities hold*

$$h^{3,0}(\Gamma_p^*, \mathbb{C}) \geq 2, \quad t = 227, 257, 269, 283, \quad \text{and} \quad h^{3,0}(\Gamma_{293}^*, \mathbb{C}) \geq 4.$$

This theorem was proven using theta-block theory of Gritsenko–Skoruppa–Zagir [GSZ] and the methods of the method of author's paper [GPY].

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