National Research University Higher School of Economics
International Laboratory of Dynamical Systems and Applications

as a manuscript

Barinova Marina Konstantinovna

CONSTRUCTION OF ENERGY FUNCTIONS FOR 2- AND 3-DIFFEOMORPHISMS WITH CHAOTIC DYNAMICS

Summary of the PhD thesis
for the purpose of obtaining academic degree
Doctor of Philosophy in Mathematics

Academic supervisor:
Doctor of Physical and Mathematical scientific,
Professor
Pochinka Olga Vitalievna

Nizhny Novgorod – 2021

An energy function for dynamical systems is a natural generalization of an energy function for dissipative physical systems. In the case of discrete dynamical systems, in contrast to continuous ones, such a function does not always exist, even for systems with regular dynamics. The dissertation is devoted to the study of the existence and construction (in the case of existence) of an energy function for Ω-stable diffeomorphisms with chaotic behavior due to the presence of non-trivial (other than a periodic orbit) basic sets.

For any dynamical system (flow or cascade) given on a metric space, one can introduce the concept of a chain-recurrent set associated with the concepts of a ε-trajectory or a pseudo-orbit. Since the dissertation deals with discrete dynamical systems on compact manifolds, we will only give the corresponding definitions; for flows, one can introduce similar ones. Let M be a smooth compact orientable n-manifold and f be a diffeomorphism on M. A *ε-chain of length n* connecting the point $x \in M$ with the point $y \in M$ for the cascade f is a sequence of points $x = x_0, \dots, x_n = y$ from M such that $d(f(x_i), x_{i+1}) < \varepsilon$ for $1 \leqslant i \leqslant n-1$ (see figure 1). A point $x \in M$ is called *chain-recurrent* if for any $\varepsilon > 0$ there is a number n and a ε-chain of length n connecting x with itself. The set of all chain-recurrent points of the cascade f is called *chain-recurrent set of f* and is denoted by $\mathcal{R}_f$. An equivalence relation can be introduced on the set $\mathcal{R}_f$ by the following rule: $x \sim y$, $x, y \in \mathcal{R}_f$, if for any $\varepsilon > 0$ there are ε-chains connecting x with y and y with x. Then the chain-recurrent set is divided into equivalence classes called *chain components* of the system.

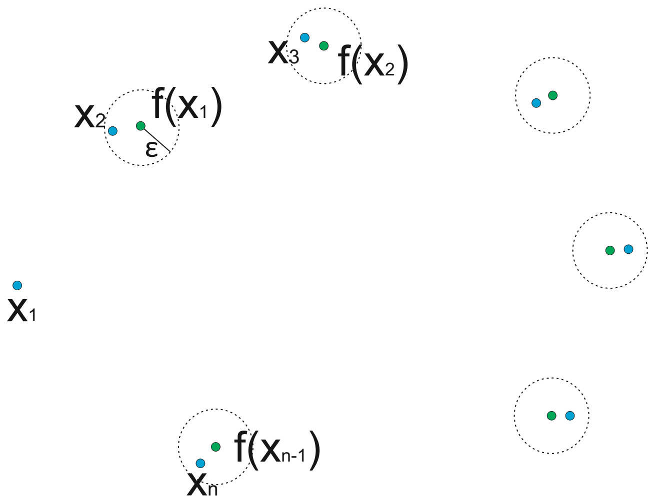

Figure 1: ε-chain

A Lyapunov function of a dynamical system (flow or cascade) given on M is a continuous function $\varphi : M \to \mathbb{R}$, which is constant on each chain component of the system and decreases along its orbits outside the chain recurrent set. By virtue of the results of C. Conley [6], such a function exists for any dynamical system, and the fact of its existence is called the "Fundamental Theorem of Dynamical Systems". It should be noted that C. Conley additionally demanded that the image of the chain recurrent set by virtue of φ is nowhere dense on the real number line, and the values of the function φ on different chain components of the chain recurrent set are different, and call such a function *a complete Lyapunov function*. The numbers belonging to the image of the chain recurrent set were called by C. Conley *critical values* of the function φ.

However, for a smooth function, its critical value is usually called the image of a critical point (a point at which the gradient of the function becomes 0), which do not belong to the chain recurrent set in general. Therefore the concept of an *energy function* is used, that is a Lyapunov function, whose set of critical points coincides with the chain recurrent set of the system. Note that in the continuous category it is also possible to introduce the concept of a critical point and define the energy function without requiring the smoothness of the Lyapunov function.

Dynamical systems with a hyperbolic chain-recurrent set are natural objects to study for the existence of an energy function. Recall that for a diffeomorphism $f : M \to M$ a compact f-invariant set $\Lambda \subset M$ is called *hyperbolic* if there exists a continuous Df-invariant decomposition of the tangent subbundle $T_\Lambda M$ into the direct sum $E^s_\Lambda \oplus E^u_\Lambda$, $x \in \Lambda$ such that

$$||Df^k(v)|| \leqslant c\lambda^k ||v||,\ v \in E^s_\Lambda,\ k > 0,$$

$$||Df^{-k}(v)|| \leqslant c\lambda^k ||v||,\ v \in E^u_\Lambda,\ k > 0$$

for some fixed $c > 0$ and $0 < \lambda < 1$. The presence of a hyperbolic structure on a chain-recurrent set is equivalent to the Ω-stability of the system, that is, such diffeomorphisms preserve the structure of a non-wandering set with small perturbations. In this case, the chain-recurrent set coincides with the non-wandering set of the system and the periodic orbits are dense in R_f. Thus, if a chain-recurrent set $\mathcal{R}_f$ of a diffeomorphism f is hyperbolic, then f is an A-diffeomorphism [1] and Smale Spectral Decomposition Theorem holds, namely: R_f has only a finite number of chain components, each of which is compact, invariant, and topologically transitive. In this case, they are called *basic sets* of the diffeomorphism f. If a basic set is a periodic orbit, then it is called *trivial*, otherwise is *nontrivial*. In a neighborhood of a hyperbolic isolated point of a chain-recurrent set, it is natural to construct the energy function in the form of a hypersurface of the second order; therefore, for classes with a finite chain-recurrent set, the problem of the existence of an energy function is usually solved in the class of Morse functions — C^2-smooth functions, all critical points of which are non-degenerate.

The first results on the construction of an energy function belong to S. Smale [33], who in 1961 proved the existence of a Morse energy function for an arbitrary gradient-like flow (structurally stable flows the chain-recurrent set of which consists of a finite number of fixed hyperbolic points). K. Meyer [27] in 1968 generalized this result by construction of a

[1] A diffeomorphism $f : M \to M$ given on a compact manifold M, is called an A-diffeomorphism if it satisfies Axiom A (C. Smale), that is, its non-wandering set $NW(f)$ is hyperbolic and periodic points are dense in $NW(f)$.

Morse-Bott energy function[2] for an arbitrary structurally stable flow, the chain-recurrent set of which consists of a finite number of fixed points and a finite number of periodic orbits.

As noted in 1985 by J. Franks [7], the application of the results of W. Wilson [36] to the construction of C. Conley gives the existence of a smooth energy function for any smooth flow with a hyperbolic chain-recurrent set. Then, using the suspension, one can construct a smooth Lyapunov function for any diffeomorphism with a hyperbolic chain recurrent set. But a function constructed in this way may have critical points that are not chain-recurrent and, therefore, the Lyapunov function is not energy. A natural question arises: what discrete dynamical systems admit energy functions? The first results in this direction were obtained by D. Pixton in 1977, in his work [30] he proved the existence of a Morse energy function for any Morse-Smale diffeomorphism on a surface. Pixton's result was generalized on Ω-stable 2-diffeomorphisms with a finite non-wandering set, the Morse energy function for such diffeomorphisms was constructed by T. Mitryakova, O. Pochinka, A. Shishenkova [28]. In the same paper [30] D. Pixton constructed a Morse-Smale diffeomorphism on a three-dimensional sphere, which does not possess a Morse energy function. In the works of V. Grines, F. Laudenbach, O. Pochinka [12], [13] and the book [15], necessary and sufficient conditions for the existence of a Morse energy function for three-dimensional Morse-Smale diffeomorphisms were obtained. There are also examples of Morse-Smale diffeomorphisms in dimension $n > 3$ which do not possess an Morse energy function (see, for example, [25]).

It follows from the results above that not all diffeomorphisms, even with regular dynamics, have an energy function. All the more surprising is the fact that some discrete dynamical systems with chaotic behavior have an energy function. In this paper, an energy function is constructed for some classes of Ω-stable 2- and 3-diffeomorphisms with non-trivial basic sets. Technically, the construction of such a function is based on the dynamic properties of basic sets and the smoothing procedure for a continuous map.

The work consists of eight chapters.

Chapter 1 is an overview of the results available on this topic.

In Chapter 2 main results of the dissertation are formulated.

In Chapter 3 a technical theorem about smoothing of a continuous function is proved, which is further used to construct smooth energy functions for the considered classes of diffeomorphisms.

Theorem 1 ([19]*[3], Lemma 2.1, [4]*, Lemma 5). *Let M^n be a smooth compact n-manifold, $K \subset M^n$ be a closed subset of M and U be some closed neighborhood of the set K such that $K \subset int\, U$. Let a continuous surjective function $\varphi : U \to [0;1]$ be C^1-smooth on $U \setminus K$ and $\varphi^{-1}(0) = K$. Then for any $\delta \in (0;1)$ there exists a C^1-smooth function $g : [0;1] \to [0;1]$ satisfying the following properties:*

[2]A C^2-smooth function is called a *Morse-Bott function* if the Hessian at each critical point is non-degenerate in the direction normal to the critical level set.

[3]Here and below, a star marks works in which one of the co-authors is a dissertation candidate and the results of which are presented in this dissertation

- $g'(0) = 0$ *and* $g'(c) > 0, \ \forall c \in (0; 1]$*;*
- $g(c) = c, \ \forall c \in [\delta; 1]$*;*
- $\psi = g \circ \varphi$ *is* C^1*-smooth on the whole set* U*.*

The idea of prof of Theorem 1 is based on the construction of the desired function g by the method of partitioning unity with the fulfillment of the conditions necessary for the differentiability of the composition $g \circ \varphi$.

In Chapter 4, the properties of non-trivial basic sets necessary for construction of energy functions are given. In addition, the class $S(M^2)$ of Ω-stable diffeomorphisms defined on a closed orientable surface M^2, all of whose non-trivial basic sets are attractors or repellers, is considered. The main result of the chapter is the following theorem.

Theorem 2 ([17]*, Theorem 1). *For any diffeomorphism* $f \in S(M^2)$*, there exists a smooth energy function that is a Morse function outside non-trivial basic sets.*

The proof of Theorem 2 essentially relies on the existence of a canonical support for one-dimensional basic sets of Ω-stable diffeomorphisms on surfaces. The ideas of construction such a support form the basis of the fundamental theory of surface basic sets, constructed in the works of V.Z. Grines [9, 10], A.Yu. Zhirov [37, 38, 39], R.V. Plykin [31]. This makes it possible to distinguish a trapping neighborhood for non-trivial basic sets in the form of a surface with a boundary, and each component of the boundary is a circle. Then the wandering part of the basin of each non-trivial attractor is foliated into circles, which makes it possible to make them the level lines of the future energy function inside some trapping neighborhood. Outside the trapping neighborhoods of the hyperbolic attractors and repellers, the diffeomorphism has a finite hyperbolic chain recurrent set. The construction of an energy function for regular components of a diffeomorphism is based on the existence of a Morse energy function for Morse-Smale diffeomorphisms on surfaces, proved by D. Pixton. The resulting energy function is a constant on one-dimensional attractors and repellers and is a Morse function on their complement. The smoothness of such a function is ensured by the technical Theorem 1.

If at least one zero-dimensional basic set appears among non-trivial basic sets of an Ω-stable diffeomorphism given on a closed orientable surface, then there is still no unambiguous answer to the question of the existence of an energy function.

In Chapter 5 a subclass of such diffeomorphisms is considered, namely, Ω-stable diffeomorphisms defined on a closed orientable surface M^2, whose non-wandering set contains at least one non-trivial zero-dimensional basic set without pairs of conjugated points (points x, y from some basic set Λ is called a *pair of conjugated points* if $W^s_x = W^s_y$, $W^u_x = W^u_y$ and open arcs of stable and unstable manifolds, bounded by points x and y do not contain points of the basic set Λ, the figure 2 shows a pair of conjugated points: at least one of the arcs (red or green) contains points of the basic set). As follows from the main results of this chapter presented below, the presence of such a basic set is an obstruction to the existence of an energy function for a diffeomorphism.

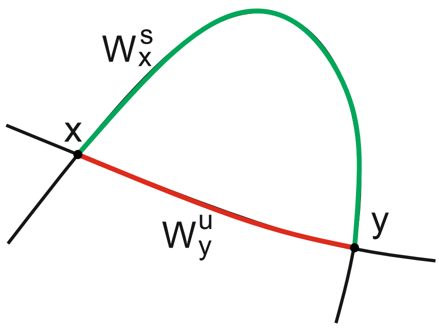

Figure 2: x, y is a pair of conjugated points

Theorem 3 ([1]*, Theorem 1). *Every Ω-stable diffeomorphism $f: M^2 \to M^2$ given on a closed orientable surface M^2, whose non-wandering set contains a zero-dimensional non-trivial basic set without pairs of conjugated points, does not possess an energy function.*

The proof of Theorem 3 is based on properties of zero-dimensional non-trivial basic sets without pairs of conjugated points. The idea of studying such sets using a universal covering by the Lobachevsky plane was developed in the works of V.Z. Grines and C. Kalai [9, 11, 23]. The absence of pairs of conjugated points in a zero-dimensional basic set allows one to single out disks whose interior consists of wandering points of the diffeomorphism, such that any Lyapunov function (even a non-smooth one) has critical points inside these disks, that is, it is not an energy function.

If a diffeomorphism $f: M \to M$ is an Ω-stable, then on the set of its basic sets one can introduce the S. Smale partial order relation as follows: $\Lambda_1 \prec \Lambda_2$, if $W^s_{\Lambda_1} \cap W^u_{\Lambda_2} \neq \emptyset$.

In Chapter 6 we obtain a partial solution to the Smale problem concerning the description of diagrams of (A, B)-diffeomorphisms (satisfying axioms A and B), constructed from S. Smale's partial order on the set of its basic sets. A Smale diagram is a special case of a Hasse diagram of a partially ordered set $(X, \prec)$ and is a graph whose vertices are elements of the set X, and the pair (x, y) forms an edge if $x \prec y$ and $\nexists z: \ x \prec z, \ z \prec y$. It was established in Lemma 6.1 that a Smale diagram of any Ω-stable diffeomorphism is a connected Hasse diagram. A Smale surgery is used to construct model diffeomorphisms of a two-dimensional torus. In Lemma 6.6, necessary and sufficient conditions for a topological conjugacy of model diffeomorphisms are obtained. Next, we introduce the class $\mathcal{H}$ of Ω-stable diffeomorphisms of surfaces, which are connected sums of model diffeomorphisms. The main result of this section is the following theorem.

Theorem 4 ([2]*, Theorem). *Any connected Hasse diagram can be realized by some diffeomorphism from the class $\mathcal{H}$.*

A labelled Smale diagram is the Smale diagram in which the topological conjugacy class of the restriction of the diffeomorphism to the corresponding basic set is additionally specified near each vertex. For diffeomorphisms $f,\ f' \in \mathcal{H}$, the isomorphism of their labelled Smale diagrams is a necessary and sufficient condition for their Ω-conjugacy. However, in general the conjugating homeomorphism does not extend from the basic sets to the ambient surface. In this paper, a subclass of $\mathcal{H}_* \subset \mathcal{H}$ diffeomorphisms is distinguished, in which any two model diffeomorphisms are connected in at most one orbit. For such diffeomorphisms, the isomorphism class of the labelled Smale diagram is a complete invariant of the ambient Ω-conjugacy.

Theorem 5 ([3]*, Theorem). *Diffeomorphisms $f, f' \in \mathcal{H}_*$ are ambient Ω-conjugate if and only if their labelled diagrams are isomorphic.*

In Chapter 7 Ω-stable 3-diffeomorphisms with non-trivial two-dimensional basic sets are discussed. If the topological dimension of the attractor (repeller) coincides with the dimension of the unstable (unstable) manifolds of its points, then it is called *expanding (contracting).* For the class $T(M^3)$ of structurally stable diffeomorphisms with two-dimensional expanding attractor or contracting repeller from the results of V.Z. Grines, E.V. Zhuzhomy and V.S. Medvedev [20, 26] knows that all other basic sets of such diffeomorphisms are trivial, a non-trivial basic set has only branches of degree two, and the ambient manifold is always homeomorphic to a three-dimensional torus. In addition, a non-trivial basic set is separated from a set with regular dynamics by a so-called characteristic sphere. This fact allows us to prove the tame embedding of saddle separatrices and construct a Morse energy function for the considered diffeomorphism outside the expanding attractor (contracting repeller), using the results of V.Z. Grines, F. Laudenbach and O.V. Pochinka [13] on the existence of a Morse energy function for Morse-Smale 3-diffeomorphisms. Theorem 1 allows us to smoothly extend the constructed function to a non-trivial basic set and, thus, to prove the following theorem.

Theorem 6 ([18]*, Theorem 1). *For any diffeomorphism $f \in T(M^3)$, there exists a smooth energy function that is a Morse function outside the non-trivial basic set.*

A similar result is obtained for the class of $Q(M^3)$ of Ω-stable 3-diffeomorphisms with two-dimensional surface basic sets.

Theorem 7 ([19]*, Theorem 1.1). *Any diffeomorphism $f \in Q(M^3)$ has a smooth energy function.*

The idea of the proof of the Theorem 7 is based on the fact that any basic set of the diffeomorphism under consideration is a torus tamely embedded into M^3, and the restriction of the diffeomorphism on each basic set is conjugated to an algebraic automorphism of a 2-torus. This fact, following from the works of V.Z. Grines, E.V. Zhuzhoma, V.S. Medvedev, Yu.A. Levchenko and O.V. Pochinka [14, 16, 21], allows to construct a smooth energy function on the wandering set of such a diffeomorphism. To continue the constructed function to the chain recurrent set, we use Theorem 1.

Chapter 8 discusses 3-diffeomorphisms with one-dimensional source-sink dynamics. In this case the attractor (the repeller) is automatically expanding (contracting). R. Williams [35] shows that the dynamics on such a basic set is conjugate to the shift on the reverse limit of a branched 1-manifold with respect to an expanding map. A construction of 3-diffeomorphisms with one-dimensional attractor-repeller dynamics firstly was suggested by J. Gibbons [8]. He construct many models on 3-sphere with Smale's solenoid basic sets and proves that all examples are not structurally stable. B. Jiang, Y. Ni and S. Wang [22] proved that a 3-manifold M^3 admits a diffeomorphism f whose non-wandering set consists of Smale's solenoid attractors and repellers if and only if M^3 is a lens space $L(p, q)$ with $p \neq 0$. They also shown that such f are not structural stable.

All generalizations of Smale's solenoid as the intersections of nested handlebodies are not surface. Moreover, all known examples of diffeomorphisms with the generalized solenoids as the attractor and the repeller are not structurally stable.

Note that such a dynamics on a surface is not structurally stable due to the results of R. Robinson and R. Williams [34]. A natural way to get a surface one-dimensional attractor for a 3-diffeomorphism f is to take an attractor A of some 2-diffeomorphism and multiply its trapping neithborhood by a contraction in transversal direction. In such a case A is called *canonically embedded surface attractor.*

In the present paper, we construct examples of 3-diffeomorphisms with canonically embedded surface one-dimensional attractor and repeller; namely, the following theorem is proved.

Theorem 8 ([4]*, Theorem 1). *There are infinitely many pairwise Ω-non-conjugated 3-diffeomorphisms whose non-wandering sets are pairwise homeomorphic, and each of them is a union of a canonically embedded one-dimensional surface attractor and repeller.*

The surface dynamics of the constructed diffeomorphisms and the result of Theorem 1 allow us to prove the existence of a smooth energy function for them.

Theorem 9 ([4]*, Theorem 2, [4]*, Theorem 1). *Each Ω-stable diffeomorphism on a closed orientable 3-manifold M^3, whose non-wandering set is a union of a connected canonically embedded one-dimensional surface attractor and repeller, has an energy function.*

C. Bonatti, N. Guelman [5] and Y. Shi [32] constructed structurally stable examples of 3-diffeomorphisms with one-dimensional attractor-repeller dynamics. But the embedding of basic sets in the ambient manifold is so non-trivial that it does not allow us to solve the problem of the existence of an energy function for such diffeomorphisms.

Conclusion. A significant part of this dissertation is devoted to the construction of energy functions for Ω-stable diffeomorphisms with chaotic dynamics defined on 2- and 3-manifolds. The main result of this work is a constructive proof of the existence of a smooth energy function for the following classes of diffeomorphisms:

- Ω-stable diffeomorphisms given on surfaces, all non-trivial basic sets of which are one-dimensional (Theorem 2);
- structurally stable 3-diffeomorphisms with two-dimensional expanding attractor or contracting repeller (Theorem 6);
- Ω-stable 3-diffeomorphisms with two-dimensional chain recurrent set (Theorem 7);

- Ω-stable 3-diffeomorphisms with dynamics one-dimensional canonically embedded surface attractor-repeller (Theorem 9).

The construction of a smooth energy function is essentially based on the dynamical properties of the considered diffeomorphisms and the technical

- theorem about smoothing a continuous function (Theorem 1)

In the class of Ω-stable 3-diffeomorphisms with dynamics one-dimensional canonically embedded surface attractor-repeller was constructed

- infinitely many pairwise Ω-non-conjugated 3-diffeomorphisms (Theorem 8).

In addition, the dissertation proved

- the fact that there is no energy function (even in a continuous category) for Ω-stable diffeomorphisms given on surfaces with a zero-dimensional non-trivial basic set without pairs of conjugated points (Theorem 3).

Also in this work the Smale problem is partially solved, concerning the description of the diagrams of Ω-stable diffeomorphisms constructed on the basis of the partial order of S. Smale on the set of its basic sets. Model diffeomorphisms on a two-dimensional torus were constructed with the help of Smale surgery and it was proved that

- any Smale diagram can be realized by an Ω-stable surface diffeomorphism, which is a connected sum of model diffeomorphisms (Theorem 4);

- a subclass of connected sums of model diffeomorphisms is distinguished for which the isomorphism class of the labeled Smale diagram is a complete invariant of the ambient Ω-conjugacy (Theorem 5).

Four articles published based on research results

- Barinova M. On Existence of an Energy Function for Ω-stable Surface Diffeomorphisms // Lobachevskii Journal of Mathematics. 2022. Vol. 43. No. 2. P. 257-263.

- Barinova M., Gogulina E., Pochinka O. Omega-classification of Surface Diffeomorphisms Realizing Smale Diagrams // Russian journal of non-linear dynamics. 2021. Vol. 17. No. 3. P. 321-334.

- Barinova M., Grines V., Pochinka O., Yu B. Existence of an energy function for three-dimensional chaotic "sink-source" cascades // Chaos. 2021. Vol. 31. No. 6. Article 063112. P. 1-8.

- Pochinka O., Grines V., Noskova (Barinova) M. The construction of an energy function for three-dimensional cascades with a two-dimensional expanding attractor // Transactions of the Moscow Mathematical Society. 2015. Vol. 76. No. 2. P. 271-286.

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