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Victor Lapshin

IMMUNIZING A MARKED-TO-MODEL OBLIGATION WITH MARKED-TO-MARKET FINANCIAL INSTRUMENTS

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Abstract

The traditional bond portfolio immunization problem statement assumes that both the obligation and the immunizing portfolio belong to the same liquidity class, i.e. both are valued either at their respective observed market prices (mark-to-market) or via the sum of discounted cash flows (mark-to-model). However, it is customary to hedge an obligation for which there is no liquid market with relatively liquid market instruments. We propose a new problem formulation, where the obligation is marked to a model via discounted cash flows while the immunizing portfolio is marked to the market via real observed prices. We solve the immunization problem in this new formulation and test the performance of its solution. The new approach performs better within the new problem formulation while the traditional approach performs better within the classical problem formulation. The differences are more pronounced if the number of actively traded bonds is small.

JEL: G12, E43.

Keywords: immunization, mark-to-market, mark-to-model, empirical test, bond portfolio.

[1]HSE University, Moscow, Russia; vlapshin@hse.ru

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1 Introduction

The immunization of a bond-like obligation with a portfolio of bonds is a classical topic in finance. Bierwag and Fooladi (2006) and Shah et al. (2020) provide a detailed review of the history of the question and the variety of approaches.

One of the main distinctions in different immunization approaches is the assumption about the possible changes in the term structure. Bierwag (1977) considered various multiplicative and additive term structure shift specifications. Cooper (1977); Willner (1996); Bravo and Fonseca (2012) considered the parametric immunization–they assumed that the term structure was defined by a specific parametric equation (e.g. that of Nelson and Siegel (1987)) and the space of possible changes was constrained to a low-dimensional manifold. Tark (1990); Barber and Copper (1996) estimated the most plausible term structure movements from the data via principal component analysis. Ingersoll et al. (1978); Cox et al. (1979) assumed the short rate governed by a specified stochastic process and inferred possible term structure changes from it.

Bierwag et al. (1983) considered the uncertainty in the said stochastic process. Fong and Vasicek (1984); Shiu (1988); Uberti (2000); Nawalkha and Chambers (1996); Barber and Copper (1998) among others considered worst-case immunization within fairly general classes of term structure shifts.

Reitano (1990); Ho (1992) introduced key rate durations–the sensitivities of instrument prices to changes in the interest rates for key terms to maturity. Dattatreya and Fabozzi (1995) considered sensitivities to the prices of the exact instruments chosen for hedging. Lapshin (2019) assumed that the term structure for pricing the immunized portfolio is inferred from the quotes of the same bonds (e.g. government bonds) which are used for immunization via smoothing splines.

Barber and Copper (1998); Balbás and Ibáñez (2002); Balbás et al. (2002) considered worst-case immunization, a concept which was later raised to infinite dimensional Banach spaces by Balbás and Romera (2007).

Most subsequent papers focused on empirical comparisons of previously developed immunization approaches. Of them, we note the papers by Bravo and Fonseca (2012); Soto (2004); Diaz et al. (2008); Ortobelli et al. (2018). However these approaches assume that the original obligation and the hedging instruments are priced via the same term structure. In many practical situations the hedging instruments will be chosen to be sufficiently liquid and will thus be marked to the market as per modern financial and risk reporting standards while the obligation will probably have to be marked to a model due to the lack of a liquid market for it, e.g., the immunization of pension and life insurance obligations which are non-traded by nature.

In this paper we describe the corresponding mathematical setup and solve the immunization problem in this new formulation, thus indirectly extending the work of Willner (1996) for classical parametric immunization in the spirit of Dattatreya and Fabozzi (1995). We also perform an empirical comparison of the new and the old approaches in the new and the old setups.

The remainder of the paper is organized as follows. Section 2 outlines the model setup and describes the proposed immunization method and the existing approach. Section 3 describes the data and the empirical analysis for comparison. Section 4 concludes.

2 Methodology

Let the original obligation be described by its cash flows $F_{0,i} \geq 0$ scheduled at terms $t_i > 0$, $i = 1..N$. The obligation is marked to a term structure r_i as the sum of discounted cash flows:

$$PV(F_0;\ r;\ l) = \sum_{i=1}^{N} F_{0,i} e^{-(r_i+l)t_i},$$

where the term structure r_i is estimated from the prices of K benchmark bonds promising cash flows $F_{k,i} \geq 0$ at terms t_i, $i = 1..N$, $k = 1..K$. Since the obligation in question is not liquid (or even not traded at all), we introduce an additional liquidity premium l. In most applications, the liquidity premium l will be determined by the asset class of the obligation and will stay fixed throughout the investment period. Note that the cash flow terms t_i can be considered the same for all bonds as we can introduce zero cash flows when necessary, i.e. set $F_{k,i} = 0$ if the bond k does not promise a cash flow at t_i.

2.1 The Model

We assume that the term structure r_i is obtained from fitting a parametric term structure curve $r_i = f(t_i,\ \theta)$ to the bond price data:

$$\|PV(F;\ r(\theta);\ 0) - P\|^2 = \sum_{k=1}^{K} \left(\sum_{i=1}^{N} F_{k,i} e^{-f(t_i,\ \theta)\cdot t_i} - P_k \right)^2 \to \min_{\theta}, \tag{1}$$

where $PV(F;\ r(\theta);\ 0)$ and P are the vectors of the calculated model prices and observed prices of all bonds respectively, and θ is the vector of the term structure parameters. Note that there is no liquidity premium here as the term structure is being estimated from liquid instruments. For example, many authors consider parametric immunization with respect to the parametric form of Nelson and Siegel (1987). It has 4 parameters $\theta = (\beta_0,\ \beta_1,\ \beta_2,\ \tau)$ and

$$f(t,\ \beta_0, \beta_1, \beta_2, \tau) = \beta_0 + (\beta_1 + \beta_2)\frac{1 - e^{-t/\tau}}{t/\tau} - \beta_2 e^{-t/\tau}.$$

Let w_k be the amounts of the benchmark bonds in the hedging (immunizing) portfolio. We do not consider trading restrictions here for the sake of brevity, however constraints $w_k \geq 0$ should be imposed if short positions are unavailable.

The immunized portfolio value is

$$-PV(F_0;\ r(P);\ l) + \sum_{k=1}^{K} w_k P_k,$$

where $r_i(P) = f(t_i,\ \theta(P))$ is the term structure fitted to observed bond prices P according to Eq. (1). Here we explicitly use that the benchmark bonds are marked-to-market. The risk factors of our model are benchmark bond prices P_k. The portfolio is immunized if its sensitivities to the risk factors are zero:

$$-\frac{\partial PV(F_0;\ r(P);\ l)}{\partial P_k}+w_k=0,\ k=1..K.$$

The first order condition for an extremum in Eq. (1) is:

$$2(PV(F;\ r;\ 0)-P)^T\frac{\partial PV(F;\ r;\ 0)}{\partial \theta}=0$$

Differentiation with respect to P yields

$$2\left(\frac{\partial PV(F;r;\ 0)}{\partial \theta}\right)^T\left(\frac{\partial PV(F;\ r;\ 0)}{\partial \theta}\frac{\partial \theta}{\partial P}-I\right)+ \\ 2\sum_{k=1}^{K}\left(PV(F_k;r;\ 0)-P_k\right)\frac{\partial^2 PV(F_k;r;\ 0)}{\partial \theta^2}\frac{\partial \theta}{\partial P}=0,$$

where I is an identity matrix of suitable dimensions. Let $\frac{\partial PV(F;r;0)}{\partial \theta}=B^TQ^T$, where $B=\left(\frac{\partial PV(F;r;0)}{\partial r}\right)^T$ is the matrix of the sensitivities of bond model prices to interest rate changes: $B_{i,k}=-t_iF_{k,i}e^{-r_it_i}$, and $Q=\left(\frac{\partial r(\theta)}{\partial \theta}\right)^T$ is the matrix of term structure sensitivities with respect to its parameters which, for our case of the Nelson-Siegel parametric estimate, are:

$$\begin{aligned}
Q_{i,1}&=\frac{\partial f(t_i,\beta_0,\beta_1,\beta_2,\tau)}{\partial \beta_0}=1,\\
Q_{i,2}&=\frac{\partial f(t_i,\beta_0,\beta_1,\beta_2,\tau)}{\partial \beta_1}=\frac{1-e^{-t_i\tau}}{t_i/\tau},\\
Q_{i,3}&=\frac{\partial f(t_i,\beta_0,\beta_1,\beta_2,\tau)}{\partial \beta_2}=\frac{1-e^{-t_i\tau}}{t_i/\tau}-e^{-t_i/\tau},\\
Q_{i,4}&=\frac{\partial f(t_i,\beta_0,\beta_1,\beta_2,\tau)}{\partial \tau}=(\beta_1+\beta_2)\left(\frac{1-e^{-t_i/\tau}}{t_i}-\frac{e^{-t_i/\tau}}{\tau}\right)-\beta_2\frac{t}{\tau^2}e^{-t_i/\tau}.
\end{aligned}$$

Then we can write

$$QB(B^TQ^T\frac{\partial \theta}{\partial P}-I)+A_\theta\frac{\partial \theta}{\partial P}=0$$

with $A_\theta=\sum_{k=1}^{K}\left(PV(F_k;r;\ 0)-P_k\right)\frac{\partial^2 PV(F_k;r;0)}{\partial \theta^2}$. We can now express

$$\begin{aligned}
\frac{\partial \theta}{\partial P}&=\left(QBB^TQ^T+A_\theta\right)^{-1}QB,\\
\frac{\partial PV(F_0;r(P);\ l)}{\partial P}&=\frac{\partial PV(F_0;r(\theta);\ l)}{\partial \theta}\frac{\partial \theta}{\partial P}=B_0(l)^TQ^T\left(QBB^TQ^T+A_\theta\right)^{-1}QB,
\end{aligned}$$

where $B_0(l)=\left(\frac{\partial PV(F_0;r;\ l)}{\partial r}\right)^T$ is the vector of the sensitivities of the original obligation model price to interest rate changes: $B_{0,k}(l)=-t_iF_{0,i}e^{-(r_i+l)t_i}$.

Now the optimal hedging coefficients w_{New} are calculated from

$$w_{New}=\frac{\partial PV(F_0;\ r(P);\ l)}{\partial P}=B^TQ^T\left(QBB^TQ^T+A_\theta\right)^{-1}QB_0(l), \tag{2}$$

where Q and A_θ are determined by the term structure model and its current parameters, B and B_0 are determined by the current term structure and the cash flow structure of the benchmark bonds and the original obligation respectively. Note that even though the term structure model has fewer degrees of freedom than the number of traded bonds, we are able to identify all hedging coefficients because our model has one risk factor per bond. This is in contrast to the traditional approach below where the risk factors are the term structure parameters (which are fewer than bonds).

Now we introduce a regularized version of this approach. Traditional Tikhonov-style regularization does not seem like a good idea here because the source of the variance is different. If we take the term structure estimation model in Eq. (1) for granted, there is no other observation error in our model–the observed bond prices enter this this equation exactly as is–so the term structure used for pricing the obligation is determined via Eq. (1) from the exact observed values P_k. However, there is another source of excess variance. Since Eq. (1) is solved numerically, the obtained values of Q, B, and A_θ are not exact, which could make the inversion of $QBB^TQ^T + A_\theta$ unstable or even impossible. Thus, regularization should be done within the inversion of $QBB^TQ^T + A_\theta$, by imposing a lower threshold on its singular values:

$$w_{R-New} = \frac{\partial PV(F_0;\ r(P);\ l)}{\partial P} = B^T Q^T \left(QBB^T Q^T + A_\theta\right)^{+}_{(\alpha)} QB_0(l), \tag{3}$$

where $(A)^{+}_{(\alpha)}$ denotes the pseudoinverse of A with the singular values less than α treated as 0. The higher the threshold α, the more pronounced the regularization. In what follows, we test this regularized version along with the unregularized one.

2.2 Traditional parametric immunization

In this section we describe the existing parametric immunization approach within the same framework to facilitate the comparison. Traditional parametric immunization requires the sensitivities of the immunizing portfolio with respect to the term structure parameters be equal to those of the original obligation:

$$w^T \frac{\partial PV(F;\, r(\theta);\, 0)}{\partial \theta} = \frac{\partial PV(F_0;\, r(\theta);\, 0)}{\partial \theta}$$

or $QBw = QB_0(0)$ in our notation. The traditional approach does not consider the liquidity premium ($l = 0$), therefore we consider the effect of introducing a nonzero liquidity premium separately. Unfortunately, this only allows us to identify the number of hedging coefficients equal to the number of parameters in the term structure model (4 in the case of the Nelson-Siegel model). The best practice is to search for the least norm solution

$$\begin{cases} \|w\|^2 \to \min, \\ QBw = QB_0(0). \end{cases}$$

The least squares parametric hedging coefficients are thus defined by

$$w_{LS} = B^T Q^T (QBB^T Q^T)^{-1} QB_0(0), \tag{4}$$

which differs from w_{New} by the absence A_θ and in assuming $l = 0$. A regularized version of this is the solution of

$$\|QBw - QB_0(0)\|^2 + \alpha\|w\|^2 \to \min,$$

which is known to be defined by

$$w_{R-LS} = (B^T Q^T QB + \alpha I)^{-1} B^T Q^T QB_0(0) = B^T Q^T (QBB^T Q^T + \alpha I)^{-1} QB_0(0), \tag{5}$$

where I denotes an identity matrix with suitable dimensions.

2.3 Comparison of the Two Approaches

We start the comparison by noting that there are two distinct differences between the traditional least-squares immunization and the proposed model–a special term A_θ appearing in place of the regularization term αI and the introduction of a nonzero liquidity premium l.

2.3.1 Differences in the Liquidity Premium

As the traditional sensitivity-based immunization does not consider the liquidity premium at all, we assess the effect of omitting the liquidity premium for the obligation–given that it exists. For this purpose, we consider the liquidity-premium-aware versions of the traditional sensitivity-based immunization and compare its performance with the original approach, which is unaware of any possible liquidity premium:

$$w_{LS-liq} = B^T Q^T (QBB^T Q^T)^{-1} QB_0(l) \tag{6}$$

for the unregularized version and

$$w_{R-LS-liq} = (B^T Q^T QB + \alpha I)^{-1} B^T Q^T QB_0(l) = B^T Q^T (QBB^T Q^T + \alpha I)^{-1} QB_0(l) \tag{7}$$

for the regularized version.

We are thus interested in whether a failure to include a known liquidity premium into the model has significant consequences. This could happen when different company divisions are responsible for pricing the obligation and for hedging it–in these circumstances a separate effort is necessary to keep the pricing and the hedging models consistent with each other. We are basically testing whether this separate effort produces any tangible outcome in terms of the immunization. Note that the liquidity premium is exogenous in our setting.

2.3.2 Difference in the Problem Statement

Another difference is due to the different problem statement–the observed term structure r is now endogenously determined from the observed bond prices P instead of fluctuating randomly. Assume the observed bond prices are determined from an unobserved random 'true' term structure r^*.

Then the chain of dependence is $r^* \to P \to r$. If we assume the true r^* to belong to the same parametric family within which we estimate r and if there are enough bonds in P to estimate r reliably, then r will be close enough to r^* and thus the endogeneity effect will be small: $r = r^* \to P$.

We now show that if the amount of data for estimating the term structure grows infinitely, the two approaches yield the same results.

Proposition 1. *Assume that the model is correct, i.e. there is a 'true' term structure r^* belonging to the chosen parametric family $r^* = r(\theta^*)$ for some choice of θ^*. Also assume that the liquidity premium is zero and the dataset is rich enough, that is,*

- *there are K zero-coupon bonds with terms to maturity t_k uniformly spread over $[0, T]$,*
- *$P_k = PV(F_k;\, r^*;\, 0) + \varepsilon_k$ with the residuals ε_k i.i.d. normal with finite variance $\sigma_k^2 = \sigma(t_k)^2$,*
- *the true term structure parameters θ^* are uniquely identified by the observed bond prices P via Eq. (1) with a regularity condition $\frac{\partial PV}{\partial \theta} = B^T Q^T$ having full column rank at $\theta = \theta^*$, that is, we do not have to resort to second order effects to identify the parameters θ^* from the observed bond prices.*

Then the two estimates converge:

$$\lim_{K \to \infty} \frac{\|w_{New} - w_{LS}\|}{\|w_{LS}\|} = 0.$$

Proof. The proof is technical. See the Appendix. □

The intuition behind Proposition 1 is simple, w_{LS-liq} assumes that the true parameters θ^* evolve independently and randomly with the bond prices determined from them; while w_{New} assumes that θ is endogenously determined from the observed bond prices. When the number of bonds goes to infinity, θ estimated from the bond prices converges to the true parameter vector θ^* and the endogeneity effect vanishes. Therefore, we expect the differences between the two methods to be more pronounced for small numbers of bonds in the dataset.

3 Data and Empirical Analysis

This section describes the nature of our empirical test. Some of our testing objectives are motivated by the discussion above (the main questions) while others are inherent in the design of the experiment itself, such as robustness testing (the design questions). There are two main questions to be answered empirically.

1. How much does it cost–in terms of the immunization performance–to ignore the liquidity premium when immunizing an illiquid obligation with liquid instruments? We expect this effect to exist, however its magnitude is of primary interest.

2. In which circumstances does the proposed approach outperform the traditional sensitivity-based immunization? We expect it to excel within a carefully tuned testing environment. However, the limits of its practical applicability are of more interest.

For the empirical test we use a dataset of daily Spanish government bonds prices from 1996 to 2020 obtained from Bloomberg. We chose this dataset because there are many actively traded bonds, some with very long terms to maturity–up to 50 years. Also, Spanish bonds are used in some other immunization studies. The bonds are coupon-bearing with the entire payment schedule known ahead. Figure 1 shows the maturities of all the bonds in the dataset. It shows whether a bond with a given maturity was traded on a given date. Figure 2 shows that the typical numbers of traded bonds was from 24 to 34 with the full range from 9 to 44.

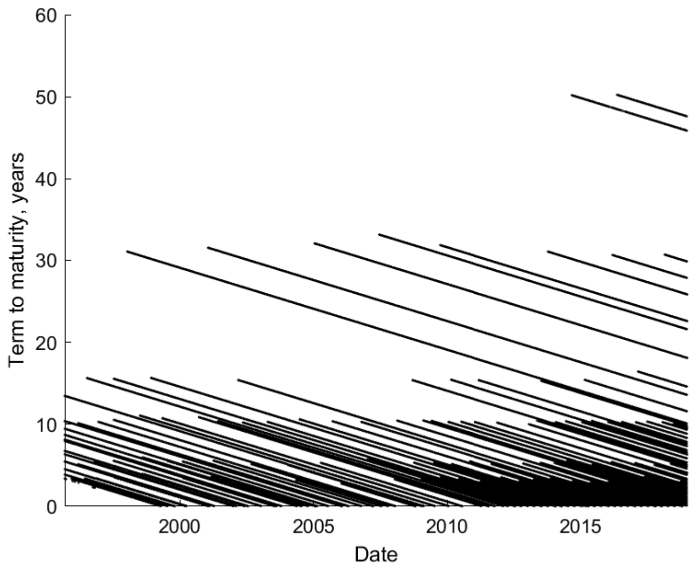

Figure 1: Maturities of all bonds in the dataset.

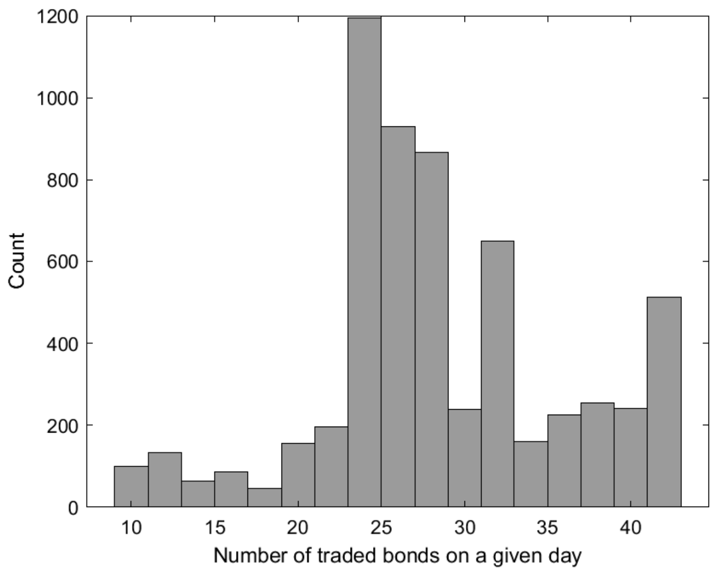

Figure 2: Number of bonds in the dataset.

We use the randomized leave-out-one cross validation procedure where we take a random trading day τ and choose a random bond k traded on that day as an obligation to be hedged–subject to it having neither the longest nor the shortest term to maturity in the dataset. Then we temporarily remove the chosen bond-now-obligation from the dataset and immunize it with all remaining bonds according to all immunization strategies. We do not consider the bonds maturing before the immunization horizon.

If at time τ we form the immunizing portfolio with the hedging vector w, then (assuming the portfolio is immunized well enough and earns only the risk-free rate and the liquidity premium if any) the financial outcome which can be reasonably expected at the reporting time $\tau+H$ is

$$FV(w, \tau, k, H) = -V_k^\tau e^{(r_H^\tau(P_{-k}^\tau)+l)H} + w^T V_{-k}^\tau e^{r_H^\tau(P_{-k}^\tau)H},$$

where V_k^τ is the value of the bond k playing the part of the obligation to be hedged at time τ, V_{-k}^τ are the values of the hedging instruments (all other bonds except k) at time τ, the bonds are valued at either the market prices or model prices as described below, H is the immunization horizon, $r_H^\tau(P_{-k}^\tau)$ is the spot rate for the term H, which can be estimated at time τ from the observed prices P_{-k}^τ of all bonds except k, and l is the fixed liquidity premium.

For a given immunization strategy, we calculate the mean absolute deviations of the actual financial outcome (portfolio value) from the corresponding expected financial outcome:

$$MAD = \frac{1}{N_{sim}} \sum_{i=1}^{N_{sim}} \left| FV(w^i, \tau^i, k^i, H) - (-V_{k^i}^{\tau^i+H} + w^T V_{-k^i}^{\tau^i+H}) \right|, \tag{8}$$

where τ^i is a randomly sampled starting time, k^i is the bond to play the role of the obligation as discussed above, w^i is the hedging vector calculated according to the chosen immunization strategy, $V_{k^i}^{\tau^i+H}$ is the value of the obligation at the reporting time τ^i+H, and $V_{-k^i}^{\tau^i+H}$ are the values of the hedging instruments (all other bonds except k^i) at the reporting time τ^i+H.

We assume a 1-step immunization with no interim portfolio rebalancing (a multi-step immunization can be considered as a sequence of 1-step problems with short horizons–and we test these). All coupon payments due within the immunization horizon are assumed to be reinvested at the corresponding observed interest rates. If a new bond was issued within the immunization horizon, it was not used for immunization purposes (as it was not there at time τ^i), but it was used to estimate the term structure at time τ^i+H for valuation purposes. These details are not reflected in Eq. (8) in order not to clutter the notation even more.

All numbers reported are based on $N_{sim} = 10,000$ portfolios.

Robustness test: we ran the simulations several times and the results were robust to the choice of the random seed.

There are 5 dimensions in our numerical experiment, for each of which we list the options and their expected influence on the hedging performance.

With respect to the problem statement (portfolio valuation V), we have 3 options:

- **Traded obligation**–the obligation is valued at its market price:

$$V_k^\tau = P_k^\tau e^{-lD_k^\tau},$$

where P_k^τ is the observed market price of the bond k playing the part of the obligation (since a real bond plays this role, its market price is known, which however does not include a liquidity premium), l is the liquidity premium and D_k^τ is the duration of the bond k at time τ. This attempts to artificially introduce the liquidity premium l to make up for a liquid bond k playing the part of an illiquid obligation. The hedging instruments are valued at their respective market prices at that time without any corrections:

$$V_{-k}^\tau = P_{-k}^\tau.$$

- **Illiquid obligation**–the obligation is valued at its model price:

$$V_k^\tau = PV(F_k^\tau;\ r^\tau(P_{-k}^\tau);\ l), \quad V_{-k}^\tau = P_{-k}^\tau.$$

  The hedging instruments are still marked-to-market. This is the problem statement we explore in this paper, therefore we expect the new approach to perform better in this setting.

- **Idealized conditions**–as an additional test, we include another problem statement, where all bonds in the portfolio are valued at their model prices:

$$V_k^\tau = PV(F_k^\tau;\ r^\tau(P_{-k}^\tau);\ l), \quad V_{-k}^\tau = PV(F_{-k}^\tau;\ r^\tau(P_{-k}^\tau);\ 0).$$

  While this approach is questionable from a practical point of view (we consider the bonds liquid enough to estimate the term structure from them but value them via a model rather than at their market prices), it is exactly this setting which is usually used to derive the traditional sensitivity-based approach. Thus, we expect this problem statement to favor the unregularized traditional approach, especially for smaller term structure changes caused by shorter immunization horizons.

Expected influence: we expect the **traded obligation** formulation to favor the regularized traditional hedging approach, the **illiquid obligation** formulation to favor our new proposed approach and the **idealized conditions** formulation to favor the unregularized traditional approach.

With respect to the immunization method, that is, the approach for determining the hedging coefficients w, we test 4 approaches:

- the least squares solution to the traditional sensitivity-based immunization problem either as is (LS–as defined by Eq. (4)) or incorporating the nonzero liquidity premium (LS-liq–as defined by Eq. (6)),

- their respective regularized versions (R-LS and R-LS-liq) as defined by Eqs. (5) and (7) respectively,

- the proposed method (New) as defined by Eq. (2),

- its regularized version (R-New) as defined by Eq. (3).

Expected influence: we expect the regularized versions to perform better than their unregularized counterparts in all problem formulations except the **idealized conditions**. We also expect the new approach to perform better under the **illiquid obligation**.

With respect to the immunization horizon H, we have 3 options:

- Short-term hedging–with a horizon of 1 day.
- Medium-term hedging–with a horizon of 2 weeks.
- Long-term hedging–with horizon of 1 year. Larger horizons are not practical because most financial reporting is done at least annually.

Expected influence: we expect the differences between the methods to be the most pronounced for shorter immunization horizons as the term structure changes will most likely be negligible.

With respect to K–the number of bonds in the dataset, we test 2 scenarios:

- Rich dataset–using all available information except the one bond designated to play the role of the original obligation for estimating the term structure.
- Scarce dataset–at each date we restrict the bond universe to $K+1=6$ randomly chosen bonds, one of which plays the role of the obligation while the other $K=5$ bonds are used to estimate the term structure and to immunize the obligation. These same bonds are used to estimate the term structure at the end of the hedging horizon to avoid extreme jumps in the term structure. As the Nelson-Siegel term structure model only has 4 parameters, we expect that with 5 bonds in the dataset the endogeneity effects will be most pronounced.

Expected influence: motivated by Proposition 1, we expect the differences between the approaches to be more pronounced for a scarce dataset.

Finally, with respect to the liquidity premium l, we have 2 options:

- Use a known liquidity premium of $l=100$ basis points.
- Use the liquidity premium $l=100$ b.p. only in valuation when assessing the performance according to Eq. (8), but not in hedging. The hedging coefficients w are calculated using $l=0$, that is, according to Eqs. (4) and (5) instead of Eqs. (6) and (7). This allows us to separately assess the influence of the two changes to the traditional immunization approach–the endogenously determined term structure and the liquidity premium.

Expected influence: we expect all methods to exhibit reduced performance for the second case, the degree of this reduction is of primary interest.

The regularization parameter α for both regularized methods was chosen to be 10^{-6} using a preliminary test run with different random dates and bonds.

Table 1 presents the results for the case of a known liquidity premium. Since the mean absolute deviations have varying orders of magnitude, we report the risk reduction ratios–the ratios of mean absolute deviations before the immunization and after it, as: $R=\frac{\mathrm{MAD}_{\mathrm{unhedged}}}{\mathrm{MAD}_{\mathrm{hedged}}}$.

There are 4 conclusions to be drawn from Table 1:

Horizon	Method	Traded Obligation $K = 5$	Traded Obligation All	Illiquid Obligation $K = 6$	Illiquid Obligation All	Idealized Conditions $K = 6$	Idealized Conditions All
1 day	LS-liq	3.2	6.6	5.5	9.2	26.0	91.4
	R-LS-liq	5.1	6.1	8.8	7.7	9.2	14.0
	New	4.0	5.7	10.6	9.2	12.1	10.5
	R-New	4.8	6.2	12.2	10.3	12.8	12.3
2 weeks	LS-liq	2.9	9.3	3.9	8.9	27.0	82.4
	R-LS-liq	7.0	8.1	13.8	7.9	14.1	19.2
	New	6.0	8.2	17.5	9.2	21.3	13.9
	R-New	6.5	8.4	18.6	9.4	21.0	14.3
1 year	LS-liq	8.4	17.8	11.2	29.4	18.2	41.0
	R-LS-liq	13.4	14.9	19.3	22.5	19.5	24.3
	New	11.0	15.6	16.6	25.1	18.1	29.4
	R-New	12.8	15.9	20.8	25.8	21.7	30.2

Table 1: Risk reduction ratio, $R = \frac{\mathrm{MAD}_{\mathrm{unhedged}}}{\mathrm{MAD}_{\mathrm{hedged}}}$. The liquidity premium $l = 1\%$ p.a. is known. More is better.

1. All the effects mentioned below are less pronounced or vanish completely for a large number of bonds. The effects become more pronounced if the number of bonds used to estimate the term structure is small ($K = 5$). This agrees with our initial hypothesis.

2. The new method performs better for the case of an illiquid obligation; the traditional method performs better for the idealized problem formulation from which it was derived. This agrees with our initial hypothesis.

3. The new method performs slightly worse for a traded obligation. Regularization improves the performance of both methods via reducing the adverse effects of model misspecification. However, if model misspecification is not an issue (ideal conditions for the traditional approach), regularization is not necessary and actually reduces performance. This also agrees with our initial hypothesis.

4. The differences outlined above have no clear dependence on the immunization horizon. Ideal conditions become more ideal with shorter horizons, with the idealized model performing better as expected. However, the overall relative effect of the immunization is otherwise higher for large immunization horizons. This is an artifact of reporting the relative risk reduction. In fact, the absolute MAD values after immunization for 1 year are about 10 times higher than for 1 day and for 14 days about 2–3 times higher than for 1 day. However, the absolute MAD values before immunization are much higher for larger horizons. This might explain the higher relative risk reduction.

Table 2 shows the effect of liquidity premium misspecification. The performance of all models drops typically 1.5–2 times and becomes more leveled compared to the known liquidity premium. We expected this effect to exist, however its magnitude suggests that accounting for a liquidity premium should probably have higher priority than choosing the immunization model.

Horizon	Method	Traded Obligation $K=5$	Traded Obligation All	Illiquid Obligation $K=6$	Illiquid Obligation All	Idealized Conditions $K=6$	Idealized Conditions All
1 day	LS	2.3	5.1	3.2	4.8	9.1	9.3
	R-LS	4.3	4.9	6.2	4.6	6.3	6.8
	New	2.7	4.7	4.2	4.8	4.2	6.2
	R-New	4.1	4.7	6.9	4.9	7.1	6.4
2 weeks	LS	2.6	6.2	3.3	4.4	9.6	8.4
	R-LS	5.7	5.8	7.7	4.3	7.8	7.4
	New	4.5	5.8	7.1	4.4	7.8	6.5
	R-New	5.2	5.8	8.2	4.5	8.6	6.6
1 year	LS	5.9	9.4	6.6	10.3	9.1	11.0
	R-LS	9.3	8.8	10.4	9.9	10.5	10.1
	New	6.1	9.0	6.8	10.0	5.4	10.2
	R-New	8.5	9.0	10.0	10.0	10.2	10.3

Table 2: Risk reduction ratio, $R = \frac{\mathrm{MAD}_{\mathrm{unhedged}}}{\mathrm{MAD}_{\mathrm{hedged}}}$. The liquidity premium $l = 1\%$ p.a. is assumed to be zero while hedging. More is better.

4 Conclusion

Traditional immunization methods usually assume that both the original obligation and the immunizing portfolio are marked-to-market (earlier studies assumed that they are all marked-to-model–mostly due to data unavailability). We study a more practice-oriented approach. The original obligation is assumed to be priced via discounted cash flows with a liquidity premium while the immunizing portfolio is assumed to be liquid enough to be priced at observed market quotes. The solution to the problem bears some resemblance to the traditional sensitivity-based parametric immunization.

An empirical test reveals that the proposed model and the traditional sensitivity-based hedging each perform better in their respective natural domains–hedging an illiquid obligation for the new approach and the 'ideal conditions' for the traditional sensitivity-based approach. If both the obligation and the immunizing portfolio are marked-to-market, regularization better accounts for imperfections. Moreover, taking the liquidity premium into account might be as (or even more) important as using the right model for immunization.

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Appendix: Proof of Proposition 1

Denote $\left.\frac{\partial f(t;\theta)}{\partial \theta_i}\right|_{\theta=\theta^*} = f_i(t)$. Note that the conditions of Proposition 1 imply that B is a diagonal matrix with $B_{k,k} = -t_k e^{-r^*(t_k)t_k}$ and $Q_{i,k} = f_i(t_k)$. Therefore,

$$\frac{T}{K}\left[QBB^TQ^T\right]_{i,j} = \sum_{k=1}^{K}\left(-t_k e^{-r^*(t_k)t_k}\right)^2 f_i(t_k)f_j(t_k)\frac{T}{K} \to \int_0^T t^2 e^{-2r^*(t)t} f_i(t)f_j(t)\,dt.$$

At the same time,

$$\sqrt{\frac{T}{K}}[A_\theta]_{i,j} = \sum_{k=1}^{K}\varepsilon_k f_{i,j}(t_k)\sqrt{\frac{T}{K}} = \sum_{k=1}^{K} f_{i,j}(t_k)\sigma(t_k)(W(t_k)-W(t_{k-1})) \to$$
$$\to \int_0^T f_{i,j}(t)\sigma(t)dW(t) \sim N\left(0, \int_0^T f_{i,j}^2(t)\sigma^2(t)dt\right),$$

where $W(t)$ is the Wiener process and $f_{i,j}(t) = \left.\frac{\partial^2 f(t,\theta)}{\partial\theta_i\partial\theta_j}\right|_{\theta=\theta^*}$. Therefore, as $K \to +\infty$, we have $QBB^TQ^T = O(K)$ while A_θ is random and is of order $\sqrt{K}$. A bit of matrix algebra gives

$$w_{New} - w_{LS} = B^TQ^T\left[(QBB^TQ^T + A_\theta)^{-1} - (QBB^TQ^T)^{-1}\right]QB_0(0) =$$
$$= -B^TQ^T(QBB^TQ^T + A_\theta)^{-1}A_\theta(QBB^TQ^T)^{-1}QB_0(0),$$

so

$$\|w_{New} - w_{LS}\|^2 = B_0^T(0)Q^T(QBB^TQ^T)^{-1}A_\theta(QBB^TQ^T + A_\theta)^{-1}\cdot$$
$$\cdot QBB^TQ^T(QBB^TQ^T + A_\theta)^{-1}A_\theta(QBB^TQ^T)^{-1}QB_0(0) = O(K^{-2}),$$

while

$$\|w_{LS}\|^2 \quad = \quad B_0^T(0)Q^T(QBB^TQ^T)^{-1}QBB^TQ^T(QBB^TQ^T)^{-1}QB_0(0) \quad = \quad O(K^{-1}).$$

Therefore,

$$\frac{\|w_{New} - w_{LS}\|}{\|w_{LS}\|} = O(K^{-\frac{1}{2}}) \to 0.$$

Victor A. Lapshin

National Research University Higher School of Economics (Moscow, Russia). School of Finance, associate professor. Laboratory for Financial Engineering and Risk Management, director; E-mail: vlapshin@hse.ru

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