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as a manuscript

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Integrable hierarchies of nonlinear differentialequations and many-body systems

Summary of the PhD thesis for the purpose of obtaining academic degree Doctor of Philosophy in Mathematics

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Moscow – 2022

Introduction

One of the most astonishing quality characterising integrable systems is their nontrivial interconnections with each other. In particular, there is a connection between integrable spin chains, integrable hierarchies of nonlinear partial equations and classical many-bodies models.

In this thesis we study poles dynamics of singular solutions of integrable hierarchies of KP type and show that it is isomorphic to dynamics of particles in many-body integrable systems on the level of hierarchies. Such connection between two different types of integrable systems has been a long known conjecture. The connection between nonlinear integrable equations and many-body systems was first study in seminal paper (Airault et al. [1977]). After that in the works such as (Krichever [1978],Krichever [1980], Krichever and Zabrodin [1995]) it was established that for the first nontrivial times dynamics of poles correspond to the motion of particles in systems of Calogero-Moser type with standard Hamiltonians. After that in papers (Shiota [1994], Haine [2007], Zabrodin [2020]) such connection was extended to the level of whole hierarchies, however it was done only for rational or trigonometric solutions which are just a limits of the most general elliptic solutions.

In a series of the articles presented in this thesis authors extend a connection between integrable hierarchies and many-body systems of Calogero type for three different hierarchies such as KP, 2D Toda lattice and matrix KP up to the most general elliptic solutions. The main results of these paper is that authors establish a connection between spectral curves of elliptic many-body systems and Hamiltonians responsible for dynamics of poles in higher times of corresponding hierarchy. Besides that methods developed in these articles could be used to discover poles dynamics for singular solutions of other hierarchies.

My thesis presents the results of five articles in which I am one of co-authors. In these articles a connection between integrable hierarchies of nonlinear differential equations and integrable many-body systems was studied. These works contain most general results for KP 2d-Toda and matrix KP hierarchies.

Chapter 1

Historical remarks

1.1 Nonlinear differential hierarchies

One of the first discovered integrable equations is a famous Korteveg-de Vris equation (1.1). It was written by (Boussinesq [1877]) and rediscovered in (Korteveg, D.J. and de Vries, G. [1895]) as an attempt to find a mathematical description of solitary waves observed by Russel and described by him in (Russel [1844]).

$$4u_t - 12uu_x - u_{xxx} = 0 \tag{1.1}$$

However, the fact that this equation contains infinitely many conserved quantities $I_i = \int\limits_{-\infty}^{\infty} Q_i(x,t)\mathrm{d}x$ was proven only almost a century after in (Miura et al. [1968]). In this paper authors presented a general formula for Q_{2m+1}'s as a graded polynomials of u, u', u'', etc., where $u' \equiv u_x \equiv \partial u$:

$$\begin{aligned} &Q_{-1}[u] = u, && Q_1[u] = \frac{u^2}{2}, \\ &Q_3[u] = \frac{u^3}{3} - \frac{u_x^2}{12}, && Q_5[u] = \frac{u^4}{4} - \frac{uu_x^2}{4} + \frac{u_{xx}^2}{360}, \\ &\ldots\ldots\ldots\ldots && \ldots\ldots\ldots \end{aligned}$$

The same year in (Lax [1968]) it was discovered that (1.1) can be rewritten through two differential operators as

$$L_t = [A_3, L] = A_3 L - L A_3. \tag{1.2}$$

This form of equations now referred as Lax form.

In (1.2) L and A_3 are:

$$L = \partial_x^2 + u \tag{1.3}$$
$$A_3 = \partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x = \partial^3 + \frac{3}{4}u\partial_x + \frac{3}{4}\partial_x u \tag{1.4}$$

where in the last formula operator written in a skew-symmetric form for the standard scalar product $(f, g) = \int\limits_{-\infty}^{\infty} f(x)g(x)\mathrm{d}x$.

Equation (1.2) indicates that $L(t) = U(t)L(0)U^{-1}(t)$ where $U(t)$ is an unitary operator. It becomes clear, that $A_3 = U^\dagger U_t = -U_t^\dagger U$ is skew-symmetric.

Lax also considered a case of higher KdV equations as a generalization of such construction. He introduced general skew-symmetric operators

$$A_{2n+1} = \partial_x^{2n+1} + \sum_{i=1}^{n}(b_i\partial_x^{2i-1} + \partial_x^{2i-1}b_i) \tag{1.5}$$

and put them instead of A_3 into equation (1.2). The fact that $L_{t_{2n+1}} = [A_{2n+1}, L]$ is a function not differential operator imposes n conditions which uniquely determine n coefficients b_i's and equality itself determines a higher order KdV equation.

$$u_{t_{2n+1}} = K_{2n+1}(u). \tag{1.6}$$

Such set of infinite equations is called hierarchy.

Later in (Zakharov and Fadeev [1971]) it was shown that KdV equation have a Hamiltonian form:

$$u_t = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta I_3[u]}{\delta u(x)}. \tag{1.7}$$

Here skew-symmetrical operator $\frac{\mathrm{d}}{\mathrm{d}x}$ is infinite dimensional analogue of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ in the theory of classical Hamiltonian systems.

Moreover higher order KdV equations can be written also as

$$u_{t_n} = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\delta I_n[u]}{\delta u(x)}. \tag{1.8}$$

It proves that KdV equation can be viewed as an infinite dimensional analogue of classical integrable system from Hamiltonian mechanics.

After these observations it becomes ambiguous to somehow connect KdV equation with some known or unknown finite-dimensional integrable system. In seminal paper (Airault et al. [1977]) connection between class of elliptic solutions of KdV and so-called Calogero-Moser system was shown. Calogero-Moser system (1.17) describes dynamics of non-relativistic particles on complex line with pairwise interaction between every particle with each other (Calogero [1971],Calogero [1975]).

However dynamics of poles was described by special locus and it appears that more natural connection arise between 3-d generalization of KdV hierarchy – Kadomtsev–Petviashvili (or simply KP) hierarchy and Calogero Moser system. KP hierarchy like KdV hierarchy is generalization of nonlinear differential equation called KP equation.

$$3u_{yy} = (4u_t - 12uu_x - u_{xxx})_x \tag{1.9}$$

Kadomtsev-Petviashvili equation originates from (Kadomtsev and Petviashvili [1970]) in which authors derived the equation as a model to study the evolution of long ion-acoustic waves of small amplitude propagating in plasmas under the effect of long transverse perturbations. In the absence of transverse dynamics, this problem is described by the KdV equation. The KP equation was soon widely accepted as a natural extension of the classical KdV equation to two spatial dimensions.

In a paper (Dryoma [1974]) Lax representation of KP equation was found:

$$L_t = [A, L] \tag{1.10}$$

with $L = \partial_y + \partial_x^2 + 2u$ and $A = \partial_x^3 + 3u\partial_x + \int^x u_y \mathrm{d}x$.

However more natural way to describe KP equation was suggested in (Sato

[1983]), where author wrote down the whole hierarchy.

The main idea was to consider a pseudo-differential operator

$$\mathcal{L} = \partial + \sum_{m=1}^{\infty} u_m \partial^{-m} \tag{1.11}$$

where ∂ is ordinary differential operator acting on x with following standard commutation relation with function $\partial f = f' + f\partial$. Multiplying both sides of this equality by ∂^{-1} from left and from right gives $\partial^{-1} f = f\partial^{-1} - \partial^{-1} f' \partial^{-1}$. The multiple application of this rule yields:

$$\partial^{-n} f = \sum_{k\geq 0} (-1)^k \begin{pmatrix} k+n-1 \\ k \end{pmatrix} f^{(k)} \partial^{-n-k} \tag{1.12}$$

which is similar to the rule for usual derivative

$$\partial^n f = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} f^{(k)} \partial^{n-k}. \tag{1.13}$$

Equations of KP hierarchy are equivalent to compatibility condition of a system of Lax equations

$$\partial_{t_n} \mathcal{L} = [\mathcal{A}_n, \mathcal{L}]. \tag{1.14}$$

Where $\mathcal{A}_n$ is monic differential operators of order n. It is clear, that the only way equation (1.14) make sense if r.h.s is pseudo-differential operator with zero coefficients at positive powers of ∂. The easiest way to impose this condition is to take $\mathcal{A}_n$ as purely differential part of $\mathcal{L}^n$. It can be written using standard notation $\mathcal{A}_n = (\mathcal{L}^n)_+$. Indeed, since $[\mathcal{L}^n, \mathcal{L}] = 0$ $[\mathcal{A}_n, \mathcal{L}] = -[\mathcal{L}^n - \mathcal{A}_n, \mathcal{L}]$ and since $\mathcal{L}^n - \mathcal{A}_n$ has zero differential part it is clear that $[\mathcal{A}_n, \mathcal{L}]$ is also have zero differential part.

Following chain of equalities aims to show, that $\partial_{t_n}\partial_{t_m}\mathcal{L} - \partial_{t_m}\partial_{t_n}\mathcal{L} = 0$.

$$\partial_{t_n}\partial_{t_m}\mathcal{L} - \partial_{t_m}\partial_{t_n}\mathcal{L} = \partial_{t_n}[(\mathcal{L}^m)_+,\mathcal{L}] - \partial_{t_m}[(\mathcal{L}^n)_+,\mathcal{L}] =$$
$$[(\mathcal{L}^n)_+,\mathcal{L}^m]_+\mathcal{L} + (\mathcal{L}^m)_+[(\mathcal{L}^n)_+,\mathcal{L}] - \mathcal{L}[(\mathcal{L}^n)_+,\mathcal{L}^m]_+ - [(\mathcal{L}^n)_+,\mathcal{L}](\mathcal{L}^m)_+ - (n \leftrightarrow m) =$$
$$(\mathcal{L}^n)_+(\mathcal{L}^m)_+\mathcal{L} + [(\mathcal{L}^n)_+,(\mathcal{L}^m)_-]_+\mathcal{L} + \mathcal{L}(\mathcal{L}^m)_+(\mathcal{L}^n)_+ - \mathcal{L}[(\mathcal{L}^n)_+,(\mathcal{L}^m)_-]_+ - (n \leftrightarrow m)$$
$$= [(\mathcal{L}^n,\mathcal{L}^m]_+\mathcal{L} - \mathcal{L}[\mathcal{L}^n,\mathcal{L}^m]_+ = 0.$$

For example in case of $n = 1$ one have $\mathcal{A}_1 = \partial$ which means that $\partial_{t_1} = \partial_x = \partial$ and dependence on x can be restored $u(x, t_1, t_2, ...) = u(t_1 + x, t_2, ...)$.

KP equation is a compatibility condition for system with $n = 2, 3$ and it can be written in zero curvature form:

$$\partial_{t_3}\mathcal{A}_2 - \partial_{t_2}\mathcal{A}_3 + [\mathcal{A}_2, \mathcal{A}_3] = 0 \tag{1.15}$$

here t_2 identifies with y.

Higher KP equations are the same for two arbitrary higher times.

$$\partial_{t_n}\mathcal{A}_m - \partial_{t_m}\mathcal{A}_n + [\mathcal{A}_m, \mathcal{A}_n] = 0. \tag{1.16}$$

In series of works (Krichever [1978], Krichever [1980]) author showed that function $u = c + 2\sum\limits_{j=1}^{n}\wp(x - x_j(y,t))$ is solution of equation (1.9) if and only if dynamics of x_i with respect to y coincide with dynamics of elliptic Calogero-Moser system:

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - 2\sum_{i\neq j}\wp(x_i - x_j) \tag{1.17}$$

The dynamics of x_i with respect to $t = t_3$ coincide with Hamiltonian flow of the same system govern by Hamiltonian which is cubic in momenta.

In (1.17) $\wp(x)$ is Weierstrass p-function which can be viewed as averaging of x^{-2} on lattice:

$$\wp(x;\omega_1,\omega_2) = \frac{1}{x^2} + \sum_{(m,n)\neq(0,0)}\left(\frac{1}{(x + 2\omega_1 m + 2\omega_2 n)^2} - \frac{1}{(2\omega_1 m + 2\omega_2 n)^2}\right). \tag{1.18}$$

It is well known fact, that Weierstrass p-function degenerates into elementary functions when one or both ω's goes to infinity. In last case it is clear, that $\wp(x)$ becomes just x^{-2}. In case when just ω_1 goes to infinity, we put $\omega_2 = \frac{\pi i}{\gamma}$ and

$$\wp(x;\omega_1,\omega_2) \to \frac{\gamma^2}{\sinh^2(\gamma x)} + \frac{1}{3}\gamma^2. \tag{1.19}$$

These limits called rational and trigonometric (hyperbolic) limits of elliptic functions.

1.2 Many body systems

The other objects of study in this thesis is a classical many body systems integrable according to Liouville i.e. containing maximal number of independent integrals of motion. The first integrable many-body system was discovered by Toda in (Toda [1967a],Toda [1967b]). Having arbitrary number of particles on the line this model consider only interaction between neighbours. With Hamiltonian

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} + \sum_{i=1}^{n-1} e^{x_i - x_{i+1}} \tag{1.20}$$

and equations of motion

$$\ddot{x}_1 = e^{x_1 - x_2} \tag{1.21}$$

$$\ddot{x}_i = e^{x_i - x_{i+1}} - e^{x_{i-1} - x_i} \quad \text{for } 1 < i < n \tag{1.22}$$

$$\ddot{x}_n = -e^{x_{n-1} - x_n}. \tag{1.23}$$

After that in (Calogero [1971]) a system with interaction between every particles with each other was found. However author consider only quantum integrability of what will be refereed as Calogero system or rational limit of Calogero-Moser system.

$$H = \frac{1}{2}\sum_{i=1}^{n} p_i^2 - 2\sum_{i\neq j} \frac{1}{(}(x_i - x_j)^2) \tag{1.24}$$

Later in (Sutherland [1972]) more general system with potential $\sin^{-2}(x_i - x_j)$

was studied but still for quantum case.

The classical analogues of these systems were proven to be integrable in a works (Calogero and Marchioro [1974], Moser, J. [1974]). In last paper author showed, that equations of motion can be rewritten in Lax form i.e. system (1.24) can be rewritten as:

$$\dot{L} = [M, L] \tag{1.25}$$

where L and M are $n \times n$ matrices with following entries

$$L_{ij} = \delta_{ij} p_i + \frac{(1-\delta_{ij})}{x_i - x_j} \tag{1.26}$$

$$M_{ij} = -2\delta_{ij} \sum_{k \neq i} \frac{1}{(x_i - x_k)^2} + \frac{2(1-\delta_{ij})}{(x_i - x_j)^2} \tag{1.27}$$

in rational and

$$L_{ij} = \delta_{ij} p_i + (1-\delta_{ij}) \coth(\gamma(x_i - x_j)) \tag{1.28}$$

$$M_{ij} = 2\delta_{ij} \sum_{k \neq i} \frac{1}{\sinh^2(\gamma(x_i - x_k))} - \frac{2(1-\delta_{ij})}{\sinh^2(\gamma(x_i - x_j))} \tag{1.29}$$

in trigonometric (or rather hyperbolic) case.

Lax matrix L becomes an important object in studies of classical integrable systems. Equation (1.25) appears almost in every known integrable system with some important exception such as double elliptic system (Braden et al. [2000]), and system, which can be obtain from BKP hierarchies (Rudneva and Zabrodin [2020]). It was shown that not only $I_m = \mathrm{tr} L^m$ are conserved quantities, which is obvious from equation (1.25), but they also commute with each other, which makes first n of them integrals of motion.

Eventually elliptic generalization (1.17) was obtained in the work (Calogero [1975]). For elliptic case Lax representation remains true but both L and M matrices now depend on additional parameter λ which is not included in equations of motion.

$$L_{ij} = \delta_{ij} p_i + (1-\delta_{ij}) \Phi(x_i - x_j, \lambda) \tag{1.30}$$

$$M_{ij} = -2\delta_{ij} \sum_{k\neq i} \wp(x_i - x_k) - 2(1-\delta_{ij})\Phi'(x_i - x_j, \lambda) \tag{1.31}$$

here $\Phi(x,\lambda)$ is Lame function and $\Phi'(x,\lambda) = \partial_x \Phi(x,\lambda)$

$$\Phi(x,\lambda) = \frac{\sigma(x)\sigma(\lambda)}{\sigma(\lambda+x)} e^{-x\zeta(\lambda)} \tag{1.32}$$

$$\sigma(x) = \sigma(x;\omega_1,\omega_2) = x \prod_{s\neq 0} \left(1 - \frac{x}{s}\right) e^{\frac{x}{s}+\frac{x^2}{2s^2}}, \quad s = 2m\omega_1 + 2n\omega_2 \tag{1.33}$$

with integers m, n. $\zeta(x) = \partial_x \log(\sigma(x))$ and $\wp(x) = -\zeta'(x)$.

λ-dependence of Lax matrix in elliptic case becomes important for investigation of correspondence between many body systems and nonlinear differential hierarchy. In trigonometric and rational limits such dependence can be easily factorized

$$L^{tr(rat)} = L^{ell}(\lambda)\Big|_{ell\to tr(rat)} + (E-I)f^{tr(rat)}(\lambda) \tag{1.34}$$

with $f^{tr}(\lambda) = \gamma(\coth(\gamma\lambda) - 1)$ and $f^{rat}(\lambda) = \dfrac{1}{\lambda}$.

Here E is a matrix consists of only unities and I is an identity matrix.

Since $\mathrm{tr}L^m(\lambda)$ in elliptic case depends on λ it cannot be integral of motion. However in (d'Hoker and Phong [1998]) authors found out following expression for spectral curve:

$$\det(z + \zeta(\lambda) - L(\lambda)) = \frac{\sigma(\lambda - \partial_z)}{\sigma(\lambda)} I(z) \tag{1.35}$$

here $I(z)$ is a polynomial of degree n with some integrals of motion as coefficients.

$$I(k) = \sum_{m=0}^{n} I_n z^{n-m} \tag{1.36}$$

$$I_m = e_m(\boldsymbol{p}) + \sum_{l=1}^{[m/2]} \sum_{\substack{|S_i\cap S_j| = 2\delta_{ij} \\ 1\leq i,j\leq l}} e_{m-2l}(\boldsymbol{p}_{(\cup_{i=1}^l S_i)^c}) \prod_{i=1}^{l} \wp(S_i) \tag{1.37}$$

We are using following notation: $e_r(\boldsymbol{p})$ is elementary symmetric polynomial in variables $\{p_i | 1 \leq i \leq n\}$, $e_r(\boldsymbol{p}_S)$ is elementary symmetric polynomial in variables

$\{p_i | i \in S\}$, S^c is a complementary of set S. $\wp(S)$ where $S = \{i, j\}$ is set of power two is just $\wp(x_i - x_j)$. First few examples:

$$I_0 = 1$$
$$I_1 = \sum p_i$$
$$I_2 = \sum{}' \left(\frac{1}{2!} p_i p_j + \frac{1}{2!} \wp(x_i - x_j) \right)$$
$$I_3 = \sum{}' \left(\frac{1}{3!} p_i p_j p_k + \frac{1}{2!} p_i \wp(x_j - x_k) \right)$$
$$I_4 = \sum{}' \left(\frac{1}{4!} p_i p_j p_k p_l + \frac{1}{2! \cdot 2!} p_i p_j \wp(x_k - x_l) + \frac{1}{2 \cdot (2!)^2} \wp(x_i - x_j) \wp(x_k - x_l) \right)$$

where $\sum{}'$ is sum for all non-repeating indices. Coefficients are chosen the way that every unique term will have coefficient 1.

In (Shiota [1994]) it was shown, that in order for function $u(x, \boldsymbol{t}) = 2 \sum_{i=1}^{n} (x - x_i(\boldsymbol{t}))^{-2}$ to be a solution of the whole KP hierarchy (1.14), the dynamics of poles x_i with respect to t_m must be the same as a dynamics of particles in rational Calogero-Moser system w.r.t. Hamiltonian $I_m = \mathrm{tr} L^m$. Later in papers (Haine [2007], Zabrodin [2020]) this result was generalize to trigonometric case in which Hamiltonians responsible to higher times are $H_m = \frac{1}{2(m+1)\gamma} \mathrm{tr} \left((L + \gamma I)^{m+1} - (L - \gamma I)^{m+1} \right)$. Result for elliptic case was obtain in (Prokofev and Zabrodin [2021b]) and in this case $H_m = \operatorname*{res}_{z=0} (z^m \lambda(z))$ where $\lambda(z)$ is defined from equation $\det(z + \zeta(\lambda) - L(\lambda)) = 0$

Chapter 2

Tau function and bilinear equation

In paper (Prokofev and Zabrodin [2021b]) one of the crucial elements of the proof is to consider an integral bilinear form of KP hierarchy. In order to make this thesis more self-contained it can be useful to proof equivalents of two forms: integral form of KP hierarchy and a standard one as an infinite set of Lax equations. This section is devoted to proving that statement. Here we also introduce an important objects such as Baker-Akhiezer function and tau function.

The content of this section follows Chapters 5 and 6 of (Dickey [2003])

2.1 Baker-Akhiezer function

We will consider pseudo-differential operator for KP hierarchy:

$$\mathcal{L} = \partial + \sum_{m=0}^{\infty} u_m \partial^{-m}. \tag{2.1}$$

It can be viewed in a dressing form:

$$\mathcal{L} = \mathcal{W} \partial \mathcal{W}^{-1}, \tag{2.2}$$

where $\mathcal{W} = \sum_{i=0}^{\infty} w_i \partial^{-i}$ and $w_0 = 1$. It is clear, that all coefficients u_n can be expressed in terms of w_n.

Equations of hierarchy (1.14) can be extended to $\mathcal{W}$

$$\partial_{t_m}\mathcal{W} = -(\mathcal{L}^m)_-\mathcal{W}. \tag{2.3}$$

Here $\mathcal{A}_+$ is a purely differential part of operator $\mathcal{A}$ and $\mathcal{A}_- = \mathcal{A} - \mathcal{A}_+$.

Action of pseudo-differential operators is not defined on functions, however we will define their action on a function $\xi(t,z) = \sum_{k=1}^{\infty} t_k z^k$ following way: $\partial^m \xi(\boldsymbol{t},z) = \partial_{t_1}^m \xi(\boldsymbol{t},z) = z^m$ and $\partial^m \exp\xi(\boldsymbol{t},z) = z^m \exp\xi(\boldsymbol{t},z)$ for both positive and negative m.

Define Baker-Akhiezer function:

$$\psi(\boldsymbol{t},z) = \mathcal{W}e^{\xi(\boldsymbol{t},z)} = e^{\xi(\boldsymbol{t},z)}w(\boldsymbol{t},z) \tag{2.4}$$

with $w(\boldsymbol{t},z) = \sum_{i=0}^{\infty} w_i(\boldsymbol{t})z^{-i}$.

Introducing conjugation: $(f\partial)^\dagger = -\partial\cdot f = -(\partial f) - f\partial$ and let $\mathcal{W}^\dagger$ be a formal adjoint to $\mathcal{W}$ define adjoint Baker-Akhiezer function

$$\psi^*(\boldsymbol{t},z) = (\mathcal{W}^{-1})^\dagger e^{-\xi(\boldsymbol{t},z)} = e^{-\xi(\boldsymbol{t},z)}w^*(\boldsymbol{t},z). \tag{2.5}$$

These functions satisfy systems:

$$\begin{cases} \mathcal{L}\psi = z\psi \\ \mathcal{A}_n\psi = \partial_n\psi \end{cases} \qquad \begin{cases} \mathcal{L}\psi^* = z\psi^* \\ \mathcal{A}_n\psi^* = -\partial_n\psi^* \end{cases} \tag{2.6}$$

here and further we put $\partial_n = \partial_{t_n}$.

Equations (1.14) can be viewed as compatibility conditions of these systems.

It is typical for both finite and infinite dimensional integrable systems to be just a compatibility conditions of overdetermined systems such as (2.6). It is often useful to study Baker-Akhiezer function instead of infinite set of $\{u_n\}$ or $\{w_n\}$ since it is just one function and it is a solution of infinitely many linear problems.

For an infinite formal series $P(z) = \sum_{-\infty}^{\infty} p_k z^k$ and an infinite series of pseudo-differential operators $\mathcal{P} = \sum_{-\infty}^{\infty} p_k\partial^k$ define operations.

Definition 1. $\operatorname*{res}_z(P(z)) = p_{-1}$

Definition 2. $\operatorname*{res}_{\partial}(\mathcal{P}(z)) = p_{-1}$

These two operations connected with useful Lemma

Lemma 1. $\operatorname*{res}_{z}\left[(\mathcal{P}e^{xz}) \cdot (\mathcal{Q}e^{-xz})\right] = \operatorname*{res}_{\partial}(\mathcal{P}\mathcal{Q}^{\dagger})$

It can be proven by simple calculation.

With this lemma it becomes easy to proof following theorem

Theorem 1. *The identity*

$$\operatorname*{res}_{z}[(\partial_1^{i_1}...\partial_m^{i_m}\psi)\psi^*] = 0$$

holds for any $(i_1, ..., i_m)$ *with arbitrary* m *if and only if* ψ *and* ψ^* *of the form* $(1 + \sum\limits_{k>0} a_k z^{-k})e^{\pm\xi}$ *are solutions of (2.6).*

Before we will proof this theorem let us show that there is an another way to rewrite it. Indeed instead of $\operatorname*{res}_{z}[(\partial_1^{i_1}...\partial_m^{i_m}\psi(\boldsymbol{t}))\psi^*(\boldsymbol{t})]$ for any $(i_1, i_2, ..., i_m)$ we can write $\operatorname*{res}_{z}[\psi(\boldsymbol{t'})\psi^*(\boldsymbol{t})]$ for any $\boldsymbol{t}, \boldsymbol{t'}$ where $f(\boldsymbol{t'})$ should be understood as a formal expansion:

$$f(\boldsymbol{t'}) = \sum \frac{1}{i_1!...i_m!}(t_1' - t_1)^{i_1}....(t_m' - t_m)^{i_m}\partial_1^{i_1}...\partial_m^{i_m} f(\boldsymbol{t}).$$

This identity can be rewritten in integral form.

$$\oint\limits_{\infty} e^{\xi(\boldsymbol{t}-\boldsymbol{t'},z)} w(\boldsymbol{t'}, z) w^*(\boldsymbol{t}, z) \mathrm{d}z = 0. \tag{2.7}$$

The integration contour is a big circle around infinity separating the singularities coming from the exponential factor from those coming from the functions w and w^*

Proof. First we will prove that if ψ and ψ^* are solutions of (2.6), then

$$\operatorname*{res}_{z}[(\partial_1^{i_1}...\partial_m^{i_m}\psi)\psi^*] = 0.$$

Since $\partial_s \psi = \mathcal{A}_s \psi$ we need a proof only for $m = 1$.

$$\operatorname*{res}_z[(\partial^i \psi)\psi^*] = \operatorname*{res}_z[(\partial^i \mathcal{W} e^{xz})(\mathcal{W}^\dagger)^{-1} e^{-xz}] =$$
$$= \operatorname*{res}_\partial[(\partial^i \mathcal{W})\mathcal{W}^{-1}] = \operatorname*{res}_\partial(\partial^i) = 0.$$

It completes the first half of the proof.

To prove the converse statement we will consider $\operatorname*{res}_z[(\partial^i w(\boldsymbol{t}, z) w^*(\boldsymbol{t}, z)] = 0$ with $\psi(z) = e^{\xi(\boldsymbol{t},z)} \sum_{i=0}^{\infty} w_i z^{-i}$ and $\psi^*(z) = e^{-\xi(\boldsymbol{t},z)} \sum_{i=0}^{\infty} w_i^* z^{-i}$. Define $\mathcal{W} = \sum_{i=0}^{\infty} w_i \partial^{-i}$ and $\mathcal{W}^* = \sum_{i=0}^{\infty} (-1)^i w_i^* \partial^{-i}$.

Using assumption one can show, that

$$0 = \operatorname*{res}_z[(\partial^i \psi)\psi^*] = \operatorname*{res}_z[(\partial^i \mathcal{W} e^{\xi})\mathcal{W}^* e^{-\xi}] = \operatorname*{res}_\partial[(\partial^i \mathcal{W})(\mathcal{W}^*)^\dagger] = \operatorname*{res}_\partial[\partial^i \mathcal{W}(\mathcal{W}^*)^\dagger].$$

It is true for every i, so if we define purely negative pseudo-different operator $\mathcal{X} = \mathcal{X}_-$ as $\mathcal{W}(\mathcal{W}^*)^\dagger = 1 + \mathcal{X}$, proven equations mean, that $\mathcal{X} = 0$ and $\mathcal{W}^* = (\mathcal{W}^\dagger)^{-1}$.

Define $\mathcal{L} = \mathcal{W}\partial\mathcal{W}^{-1}$ for which we have

$$(\partial_m \mathcal{W} + (\mathcal{L}^m)_- \mathcal{W})e^{\xi} = (\partial_m \cdot \mathcal{W} - \mathcal{W}\partial_m + (\mathcal{L}^m)_- \mathcal{W})e^{\xi} =$$
$$= (\partial_m \cdot \mathcal{W} - \mathcal{L}^m \mathcal{W} + (\mathcal{L}^m)_- \mathcal{W})e^{\xi} = (\partial_m - (\mathcal{L}^m)_+)\mathcal{W} e^{\xi}.$$

From our assumption we know, that

$$0 = \operatorname*{res}_z[(\partial^i (\partial_m - (\mathcal{L}^m)_+)\psi)\psi^*] = \operatorname*{res}_z[(\partial^i (\partial_m - (\mathcal{L}^m)_+)\mathcal{W} e^{\xi})((\mathcal{W}^\dagger)^{-1} e^{-\xi})] =$$
$$\operatorname*{res}_z[(\partial^i (\partial_m \mathcal{W} + (\mathcal{L}^m)_- \mathcal{W})e^{\xi})((\mathcal{W}^\dagger)^{-1} e^{-\xi})] = \operatorname*{res}_\partial[(\partial^i (\partial_m \mathcal{W} + (\mathcal{L}^m)_- \mathcal{W})(\mathcal{W})^{-1})]$$

This yields $\partial_m \mathcal{W} + (\mathcal{L}^m)_- \mathcal{W} = 0$ which is nothing but equation of KP hierarchy.

□

2.2 Tau function

In the last section it was shown, that the whole KP hierarchy can be rewritten as an integral equation (2.7). However it is possible to simplify it by factorizing z-dependence. In order to do so we will use following easy to prove lemma:

Lemma 2. *If* $f(z) = \sum_{i=0}^{\infty} a_i z^{-i}$ *is a formal series where* $a_0 = 1$ *then*

$$\operatorname*{res}_z f(z)(1 - z/\zeta)^{-1} = \zeta(f(\zeta) - 1).$$

More general if $g(z,\zeta) = \sum_{i=-\infty}^{\infty} b_i(\zeta) z^{-i}$ *then*

$$\operatorname*{res}_z [(1 - z/\zeta)^{-1}] g(z) = \zeta g_-(\zeta, z)|_{z=\zeta}$$

where $g_-(z,\zeta) = \sum_{i=1}^{\infty} b_i(\zeta) z^{-i}$.

Here $(1 - z/\zeta)^{-1}$ is understood as series in ζ^{-1}.

Let $D(\zeta)$ be an operator acting on series in z^{-1} with coefficients depending on $\boldsymbol{t}$ as

$$D(\zeta) f(\boldsymbol{t}, z) = f(\boldsymbol{t} - [\zeta^{-1}], z). \tag{2.8}$$

Here $[\zeta^{-1}] = (\zeta^{-1}, \zeta^{-2}/2, \zeta^{-3}/3, ...)$.

Lemma 3. *Following identities hold:*

$$w^{-1}(\boldsymbol{t}, z) = D(z) w^*(\boldsymbol{t}, z)$$

and

$$\partial \log w(\boldsymbol{t}, z) = (-D(z) + 1) w_1(\boldsymbol{t}).$$

Proof. Equation $\operatorname*{res}_z [\psi(\boldsymbol{t})\psi^*(\boldsymbol{t'})] = 0$ with $\boldsymbol{t'} = \boldsymbol{t} - [\zeta]^{-1}$ and identity

$$\exp \sum_{k=1}^{\infty} \frac{z^k}{k\zeta^k} = (1 - z/\zeta)^{-1}$$

results in

$$\operatorname*{res}_z[w(\boldsymbol{t})D(\zeta)w^*(\boldsymbol{t})(1-z/\zeta)^{-1}]=0.$$

First part of the lemma 2 allows one to transform it into

$$\zeta(w(\boldsymbol{t},\zeta)D(\zeta)w^*(\boldsymbol{t},\zeta)-1)=0.$$

Which immediately gives us first equation.

Similarly

$$0=\operatorname*{res}_z[\partial\psi(z)D(\zeta)\psi^*(z)]=\operatorname*{res}_z[(\partial w(z)+zw(z))(D(\zeta)w^*(z))(1-z/\zeta)^{-1}].$$

Second part of lemma 2 implies

$$\begin{aligned}0=[(\partial w(z)+zw(x))D(\zeta)w^*(z)]_-|_{z=\zeta}=(\partial w(\zeta)+\zeta w(\zeta))D(\zeta)w^*(\zeta)-\\ -\zeta-w_1+D(\zeta)w_1=(\partial w(\zeta))w^{-1}(\zeta)-(1-D(\zeta))w_1\end{aligned}$$

which results in second equality. □

We have shown that derivative of $w(\boldsymbol{t},z)$ with respect to t_1 can be expressed in terms of one function which is not depending on z and the whole z dependence can be hidden inside shift of an arguments. It is remarkable discovery that $w(\boldsymbol{t},z)$ itself can be expressed that way.

Theorem 2. *There is a function $\tau(\boldsymbol{t})$ such that*

$$\log w(\boldsymbol{t},z)=(D(z)-1)\log\tau(\boldsymbol{t})$$

or, in more detail

$$w(\boldsymbol{t},z)=\frac{\tau(\boldsymbol{t}-[z^{-1}])}{\tau(\boldsymbol{t})}. \tag{2.9}$$

It is clear, that since solution of (2.6) can be multiplied by any function depending on z, τ-function also determined up to $c\exp\sum_{i=1}^{\infty}c_it_i$ with $c,c_1,c_2,\ldots$ arbitrary constants.

Proof. We will consider operator $N(z) = \partial_z - \sum_{j=1}^{\infty} z^{-j-1}\partial_j$ which annihilates all functions of the form $D(z)f(\boldsymbol{t})$ moreover for functions $f = \sum_{i=0}^{\infty} f_i z^{-i-1}$ $N(z)f = 0$ implies $f = 0$.

Applying $N(z)$ to $\log w(\boldsymbol{t}, z) = (D(z) - 1)\log \tau(\boldsymbol{t})$ we obtain series of equalities:

$$a_i = \partial_i \log \tau = \operatorname*{res}_z z^i \left(-\sum_{j=1}^{\infty} z^{-j-1}\partial_j + \partial_z \right) \log w.$$

In order for this system to be compatible and agreed with last equation from lemma 3 we need $\partial a_i = -\partial_i w_1$ and $\partial_j a_i = \partial_i a_j$.

First part can be verified by simple calculation. It results in $\partial(\partial_j a_i - \partial_i a_j) = 0$ however $(\partial_j a_i - \partial_i a_j)$ itself should be differential polynomial of w_i which are independent functions. The only way it derivative can be zero is for him be only a constant term. But simple calculation, for all $w_i = 0$ shows that constant term is absent.

□

Equality (2.9) together with first part of lemma 3 gives us following representation of $w^*(\boldsymbol{t}, z)$:

$$w^*(\boldsymbol{t}, z) = \frac{\tau(\boldsymbol{t} + [z^{-1}])}{\tau(\boldsymbol{t})} \tag{2.10}$$

Eventually we can rewrite (2.7) as integral equation on function $\tau(\boldsymbol{t})$ in the form also known as bilinear relation for tau-function.

$$\oint_{\infty} e^{\xi(\boldsymbol{t}-\boldsymbol{t}', z)} \tau(\boldsymbol{t} - [z^{-1}])\tau(\boldsymbol{t} + [z^{-1}]) \mathrm{d}z = 0 \tag{2.11}$$

Chapter 3

Further Generalizations

In this section there will be shown ways to generalize KP hierarchy to 2d Toda hierarchy and matrix KP for both these cases pole dynamics of singular solutions was obtained in Appendices **??** and **??** for trigonometric solutions and in Appendices **??** and **??** these results are generalized to elliptic case.

3.1 Modified KP

Content of this section follows Chapter 13 of (Dickey [2003]).

We start with $\mathcal{L}$ pseudo-differential operator of KP hierarchy, then add infinitely many functions v_i for $i \in \mathbb{Z}$ and determine

$$\mathcal{L}_i = (\partial + v_{i-1})...(\partial + v_0)\mathcal{L}(\partial + v_0)^{-1}...(\partial + v_{i-1})^{-1} \quad \text{for } i > 0 \tag{3.1}$$

$$\mathcal{L}_{-i} = (\partial + v_{-i})^{-1}...(\partial + v_{-1})^{-1}\mathcal{L}(\partial + v_{-1})...(\partial + v_{-i}) \quad \text{for } i > 0 \tag{3.2}$$

$$\mathcal{L}_0 = \mathcal{L}. \tag{3.3}$$

This way we have evident recursion:

$$\mathcal{L}_{i+1}(\partial + v_i) = (\partial + v_i)\mathcal{L}_i. \tag{3.4}$$

Determine dynamics of v_i with respect to t_k following way.

$$\partial_k v_i = (\mathcal{L}_{i+1}^k)_+(\partial + v_i) - (\partial + v_i)(\mathcal{L}_i^k)_+. \tag{3.5}$$

In this case it is easy to show, that

$$\partial_k \mathcal{L}_i = [(\mathcal{L}_i^k)_+, \mathcal{L}_i]. \tag{3.6}$$

One can introduce dressing operators for each $\mathcal{L}_i$

$$\mathcal{W}_i \mathcal{L}_i \mathcal{W}_i^{-1} = \partial \tag{3.7}$$

where $\mathcal{W}_i = \sum\limits_{\alpha=0}^{\infty} w_{i\alpha}\partial^{-\alpha}$, with $w_{i0} = 1$. It is clear that

$$(\partial + v_i)\mathcal{W}_i = \mathcal{W}_{i+1} \cdot \partial. \tag{3.8}$$

Taking similar approach as in section 2.1 we introduce Baker-Akhiezer functions

$$\psi_i(\boldsymbol{t}, z) = \mathcal{W}_i e^{\xi}$$
$$\mathcal{L}_i \psi_i = z\psi_i, \quad (\partial + v_i)\psi_i = z\psi_{i+1}$$

and adjoint Baker-Akhiezer functions

$$\psi_i^*(\boldsymbol{t}, z) = (\mathcal{W}_i^\dagger)^{-1} e^{-\xi}$$
$$\mathcal{L}_i^\dagger \psi_i^* = z\psi_i^*, \quad (\partial - v_i)\psi_{i+1}^* = -z\psi_i^*.$$

Analogically to Lemma 3 we have

Lemma 4. *For two formal series*

$$\psi_i = \sum_\alpha w_{i\alpha} z^{-\alpha} e^{\xi}, \quad \psi^* = \sum_\alpha w_{i\alpha}^* z^{-\alpha} e^{\xi}$$

with $w_{i0} = w_{i0}^* = 1$ *the following two statements are met simultaneously.*

1) ψ_i *and* ψ_i^* *are Baker Akhiezer functions of mKP hierarchy.*

2) $\operatorname*{res}_z[z^{i-j}(\partial_1^{k_1}...\partial_m^{k_m}\psi_i)\psi_j^*] = 0$ *for* $i \geq j$ *and any* $(k_1, ..., k_m)$.

Proof. First we will show, that if condition 1 is satisfied, then condition 2 is also satisfied.

Like in KP case we need to consider only $(k, 0, 0, ..., 0)$ since $\partial_s\psi_i = (\mathcal{L}_i^s)_+\psi_i$.

$$\operatorname*{res}_z[z^{i-j}(\partial^k\psi_i)\psi_j^*] = \operatorname*{res}_z[(\partial^k\mathcal{W}_i\partial^{i-j}e^\xi)((\mathcal{W}_j^\dagger)^{-1}e^{-\xi})] = \operatorname*{res}_\partial[(\partial^k\mathcal{W}_i\partial^{i-j})\mathcal{W}_j^{-1}] =$$
$$= \operatorname*{res}_\partial[(\partial^k(\partial + v_{i-1})...(\partial + v_j)\mathcal{W}_j\mathcal{W}_j^{-1}] = 0.$$

Now we will proof reverse statement. For $i = j$ we have a case of KP hierarchy and it already has been proven, that if $\operatorname*{res}_z[(\partial_1^{k_1}...\partial_m^{k_m}\psi_i)\psi_i^*] = 0$, then ψ_i and ψ_i^* are Baker-Akhiezer functions and adjoint BA functions of KP hierarchies with $\mathcal{L}_i$ operators. We are left to prove, that these operators connected through equation (3.4). In order to do so we will consider the case $i = j + 1$ and $(k, 0, 0, ..., 0)$.

$$0 = \operatorname*{res}_z[z(\partial^k\psi_{j+1})\psi_j^*] = \operatorname*{res}_z[(\partial^k\mathcal{W}_{j+1}\partial e^\xi)((\mathcal{W}_j^\dagger)^{-1}e^{-\xi})] = \operatorname*{res}_\partial[\partial^k\mathcal{W}_{j+1}\partial\mathcal{W}_j^{-1}].$$

which means, that $\mathcal{W}_{j+1}\partial\mathcal{W}_j^{-1}$ is purely differential first order monic operator and we can put $\mathcal{W}_{j+1}\partial\mathcal{W}_j^{-1} = \partial + v_j$. From that equation (3.4) follows immediately.

□

Since every ψ_i is solution of KP Theorem 2 combined with proven Lemma means that whole mKP hierarchy is equivalent to series of bilinear equations:

$$\oint_\infty z^{n-m}e^{\xi(\boldsymbol{t}-\boldsymbol{t'},z)}\tau_n(\boldsymbol{t} - [z^{-1}])\tau_m(\boldsymbol{t} + [z]^{-1})\mathrm{d}z = 0 \tag{3.9}$$

for $n \geq m$.

But for our purposes it will be convenient to take a different look at mKP hierarchy as a half of more general 2d Toda hierarchy.

3.2 $\mathfrak{gl}((\infty))$ algebra and 2d Toda hierarchy

The content of this section is based on (Ueno and Takasaki [1984]).

We will consider formal Lie algebra $\mathfrak{gl}((\infty))$

Let Λ^j be a j-th shift matrix $\Lambda^j = (\delta_{\mu+j,\nu})_{\mu,\nu\in\mathbb{Z}}$ and E_{ij} be the (i,j)-matrix unit $E_{ij} = (\delta_{\mu i}\delta_{\nu j})_{\mu,\nu\in\mathbb{Z}}$. Let $\mathfrak{gl}((\infty))$ be a formal Lie algebra consisting of all $\mathbb{Z}\times\mathbb{Z}$ matrices

$$\mathfrak{gl}((\infty)) = \left\{\sum_{i,j\in\mathbb{Z}} a_{ij}E_{ij}|a_{ij}\in\mathbb{C}\right\}. \tag{3.10}$$

A matrix $A\in\mathfrak{gl}((\infty))$ is written in a form

$$A = \sum_{j\in\mathbb{Z}} \mathrm{diag}[a_j(s)]\Lambda^j \tag{3.11}$$

here $\mathrm{diag}[a_j(s)]$ denotes a diagonal matrix $\mathrm{diag}(..., a_j(-1), a_j(0).a_j(1), ...)$ we can define a positive/negative part of matrix of matrix $A:$ $(A)_+ = \sum_{j\geq 0}\mathrm{diag}[a_j(s)]\Lambda^j$ and $(A)_- = \sum_{j<0}\mathrm{diag}[a_j(s)]\Lambda^j$.

If $a_j(s) = 0$ for all $j > m$ we call A is order less than m. If $a_j(s) = 0$ for all $j < m$ we call A is order grater than m. If matrices A and B both less or larger than some m, then product of AB is well-defined.

There is natural correspondence between matrix A and difference operator

$$\mathcal{A}(x) = \sum_{j\in\mathbb{Z}} a_j(x)e^{j\eta\partial_x} \tag{3.12}$$

where operator $e^{j\partial_s}$ define by it action $e^{j\eta\partial_s}f(x) = f(x+j\eta)$.

Definition 3. *Set two copies of time flows $\boldsymbol{t}_+$ and $\boldsymbol{t}_-$. Let $L, \bar{L}$ and M_n for $n\in\mathbb{Z}/\{0\}$ be elements of $\mathfrak{gl}((\infty))$ where*

$$L = \sum_{j\leq 1}\mathrm{diag}[l_j(s)]\Lambda^j \quad \textit{with } l_1(s) = 1 \textit{ for any } s \tag{3.13}$$

$$\bar{L} = \sum_{-1\leq j}\mathrm{diag}[\bar{l}_j(s)]\Lambda^j \quad \textit{with } \bar{l}_{-1}(s) \neq 0 \textit{ for any } s \tag{3.14}$$

$$M_{n>0} = (L^n)_+ \qquad B_{n<0} = (\bar{L}^{-n})_-. \tag{3.15}$$

The Toda lattice hierarchy is a system of equations

$$\partial_n L = [M_n, L] \quad \partial_n \bar{L} = [M_n, \bar{L}]. \tag{3.16}$$

It can be proven the same way it was proven for KP hierarchy that second derivatives commute. In a case of both m and n positive or negative the proof is the same as in KP case. Following chain of equalities will proof it for times from different flows:

$$\begin{aligned}\partial_n\partial_{-m}L - \partial_{-m}\partial_n L = \partial_n[\bar{L}^m_-, L] - \partial_{-m}[L^n_+, L] = \\ [L^n_+, \bar{L}^m]_- L + \bar{L}^m_-[L^n_+, L] - L[L^n_+, \bar{L}^m]_- - [L^n_+, L]\bar{L}^m_- - \\ -[\bar{L}^m_-, L^n]_+ L - L^n_+[\bar{L}^m_-, L] + L[\bar{L}^m_-, L^n]_+ + [\bar{L}^m_-, L]L^n_+ \\ = [L^n_+, \bar{L}^m]L + (\bar{L}^m_- L^n_+ - L^n_+\bar{L}^m_-)L - L[L^n_+, \bar{L}^m] + L(L^n_+\bar{L}^m_- - \bar{L}^m_- L^n_+) = 0.\end{aligned}$$

Now we will proof some lemmas which help us to obtain the whole Toda lattice hierarchy in bilinear form:

Lemma 5. *There are exist two matrices W and $\bar{W}$ of form*

$$W = \sum_{j=0}^{\infty} \mathrm{diag}[w_j(s)]\Lambda^{-j} \tag{3.17}$$

$$\bar{W} = \sum_{j=0}^{\infty} \mathrm{diag}[\bar{w}_j(s)]\Lambda^{j} \tag{3.18}$$

with $w_0(s) = 1$ and $\bar{w}_0(s) \neq 0$ for any s such that $L = W\Lambda W^{-1}$, $\bar{L} = \bar{W}\bar{\Lambda}^{-1}\bar{W}^{-1}$ and they satisfy equations

$$\partial_n W = -L^n_- W \quad \partial_{-n} W = \bar{L}^n_- W \quad n > 0 \tag{3.19}$$

$$\partial_n \bar{W} = L^n_+ \bar{W} \quad \partial_{-n} W = -\bar{L}^n_+ \bar{W} \quad n > 0. \tag{3.20}$$

Moreover they both defined up to arbitrariness

$$W \to WF^-(\Lambda) \quad \bar{W} \to \bar{W}F^+(\Lambda) \tag{3.21}$$

where $F^{\pm}(\Lambda) = \sum\limits_{j\geq 0} f_j^{\pm}\Lambda^{\pm j}$.

Proof. First of all simple calculation shows that system (3.19)-(3.20) is compatible.

It is clear that there are exist a some constant matrices W_0 and $\bar{W}_0$ of forms (3.17) and (3.18) respectfully such that

$$L = W_0\Lambda W_0^{-1} \quad \bar{L} = \bar{W}_0\bar{\Lambda}\bar{W}_0^{-1}.$$

We can consider Cauchy problem for (3.19) and (3.20) with initial conditions W_0, $\bar{W}_0$.

Straightforward calculations show, that $LW - WL$ and $\bar{L}\bar{W} - \bar{W}\bar{L}$ are both solutions of the same systems with zero initial conditions, so the uniqueness of solution obliges them be a null solutions, which means, that we have constructed W and $\bar{W}$ from lemma. □

Using that lemma it is easy to show, that matrices $\Psi = We^{\xi(\boldsymbol{t}_+,\Lambda)}$ and $\bar{\Psi} = \bar{W}e^{\xi(\boldsymbol{t}_-,\Lambda^{-1})}$ are solutions of following linear problems:

$$L\Psi = \Psi\Lambda, \quad \partial_n\Psi = M_n\Psi \tag{3.22}$$

$$\bar{L}\bar{\Psi} = \bar{\Psi}\Lambda^{-1}, \quad \partial_n\bar{\Psi} = M_n\bar{\Psi}. \tag{3.23}$$

$L\Psi = LWe^{\xi(\boldsymbol{t}_+,\Lambda)} = W\Lambda e^{\xi(\boldsymbol{t}_+.\Lambda)} = \Psi\Lambda$ and for $n > 0$ $\partial_n\Psi = -L_-^n\Psi + \Psi\Lambda^n = (-L_-^n + L^n)\Psi = M_n\Psi$ the rest calculations are similar.

Now we have proven, that

$$M_n = (\partial_n\Psi)\Psi^{-1} = (\partial_n\bar{\Psi})\bar{\Psi}^{-1} \tag{3.24}$$

or even

$$(\partial_{i_k}^{n_k}...\partial_{i_1}^{n_1}\Psi)\Psi^{-1} = (\partial_{i_k}^{n_k}...\partial_{i_1}^{n_1}\bar{\Psi})\bar{\Psi}^{-1} \tag{3.25}$$

for any $(i_1, ...i_k) \in (\mathbb{Z}/\{0\})^k$ and $(n_1, ...n_k) \in (\mathbb{N}^*)^k$.

It can be written as

$$\Psi(\boldsymbol{t}_+, \boldsymbol{t}_-)\Psi^{-1}(\boldsymbol{t}'_+, \boldsymbol{t}'_-) = \bar{\Psi}(\boldsymbol{t}_+, \boldsymbol{t}_-)\bar{\Psi}^{-1}(\boldsymbol{t}'_+, \boldsymbol{t}'_-) \tag{3.26}$$

for any $s, s', \boldsymbol{t}_+.\boldsymbol{t}_-, \boldsymbol{t'}_-, \boldsymbol{t'}_+$,

This equation resembles similar ones for KP and mKP case. It can be proven similarly to KP and mKP cases that equation (3.26) defines the whole hierarchy. Resemblance with mKP and KP hierarchies can be continued further. We have

$$W = \sum_{j=0}^{\infty} \mathrm{diag}[w_j(s)]\Lambda^{-j}, \quad W^{-1} = \sum_{j=0}^{\infty} \Lambda^{-j}\mathrm{diag}[w_j^*(s+1)],$$
$$\bar{W} = \sum_{j=0}^{\infty} \mathrm{diag}[\bar{w}_j(s)]\Lambda^{j}, \quad \bar{W}^{-1} = \sum_{j=0}^{\infty} \Lambda^{j}\mathrm{diag}[\bar{w}_j^*(s+1)].$$

And define

$$\psi(s,z) = \sum_{j=0}^{\infty} w_j(s) z^{s-j} e^{\xi(\boldsymbol{t}_+,z)}, \quad \psi^*(s,z) = \sum_{j=0}^{\infty} w_j^*(s) z^{-j-s} e^{-\xi(\boldsymbol{t}_+,z)},$$
$$\bar{\psi}(s,z) = \sum_{j=0}^{\infty} \bar{w}_j(s) z^{j+s} e^{\xi(\boldsymbol{t}_-,z^{-1})}, \quad \bar{\psi}^*(s,z) = \sum_{j=0}^{\infty} \bar{w}_j^*(s) z^{j-s} e^{-\xi(\boldsymbol{t}_-,z^{-1})}.$$

After that equation (3.26) can be rewritten as

$$\oint_{\infty} \psi(s,z;\boldsymbol{t}_+,\boldsymbol{t}_-)\psi^*(s',z;\boldsymbol{t'}_+,\boldsymbol{t'}_-)\frac{\mathrm{d}z}{2\pi i} = \oint_{0} \bar{\psi}(s,z;\boldsymbol{t}_+,\boldsymbol{t}_-)\bar{\psi}^*(s',z;\boldsymbol{t'}_+,\boldsymbol{t'}_-)\frac{\mathrm{d}z}{2\pi i}. \quad (3.27)$$

When $s \geq s'$ and $\boldsymbol{t}_- = \boldsymbol{t'}_-$ right hand side is equal to zero and we obtain mKP equation in bilinear form. We can introduce tau-functions.

$$\psi(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_+,z)} \frac{\tau_s(\boldsymbol{t}_+ - [z^{-1}],\boldsymbol{t}_-)}{\tau_s(\boldsymbol{t}_+,\boldsymbol{t}_-)} \quad (3.28)$$
$$\psi^*(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^{-s} e^{\xi(\boldsymbol{t}_+,z)} \frac{\tau_s(\boldsymbol{t}_+ + [z^{-1}],\boldsymbol{t}_-)}{\tau_s(\boldsymbol{t}_+,\boldsymbol{t}_-)}. \quad (3.29)$$

We can also introduce r_n and r_n^* such that

$$\bar{\psi}(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_-,z^{-1})} r_s(z,;\boldsymbol{t}_+,\boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_-,z^{-1})} \sum_{j\geq 0} r_{s,j} z^j \quad (3.30)$$
$$\bar{\psi}^*(s,z,\boldsymbol{t}_+,\boldsymbol{t}_-) = z^{-s} e^{-\xi(\boldsymbol{t}_-,z^{-1})} r_s^*(z,;\boldsymbol{t}_+,\boldsymbol{t}_-) = z^{-s} e^{-\xi(\boldsymbol{t}_-,z^{-1})} \sum_{j\geq 0} r_{s,j}^* z^j. \quad (3.31)$$

Applying calculations similar to one used in Lemma 3 for different choice of

$s - s', \boldsymbol{t}'_- - \boldsymbol{t}_-, \boldsymbol{t}'_+ - \boldsymbol{t}_+$ we can proof following equalities:

$$
\begin{aligned}
r_s^{-1}(z) &= D_-(z^{-1}) r^*_{s+1}(z), && \text{for } \boldsymbol{t}'_+ = \boldsymbol{t}_+,\ \boldsymbol{t}'_- = \boldsymbol{t}_- + [z],\ s' = s+1 \\
\frac{r_s(z)}{r_{s-1}(z)} &= \frac{D_-(z^{-1}) r_{s,0}}{r_{s-1,0}} && \text{for } \boldsymbol{t}'_+ = \boldsymbol{t}_+,\ \boldsymbol{t}'_- = \boldsymbol{t}_- + [z],\ s' = s+2 \\
\frac{D_+(\zeta)\tau_s D_-(z^{-1})\tau_{s+1}}{\tau_s D_+(\zeta) D_-(z^{-1})\tau_{s+1}} &= \frac{r_s(z)}{D_+(\zeta) r_s(z)} && \text{for } \boldsymbol{t}'_+ = \boldsymbol{t}_+ + [\zeta^{-1}],\ \boldsymbol{t}'_- = \boldsymbol{t}_- + [z],\ s' = s+1.
\end{aligned}
$$

Combine last two equations one can find that

$$
\bar{\psi}(s, z, \boldsymbol{t}_+, \boldsymbol{t}_-) = z^s e^{\xi(\boldsymbol{t}_-, z^{-1})} \frac{\tau_{s+1}(\boldsymbol{t}_+, \boldsymbol{t}_- - [z])}{\tau(\boldsymbol{t}_+, \boldsymbol{t}_-)} \tag{3.32}
$$

$$
\bar{\psi}^*(s, z, \boldsymbol{t}_+, \boldsymbol{t}_-) = z^{-s} e^{-\xi(\boldsymbol{t}_-, z^{-1})} \frac{\tau_{s-1}(\boldsymbol{t}_+, \boldsymbol{t}_- + [z])}{\tau(\boldsymbol{t}_+, \boldsymbol{t}_-)}. \tag{3.33}
$$

Eventually it results in integral bilinear equation on tau functions of Toda lattice hierarchy:

$$
\begin{aligned}
&\oint_\infty z^{s'-s} e^{\xi(\boldsymbol{t}_+, z) - \xi(\boldsymbol{t}'_+, z)} \tau_s(\boldsymbol{t}_+ - [z^{-1}], \boldsymbol{t}_-) \tau_{s'}(\boldsymbol{t}'_+ + [z^{-1}], \boldsymbol{t}'_-) \frac{\mathrm{d}z}{2\pi i} = \\
&= \oint_0 z^{s'-s} e^{\xi(\boldsymbol{t}_-, z^{-1}) - \xi(\boldsymbol{t}'_-, z^{-1})} \tau_{s+1}(\boldsymbol{t}_+, \boldsymbol{t}_- - [z]) \tau_{s'-1}(\boldsymbol{t}'_+, \boldsymbol{t}'_- + [z]) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{3.34}
$$

Or if one consider n not discrete but continues variable and introduce $x = \eta n$, then it can written as

$$
\begin{aligned}
&\oint_\infty z^{\eta(x'-x)} e^{\xi(\boldsymbol{t}_+, z) - \xi(\boldsymbol{t}'_+, z)} \tau(x, \boldsymbol{t}_+ - [z^{-1}], \boldsymbol{t}_-) \tau(x', \boldsymbol{t}'_+ + [z^{-1}], \boldsymbol{t}'_-) \frac{\mathrm{d}z}{2\pi i} = \\
&= \oint_0 z^{\eta(x'-x)} e^{\xi(\boldsymbol{t}_-, z^{-1}) - \xi(\boldsymbol{t}'_-, z^{-1})} \tau(x+\eta, \boldsymbol{t}_+, \boldsymbol{t}_- - [z]) \tau(x'-\eta, \boldsymbol{t}'_+, \boldsymbol{t}'_- + [z]) \frac{\mathrm{d}z}{2\pi i}.
\end{aligned} \tag{3.35}
$$

3.3 Multi-component KP hierarchy

Content of this section is based on (Dickey [1997]).

Another generalization of KP hierarchy comes when we consider elements u_m of operator $\mathcal{L}$ in (2.1) as $n \times n$ matrices

$$\boldsymbol{L} = \partial + U_1\partial^{-1} + U_2\partial^{-2} + ... \tag{3.36}$$

We can introduce times : $t_{k\alpha}$, where $k > 0$ and $1 \leq \alpha \leq n$.

To define dynamics of these elements we introduce operators $\boldsymbol{R}_\alpha$ such that $\boldsymbol{R}_\alpha \boldsymbol{R}_\beta = \delta_{\alpha\beta}\boldsymbol{R}_\alpha$, $[\boldsymbol{R}_\alpha, \boldsymbol{L}] = 0$ and $\sum\limits_\alpha \boldsymbol{R}_\alpha = 1$ (from here and further summations over Greek indices goes from 1 to n).

Now define operators $\boldsymbol{B}_{k\alpha} = (\boldsymbol{L}^k \boldsymbol{R}_\alpha)_+$ where as in regular KP $()_+$ means taking purely differential part. We introduce dynamics

$$\partial_{k\alpha}\boldsymbol{L} = [\boldsymbol{B}_{k\alpha}, L]. \tag{3.37}$$

It is clear from definition, that $\sum\limits_\alpha \partial_{1\alpha} = \partial$.

Similarly to KP case we may introduce dressing operator $\boldsymbol{W} = I + \sum\limits_{k>0} W_k \partial^{-k}$: $\boldsymbol{L} = \boldsymbol{W}\partial\boldsymbol{W}^{-1}$. It is clear, that defined as $\boldsymbol{R}_\alpha = \boldsymbol{W}E_\alpha\boldsymbol{W}^{-1}$, R_α operators are indeed satisfy all requirements . Here $(E_\alpha)_{ij} = \delta_{i\alpha}\delta_{j\alpha}$.

Introducing matrix Baker-Akhiezer and adjoint Baker-Akhiezer functions:

$$\Psi = \boldsymbol{W}e^{\xi(\boldsymbol{t},z)} = e^{\xi(\boldsymbol{t},z)}W \tag{3.38}$$

$$\Psi^* = (\boldsymbol{W}^\dagger)^{-1}e^{-\xi(\boldsymbol{t},z)} = e^{-\xi(\boldsymbol{t},z)}(W^*)^{-1} \tag{3.39}$$

where $\xi(\boldsymbol{t}, z) = \sum\limits_{k\geq 0}\sum\limits_\alpha z^k E_\alpha t_{k\alpha}$.

They are solutions of corresponding generalization of linear problems:

$$\boldsymbol{L}\Psi = z\Psi \qquad \boldsymbol{L}^\dagger\Psi^* = z\Psi^* \tag{3.40}$$

$$\partial_{n\alpha}\Psi = \boldsymbol{B}_{n\alpha}\Psi \quad \partial_{n\alpha}\Psi^* = -\boldsymbol{B}^\dagger_{n\alpha}\Psi^*. \tag{3.41}$$

It can be proven the same way as in scalar case, analogue of Theorem 1:

Theorem 3. *The identity*

$$\operatorname*{res}_z[(\partial^{i_1}_{k_1\alpha_1}...\partial^{i_m}_{k_m\alpha_m}\Psi)\Psi^*] = 0$$

holds for any $(i_1,...,i_m) \in (\mathbb{N}^*)^m$, $(k_1,...,k_m) \in (\mathbb{N}^*)^m$ *and* $(\alpha_1,...,\alpha_m) \in [1,n]^m$ *if and only if* Ψ *and* Ψ^* *of form* $(I + \sum_{k>0} A_i z^{-i})e^{\pm\xi}$ *are solutions of (3.38).*

Or it can be written in integral form

$$\oint_{\infty} \Psi(z;\boldsymbol{t})\Psi^*(z;\boldsymbol{t'})\mathrm{d}z = 0. \tag{3.42}$$

It is possible to generalize notion of τ-function.

First of all we introduce operators $D_\alpha(z) = \exp\left(-\sum_{k>1} \frac{\partial_{k\alpha}}{z^k k}\right)$ which is act by shifting α's times by $[z^{-1}]$ vector. $D_\alpha(z)f(\boldsymbol{t}) = f(..., t_{k\gamma} - \delta_{\alpha\gamma}(1/k)z^{-k}, ...)$.

As in a proof of existence of τ-function for KP it is useful to consider following identities $D_\alpha(\zeta)e^{-\xi(\boldsymbol{t},z)} = (I - E_\alpha + (1-\zeta/z)^{-1}E_\alpha)e^{-\xi(\boldsymbol{t},z)}$.

Taking (β,β)th and (α,β)th elements of equation (3.42) with $t'_{k\gamma} = t_{k\gamma} + \delta_{\beta\gamma}\frac{1}{k\zeta^k}$ we obtain equations

$$W_{\beta\beta}D_\alpha W^*_{\beta\beta} = 1 \tag{3.43}$$

$$\zeta\frac{W_{\alpha\beta}(\zeta)}{W_{\beta\beta}(\zeta)} = D_\beta(\zeta)W_{1,\alpha\beta}. \tag{3.44}$$

Taking (β,β)th element of equation (3.42) with $t'_{k\gamma} = t_{k\gamma} + \delta_{\beta\gamma}\left(\frac{1}{k\zeta_1^k} + \frac{1}{k\zeta_2^k}\right)$ results in

$$\frac{D_\beta(\zeta_1)W_{\beta\beta}(\zeta_2)}{W_{\beta\beta}(\zeta_2)} = \frac{D_\beta(\zeta_2)W_{\beta\beta}(\zeta_1)}{W_{\beta\beta}(\zeta_1)}. \tag{3.45}$$

Introducing $f_\beta = \log W_{\beta\beta}$ it can be rewritten as

$$(D_\beta(\zeta_1) - 1)f_\beta(\zeta_2) = (D_\beta(\zeta_2) - 1)f_\beta(\zeta_1) \tag{3.46}$$

Combine equations which come from (α,α)th, (α,β)th, (β,β)th and (β,α)th with $t'_{k\gamma} = t_{k\gamma} + \left(\delta_{\beta\gamma}\frac{1}{k\zeta_1^k} + \delta_{\alpha\gamma}\frac{1}{k\zeta_2^k}\right)$ one can show that

$$(D_\alpha(\zeta_1) - 1)f_\beta(\zeta_2) = (D_\beta(\zeta_2) - 1)f_\alpha(\zeta_1). \tag{3.47}$$

As in KP case we will proof that there is function τ such that $f_\alpha(z) = (D_\alpha(z) - 1)\log\tau$.

Introducing operator $N_\alpha(z) = \sum\limits_{j\geq 0} z^{-j-1}\partial_{j\alpha} + \partial_z$ such that $N_\alpha(z)D_\alpha(z)f(\boldsymbol{t}, z) = 0$ and applying it to (3.47) one obtain

$$D_\beta(\zeta_2)N_\alpha(\zeta_1)f_\alpha(\zeta_1) - N_\alpha f_\alpha(\zeta_1) = -\sum_{j\geq 0}\zeta^{-j-1}\partial_{j\alpha}f_\beta(\zeta_2). \tag{3.48}$$

Then multiply this by ζ_1^i and take $\operatorname*{res}\limits_{\zeta_1}$

$$b_{i\alpha} \equiv \operatorname*{res}_{\zeta_1}\zeta_1^i N_\alpha(\zeta_1)f_\alpha(\zeta_1) = D_\beta(\zeta_2)\operatorname*{res}_{\zeta_1}\zeta_1^i N_\alpha(\zeta_1)f_\alpha(\zeta_1) + \partial_{i\alpha}f_\beta(\zeta_2) \tag{3.49}$$

i.e

$$b_{i\alpha} = D_\beta(\zeta_2)b_{i\alpha} + \partial_{i\alpha}f_\beta(\zeta_2). \tag{3.50}$$

Since (i, α) is arbitrary we can differentiate this equality with respect to $t_{j\gamma}$. change indices and substitute one equation from other to obtain equation

$$(D_\beta(\zeta_2) - 1)(\partial_{j\gamma}b_{i\alpha} - \partial_{i\alpha}b_{j\gamma}). \tag{3.51}$$

Since $(D_\beta(\zeta_2) - 1)$ null only functions which are constant for all times, the same argument as in KP case can be applied here to show, that $\partial_{j\gamma}b_{i\alpha} - \partial_{i\alpha}b_{j\gamma} = 0$, which means, that one can introduce τ such that $b_{i\alpha} = \partial_{i\alpha}\log\tau$. Tau function is defined up to multiplication by $c(z)$, however this ambiguity can be hidden inside definition of Baker-Akhiezer functions, which also can be defined up to multiplication by some matrix which depend only on z.

Using equation 3.44 we define $\tau_{\alpha\beta} = \tau W_{1,\alpha\beta}$ and

$$W_{\alpha\beta}(z; \boldsymbol{t}) = \frac{1}{z}\frac{D_\beta(z)\tau_{\alpha\beta}(\boldsymbol{t})}{\tau(\boldsymbol{t})},\ \alpha \neq \beta. \tag{3.52}$$

If we introduce $t_i = \frac{1}{n}\sum\limits_\alpha t_{i\alpha}$ such that $\partial_n = \sum\limits_\alpha \partial_{n\alpha}$ and consider dependence only on t_n variables, we obtain Matrix KP hierarchy.

Chapter 4

Main Results

4.1 KP hierarchy

This thesis continues a series of works started with (Airault et al. [1977]) where authors have considered a singular solutions of KdV equation and shown, that its poles is governed by dynamics of cubic Hamiltonian of Calogero-Moser system in special locus, where H_2 is equal to zero. Following by famous Krichever results (Krichever [1978]) and (Krichever [1980]) where he has shown that connection between pole solutions of nonlinear partial equations and many body systems becomes more natural for KP equation. Shiota in (Shiota [1994]) have extended that correspondence to the whole hierarchy for rational case. He have shown that poles of rational solutions of KP hierarchy evolve with respect to t_m's KP time like particles of rational Calogero-Moser model governed by Hamiltonian $H_m = \mathrm{tr} L^m$ with Calogero-Moser matrix L (1.26).

Later this result was generalized in (Haine [2007]) and (Zabrodin [2020]) for trigonometric solutions with corresponding Hamiltonians

$H_m = \dfrac{1}{2(m+1)\gamma}\mathrm{tr}\left((L+\gamma I)^{m+1} - (L-\gamma I)^{m+1}\right)$ where L is Lax matrix for trigonometric Calogero-Moser system (1.28).

Article (Prokofev and Zabrodin [2021b]) contains the most general version of this statement. It considers elliptic solution of whole hierarchy in form of

$$\tau(x, \boldsymbol{t}) = \prod_{i=1}^{N} \sigma(x - x_i(\boldsymbol{t})). \tag{4.1}$$

It is proven, that (4.1) gives solution of (2.11) if and only if evolution of x_i's with respect to t_m is governed by

$$H_m = \operatorname*{res}_{z=\infty} (z^m \lambda(z)) \tag{4.2}$$

where $\lambda(z)$ solves

$$\det(L(\lambda) - (z + \zeta(\lambda)I) = 0 \tag{4.3}$$

with elliptic Lax matrix

$$L_{jk} = -p_j \delta_{jk} - (1 - \delta_{jk})\Phi(x_j - x_k, \lambda). \tag{4.4}$$

It appears that there is only one unique solution of (4.3) when $z \to \infty$.

This article also include nontrivial calculations connecting this solution in the limit when one or both periods of elliptic curve goes to infinity with results of previous works.

4.2 2d Toda hierarchy

Dynamics of poles of elliptic solutions to the 2DTL and mKP hierarchies was studied in (Krichever and Zabrodin [1995]). It was proved that the poles move as particles of the integrable Ruijsenaars–Schneider many-body system (Ruijsenaars and Schneider [1986]) which is a relativistic generalization of the Calogero–Moser system. The extension to the level of hierarchies for rational solutions to the mKP equation was made in (Iliev [2007]): again, the evolution of poles with respect to the higher times t_k of the mKP hierarchy is governed by the higher Hamiltonians $-\mathrm{tr}L^k$ of the Ruijsenaars–Schneider system.

Article (Prokofev and Zabrodin [2019]) generalize that result. It contains direct solutions of bilinear relation for the whole 2d Toda lattice (3.35) with trigonometric tau-function of the form

$$\tau(x, \boldsymbol{t}_+, \boldsymbol{t}_-) = \exp\left(-\sum_{k\geq 1} k t_k t_{-k}\right) \prod_{i=1}^{N} \left(e^{2\gamma x} - e^{2\gamma x_i(\boldsymbol{t}_+, \boldsymbol{t}_-)}\right). \tag{4.5}$$

It is shown, that evolution of the x_i's with respect to the time t_m govern by Hamiltonian

$$H_m = -\frac{\sinh(m\gamma\eta)}{m\gamma\eta} \mathrm{tr}(L)^m \tag{4.6}$$

for both positive and negative m. Here

$$L_{ij} = \frac{\gamma\eta e^{\eta p_i}}{\sinh(\gamma(x_i - x_j - \eta))} \prod_{l\neq i} \frac{\sinh(\gamma(x_i - x_l + \eta))}{\sinh(\gamma(x_i - x_l))} \tag{4.7}$$

is the Lax matrix of trigonometric Ruijsenaars–Schneider system.

Generalization to elliptic case is given in (Prokofev and Zabrodin [2021a]) where we consider solutions of 2d Toda lattice hierarchy of the form

$$\tau(x, \boldsymbol{t}_+, \boldsymbol{t}_-) = \exp\left(-\sum_{k\geq 1} k t_k t_{-k}\right) \prod_{i=1}^{N} \sigma\left(x - x_i(\boldsymbol{t}_+, \boldsymbol{t}_-)\right). \tag{4.8}$$

In order for 4.8 to be solution, evolution of x_i with respect to time t_m should be governed by Hamiltonian

$$H_m = \operatorname*{res}_{z=\infty} \left(z^{m-1}\lambda(z)\right) \tag{4.9}$$

for $m > 0$ and

$$H_m = \operatorname*{res}_{z=0}(z^{m-1}\lambda(z)) \tag{4.10}$$

for $m < 0$.

$\lambda(z)$ can be found from the equation

$$\det(L(\lambda) - z^{\eta\zeta(\lambda)}) = 0 \tag{4.11}$$

with elliptic Lax matrix L

$$L_{ij}(\lambda) = e^{p_i}\Phi(x_i - x_j - \eta, \lambda)\prod_{l\neq i}\frac{\sigma(x_i - x_l + \eta)}{\sigma(x_i - x_l)}. \tag{4.12}$$

Equation (4.11) have unique solution near $z = \infty$.

Nontrivial calculations conducted in this paper proof that, degeneration of elliptic curve to its rational or trigonometric limits gives the same results as ones obtained before.

4.3 Matrix KP

The singular (in general, elliptic) solutions to the matrix KP equation were investigated in (Krichever and Zabrodin [1995])). It was shown that the evolution of data of such solutions (positions of poles and some internal degrees of freedom) with respect to the time t_2 is isomorphic to the dynamics of a spin generalization of the Calogero–Moser system (the Gibbons–Hermsen system (Gibbons and Hermsen [1984]). The generalization of this connection to whole hierarchy was studied in (Pashkov and Zabrodin [2018]) for rational solutions. It appears, that dynamics in t_m is governed by Hamiltonian $H_m = \mathrm{tr}L^m$.

Trigonometric version of this result is considered in (Prokofev and Zabrodin [2020]). There are trigonometric solutions of matrix KP hierarchy constructed in this paper. It is proven, that

$$\tau = \prod_{i=1}^{N}(e^{2\gamma x} - e^{2\gamma x_i(\boldsymbol{t})}) \tag{4.13}$$

with

$$W_{1,\alpha\beta} = S_{\alpha\beta} - \sum_i \frac{2\gamma e^{2\gamma x_i(\boldsymbol{t})}a_i^\alpha(\boldsymbol{t})b_i^\beta(\boldsymbol{t})}{e^{2\gamma x} - e^{2\gamma x_i(\boldsymbol{t})}} \tag{4.14}$$

Are solutions to whole matrix KP hierarchy if and only if dynamics of $x_i(\boldsymbol{t})$, $a_i^\alpha(\boldsymbol{t})$, $b_i^\alpha(\boldsymbol{t})$ in t_m is governed by Hamiltonian

$$H_m = \frac{1}{2(m+1)\gamma}\mathrm{tr}\left((L+\gamma I)^{m+1} - (L-\gamma I)^{m+1}\right) \tag{4.15}$$

where

$$L_{jk} = -p_j\delta_{jk} - (1-\delta_{jk})\frac{\gamma\sum\limits_{\alpha} b_j^\alpha a_k^\alpha}{\sinh(\gamma(x_j - x_k))} \tag{4.16}$$

with nonzero Poisson brackets $\{x_i, p_j\} = \delta_{ij}$ and $\{a_i^\alpha, b_j^\beta\} = \delta_{\alpha\beta}\delta_{ij}$.

Article (Prokofev and Zabrodin [2021c]) contains further generalization to the elliptic level.

$$\tau = \prod_{i=1}^{N} \sigma(x - x_i(\boldsymbol{t})) \tag{4.17}$$

with

$$W_{1,\alpha\beta} = S_{\alpha\beta} - \sum_i a_i^\alpha(\boldsymbol{t}) b_i^\beta(\boldsymbol{t})\zeta(x - x_i(\boldsymbol{t})) \tag{4.18}$$

is solution of matrix KP if dynamics of poles and spins in t_m is governed by

$$H_m = \operatorname*{res}_{z=\infty}(z^m\lambda(z)) \tag{4.19}$$

where $\lambda(z) = \sum\limits_{\alpha}\lambda_\alpha(z)$ and each $\lambda_\alpha(z)$ is different solution of

$$\det(L(\lambda_\alpha) - (z + \zeta(\lambda_\alpha))I) = 0 \tag{4.20}$$

with elliptic Lax matrix

$$L_{jk} = -p_j\delta_{jk} - (1-\delta_{jk})\Phi(x_j - x_k, \lambda)\sum_\nu b_\nu^j a_\nu^k. \tag{4.21}$$

It appears, that equation (4.20) has n different solutions near $z = \infty$ and each $\lambda_\alpha(z)$ is generating functions of $H_{n\alpha}$-Hamiltonians corresponding $t_{n\alpha}$ flow. So we obtained not only correspondence between Matrix KP and spin Calogero-Moser, but between multi-component KP and spin Calogero-Moser.

Rational and trigonometric limits are also found and they match with results from previous papers.

The results of the thesis are published in five papers

1. V. Prokofev and A. Zabrodin. *Toda lattice hierarchy and trigonometric Ruijsenaars–Schneider hierarchy.* Journal of Physics A: Mathematical and Theoretical , 2019. doi:10.1088/1751-8121/ab520c

2. V. Prokofev and A. Zabrodin. *Matrix Kadomtsev Petviashvili Hierarchy and Spin Generalization of Trigonometric Calogero Moser Hierarchy.* Proceedings of the Steklov Institute of Mathematics , 2020. doi:10.1134/S0081543820030177

3. V. Prokofev and A. Zabrodin. *Elliptic solutions to the KP hierarchy and elliptic Calogero–Moser model.* Journal of Physics A: Mathematical and Theoretical, 2021b. doi:10.1088/1751-8121/ac0a3

4. V. Prokofev and A. Zabrodin. *Elliptic solutions to Toda lattice hierarchy and elliptic Ruijsenaars-Schneider model.* Theoretical and Mathematical Physics , 2021a. doi:10.1134/S0040577921080080

5. V. Prokofev and A. Zabrodin.*Elliptic solutions to matrix KP hierarchy and spin generalization of elliptic Calogero-Moser model.* Journal of Mathematical Physics , 2021c. doi:10.1063/5.0051713

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