National Research University Higher School of Economics

Faculty of Mathematics

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Anastasiia Trofimova

# Stochastic integrable processes andrepresentations of the braid group

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Academic supervisor
Doctor of Science, professor
Pavel Pyatov

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## Abstract

Integrability is an important property of models of quantum mechanics and statistical physics that makes an exact solution possible. One of the main methods that is used to study integrable systems is the Bethe ansatz approach based on the Yang-Baxter equation. The study and classification of existing R-matrices as well as the search of the new ones become a problem of mathematical physics. One of the major sources of R-matrix representations is the representations of the braid group on three strings and its finite-dimensional quotient algebras. New results were obtained in two directions.

The first direction is the classification of low dimensional irreducible representations of the braid group on three strings. We consider families of the finite dimensional quotient algebras of the group algebra of the braid group on three strings by a $p$-th order generic monic polynomial relation on the elementary braids (also known as Artin generators). These are the cases of power $p = 2, 3, 4, 5$ polynomial relation with the corresponding dimensions of quotient algebras equal to $6, 24, 96$, and $600$, respectively. We construct a series of representations of dimension $\leq 6$ (Proposition 1.2) and find the conditions under which they are irreducible (Proposition 1.3). For the considered quotient algebras we formulate semisimplicity criteria, and if those criteria are satisfied we give a complete classification of irreducible representations (Theorem 1.1). Our classification complements the list of all the irreducible representations of the braid group on three strings of dimensions $\leq 5$ found by I. Tuba and H. Wenzl (2001) by adding irreducible representations of dimension 6. The study of these 6-dimensional representations brings new criteria.

Another subject of the research is stochastic integrable particle systems. Our interest concerns the statistics of a particle flow in the $q$-boson zero range process (ZRP) on a ring. With the use of Bethe ansatz and TQ-relation methods we calculate the first two cumulants of the particle current. The exact formula for the second cumulant is obtained in the form of an infinite sum of double contour integrals (Theorem 2.1). This representation allows us to perform the asymptotic analysis of the large system size limit. In this limit we find that at generic values of parameter controlling interparticle interaction the second cumulant reproduces the scaling expected for the models in the Kardar-Parisi-Zhang universality class (Theorem 2.2). Another result is the universal scaling function describing the crossover between the Kardar-Parisi-Zhang and Edwards-Wilkinson universality classes. It is obtained from the exact formula for the second cumulant in a scaling limit corresponding to the KPZ-EW transition (Theorem 2.3). It agrees with the scaling function first found for the asymmetric simple exclusion process and conjectured to be universal.

## Introduction

Integrable models of quantum systems, lattice models of statistical physics and stochastic interacting particle systems play an important role in mathematics and theoretical physics. The property of a model to be integrable suggests a special mathematical structure, which makes an exact solution possible. In its core there is the Yang-Baxter equation, which is an essential ingredient of the quantum inverse scattering method (algebraic Bethe ansatz), (see [1, 2] and references therein).

The quantum inverse scattering method also gave birth to the theory of quantum groups that became a separate area of modern mathematics. An important problem in this theory is the classification and study of R-matrices satisfying the Yang-Baxter equation and of related quantum matrix algebras such as reflection equation and RTT algebras. One of the major sources of $R$-matrices are the R-matrix representations of the braid group. By definition, all these representations are constructed starting from $R$-matrix representations of the free 3-string braid group $B_3$. They necessarily factorize through finite dimensional quotient algebras of the group algebra $\mathbb{C}[B_3]$.

### Quotient algebras of the group algebra of the braid group

A classical theorem by H.S.M. Coxeter [3] states that the quotient of the $n$-string braid group $B_n$ by the $p$-th order relation $b^p = 1$ on the elementary braid $b$ is finite if and only if

$$1/n + 1/p > 1/2\,. \tag{1}$$

In case of $B_3$ we obtain finite quotient groups of orders $6, 24, 96$, and $600$, for $p = 2, 3, 4$, and 5, respectively [3]. Generalizing this setting one can consider quotients of the group algebra $\mathbb{C}[B_n]$ obtained by imposing a $p$-th order polynomial relation on the elementary braids. Under condition (1) the resulting quotient algebras are finite dimensional and, by the Tits deformation theorem (see [4], §68, or [5], section 5), in a generic situation these algebras are isomorphic to the group algebras of the corresponding Coxeter's quotient groups. In particular, they are semisimple. As a next step it would be interesting to find the semisimplicity conditions and to describe explicitly irreducible representations of these finite dimensional quotients.

The significant contribution in this direction was made by I. Tuba and H. Wenzl. In the paper [9] they classified all the irreducible representations of $B_3$ in dimensions $d \leq 5$. Their classification scheme in dimensions $d \leq 4$ yields all the irreducible representations for the quotients in cases $n = 3$, $p = 2, 3, 4$ and describes their semisimplicity conditions. However, for $p = 5$ the above-mentioned quotients of $\mathbb{C}[B_3]$ admit irreducible representations of dimensions up to 6 and the classification in [9] does not include them. In the thesis we continue the search and study of irreducible representations of the braid group $B_3$. We construct all the irreducible representations of these algebras of dimension $d \leq 6$ and find criteria for their semisimplicity conditions. In dimensions $d \leq 5$ we reproduce the classification of the irreducible representations of $B_3$ from [9]. In dimension $d = 6$ our list gives all the irreducible representations of $B_3$ that factor through representations of the quotients of $\mathbb{C}[B_3]$. The latter factorization for the $d = 6$ dimensional representations means that their spectrum contains 5 different values, one of them with multiplicity 2. We are working in the diagonal basis for the first elementary braid generator $g_1$, and we restrict our considerations to the case where all $p$ roots of its minimal polynomial are distinct. For the sake of completeness we present formulae for representations from I. Tuba and H. Wenzl list in this basis too.

Let us describe briefly some related approaches and results.

B. Westbury suggested an approach to representation theory of $B_3$ that uses representations of a particular quiver[10]. It was subsequently used by L.Le Bruyn who constructed Zariski dense rational parameterizations of the irreducible representations of $B_3$ of any dimension [11, 12]. The method was effective for solving a problem of braid reversion, however, it does not give semisimplicity criteria for the representations constructed. A 5-dimensional variety of the irreducible 6-dimensional representations of $B_3$

that we present below is contained in an 8-dimensional family of $B_3$-representations of type 6b (see Fig.1 in [11]).

For the more general case of $B_n$, $n > 3$, series of irreducible representations related to Iwahori-Hecke algebras $H_n(q)$ ($p = 2$ case) are well investigated. With an isomorphism $H_n(q) \simeq \mathbb{C}[B_n]$ for generic values of the parameter $q$, they are labelled by Young diagrams of size $n$. The other family of finite-dimensional quotient algebras of $\mathbb{C}[B_n]$ is Birman-Murakami-Wenzl algebras. They are constructed by imposing $p = 3$ polynomial relation on elementary braid and polynomial relations containing several neighbouring generators. The complete classification of irreducible representations and their irreducibility conditions are known. Every irreducible representation is labeled by a sequence of partitions in which each partition differs from the previous one by either adding or removing a box. (for a review see [6]). Some other particular families of the $B_n$-representations were also found [7, 8].

In another direction of research Broué, C. Malle and R. Rouquier [13, 14] generalized the notions of the braid group and of the Hecke algebra associated not only to Coxeter group, but to an arbitrary finite complex reflection group $W$. They defined generic Hecke algebra over certain polynomial ring $R = \mathbb{Z}[\{u_i\}]$ and conjectured that it is a free module of rank $|W|$. This conjecture is now proved [15, 16, 17, 18], [19] (Theorem 3.5 and references therein). The quotient algebras of $\mathbb{C}[B_3]$, we deal with, are specializations of generic Hecke algebras of the groups $S_3$ and $G_4$, $G_8$, $G_{16}$ (the notation used corresponds to the complete classification of finite complex reflection groups constructed by Shepard and Todd [20]) under homomorphism $R \to \mathbb{C}$ that assigns certain complex values to the variables $u_i$. The freeness conjecture in these cases is proved in [21, 22, 23], so the dimensions of the algebras $Q_X$ coincide with the cardinalities of their corresponding Coxeter groups.

**Stochastic integrable systems of interacting particles**

An important application of the integrability is to the stochastic systems of interacting particles. We consider stochastic diffusive or driven diffusive particle systems in $1 + 1$ dimension, which are systems of particles with local inter-particle interactions subject to uncorrelated random force. The particle density field in such models can be associated with the gradient of the interface height function. Conversely the height function of a one-dimensional interface is related to the time-integrated particle current. Thus, the results for a particle system can be translated to the interface language and vice versa. Integrability of particle models, in turn, implies a possibility of obtaining exact results to be used as a source of results about interface random growth behaviour.

Two broad universality classes, Edwards-Wilkinson(EW) [24] and Kardar-Parisi-Zhang(KPZ) [25], are believed to capture the large scale behavior of various natural phenomena, including the interface random growth. These two classes unify plenty mathematical models and diverse phenomena: random growth of interfaces like borders of bacterial colonies, wetting, crystallization and combustion fronts, polymers in random media, traffic flows, etc. [26]. The common feature of these random systems is the universal behaviour at large scales characterized by universal scaling exponents and universal scaling functions. Each of EW and KPZ universality classes includes the continuous interface growth model defined by the eponymous stochastic PDE for the the height function. The analysis of the properties of their solutions was the first analytic effort in characterization of these universality classes.

The stochastic linear Edward-Wilkinson PDE was solved right after the introduction [24]. The analytic solution predicts the universal behavior of interface defined by two independent critical exponents, namely, the roughness exponent $\zeta$ and the dynamical exponent $z$ and gives the exact expression for the scaling functions. In 1+1 dimension the critical exponents are $\zeta = 1/2$ and $z = 2$. They characterize the large scale behaviour of the interface width, which scales as $N^{\zeta}$ in the stationary state of the system of size $N$ and saturates to this stationary value after the relaxation time of order of $N^{z}$. The first exponent indicates that the stationary interface looks like Brownian motion, while the second exponent suggests that the propagation of fluctuations is purely diffusive. The exponents $\zeta$ and $z$ can be translated to another pair of exponents $\alpha = \zeta/z$ and $\beta = 1/z$ responsible for fluctuation and correlation scales respectively. Specifically, the interface width grows as $t^{\alpha}$ with time $t$ in the non-stationary regime, while the correlation length scales as the $t^{\beta}$. For EW universality class they are $\alpha = 1/4$ and $\beta = 1/2$.

The stochastic nonlinear KPZ equation was introduced in 1986 [25]. The novelty of this equation was the non-linearity that made its analysis much more involved. Though the analytic solution had not been found for other 25 years, the scaling exponents and the scaling hypotheses about the form of the model dependent constants in 1+1 dimension were predicted heuristically from the dimensional analysis, mode coupling theory and renormalization group [25, 27, 28]. Both the roughness exponent $\zeta = 1/2$ and the dynamical exponent $z = 3/2$ are used to describe the height function behaviour of the corresponding interface. These exponents can also be translated to $\alpha = 1/3$, $\beta = 2/3$. However, finer characteristics of the large scale behaviour, e.g. universal scaling functions, were beyond the mentioned approaches. This is when stochastic integrable systems were proven to be useful.

The most prominent integrable interacting particle models are symmetric and asymmetric simple exclusion processes, abbreviated as SSEP and ASEP [29] respectively. They are systems of particles performing random walks on 1D lattice subject to the exclusion rule, which prevents two particles from coming to the same site. They are paradigmatic models belonging to the EW and KPZ universality classes, respectively. The results about these models can be divided into two groups.

The first group includes results about the stationary state and the large time behaviour of finite systems with periodic boundary conditions or with particle reservoirs attached to the ends. These are stationary state density and current profiles [30, 31], correlation functions [32], the large deviation functions for particle current and density on a ring [33, 34] and on a segment [35, 36, 37, 38, 39, 40], e.t.c. In particular, the two first cumulants of time integrated particle current, which we refer to below, were found in [41] (for more references see [42, 43, 44]).

The second group consists of results about the transient dynamics in the infinite system. They are the one point current distributions for several types of initial conditions [45, 46, 47, 48]. In the case of TASEP, the totally asymmetric version of ASEP, it was also possible to find all the equal-time multipoint current distributions [49, 50] as well as those for unequal times [51, 52] and space-time points on space-like paths [53, 54, 55]. Some progress was also achieved for the time-like correlation functions [56, 57].

The transition between transient and stationary regimes has also received attention recently. Some finite size results for TASEP with periodic boundary conditions far from the equilibrium were obtained and analysed asymptotically at different time scales [58, 59, 60, 61, 62, 63].

Although the mentioned results for interacting particle models make a significant contribution to the theory of EW and KPZ universality in 1+1 dimensions, it is still far from being complete. That is why testing the results obtained and hypotheses made against other interacting particle systems with richer dynamics is of interest. In the thesis we address probably the next simplest integrable interacting particle system, the zero range process (ZRP) [29]. It is a continuous time particle system on a one-dimensional lattice with jump probabilities depending only on the number of particles at the site of departure. Like in SSEP and ASEP, its stationary measure has the factorized form [64] in the infinite and periodic cases. We consider a particular case of this model known as the q-boson ZRP, whose jump rates ensure the Bethe ansatz integrability of the stochastic generator. The model received its name from the fact that algebraically its evolution operator can be realized in terms of the representation of q-boson algebra. It was initially introduced by Bogoliubov and Bullough [65] and then adapted by Sasamoto and Wadati [66] to be considered as the stochastic interacting particle model. It was also rediscovered as a ZRP solvable by the coordinate Bethe ansatz [67]. An intriguing fact is the duality under particle-hole transformation between q-boson ZRP and the q-TASEP model appeared much later as a degeneration of the Macdonald process [68]. On the one hand, there are much less results on q-boson ZRP, than for ASEP and SSEP. On the other hand, that model is a good candidate for testing the universality of results obtained from ASEP and SSEP.

The hopping rates of the model are parameterized by a real parameter $q \in \mathbb{R}$. The model is believed to belong to the KPZ universality class when $q \neq 1$ and to the EW universality class when $q = 1$. Several results were obtained on the q-boson ZRP. The mean group velocity and the diffusion coefficient for two particles on the infinite lattice were calculated [66]. The scaling form of the large deviation function of the particle current was obtained for the periodic lattice in the large system size limit for $q \neq 1$ [67].

We undertake the further research of the large time regime of q-Boson ZRP on the ring of size $N$ evaluating the exact diffusion coefficient for the particle position and the associated interface height. The problem of calculation of the current cumulants in exclusion processes has a long history. The diffusion coefficient was first obtained for TASEP on a ring [69] and on a segment [70] using the matrix product ansatz [31]. Later the matrix product technique was extended to ASEP [41]. The whole large deviation function of the distance traveled by a particle in TASEP, which in particular yielded all the exact scaled cumulants including the diffusion coefficient, was derived using the Bethe ansatz in [33]. This solution used significantly a special structure of the TASEP Bethe equations, which is not present in the more general ASEP case. The large deviation function for the ASEP was constructed in the large system size limit under a special KPZ scaling by the method of asymptotic solution of the Bethe equations proposed in [71, 72]. The universal current cumulants in SSEP on the ring were obtained asymptotically both from the Bethe ansatz and from the fluctuating hydrodynamics in [34]. Technique based on asymptotic solutions of the Bethe equations was also applied to evaluate the current large deviation function in the ASEP on the segment with open ends [37]. Finally, the approach to finding the exact expressions of current cumulants based on the functional Bethe ansatz or T-Q Baxter equation was developed for ASEP on the finite ring by Prolhac and Mallick in [74]. The exact large deviation function for the ASEP on a segment was also found by adapting the matrix product ansatz [38, 39] and using the T-Q Baxter equation [40, 75].

In the thesis we apply the method by Prolhac and Mallick [74] to q-boson ZRP on a ring. Our interest is two-fold. First, it is a technical aspect of a reformulation of the Bethe equations in the form of T-Q Baxter polynomial that can be solved perturbatively order by order in powers of the parameter of the moment generating function of particle current. It delivers the exact representation of the diffusion coefficient (Theorem 2.1). Comparing q-boson ZRP with the ASEP one notices that the similar Bethe ansatz leads to very different TQ-polynomial solutions. This can be explained by the relation of q-boson ZRP to the infinite-dimensional q-boson representation while ASEP is related to the two-dimensional representation of the quantum affine algebra $U_q(\widehat{sl_2})$. In both cases the solution of TQ equations are given in terms of polynomial truncation of the generating function of stationary weights. Complexity of this function seems to depend crucially on the dimension of underlying representation. While in the ASEP the single site weight is a simple binomial, in the q-boson ZRP we show that the single site function is an infinite sum representing the entire or meromorphic q-exponential function. Consequently, the exact expression of the diffusion coefficient obtained in [41] is an explicit sum of quantities constructed of binomial coefficients. In our case we were able to represent the final result (Theorem 2.1) in the form of an infinite sum of double contour integrals, which being less explicit, however, is well suited for asymptotic analysis.

Our second interest is the study of the scaling limit of the formulae obtained and, thus, the check of the scaling hypotheses made before on the basis of the analysis of EW and KPZ equations and ASEP. It is expected that in the infinite system of KPZ and EW universality classes a particle moves subdiffusively, with fluctuations growing with time $t$ as $t^{\alpha}$. So does the height of the associated interface. However, in the finite periodic system of size $N$ at large time $t \gg N^z$ particles move diffusively, i.e. the variance of the fluctuations grows linearly with time with the proportionality factor, named diffusion coefficient, that vanishes in the infinite system size limit as $N^{2\zeta - z}$, i.e. as $1/N$ for EW and as $1/\sqrt{N}$ for KPZ universality class (for details see review [27] and discussion in the text). The latter power law was one of the first demonstrations of KPZ scaling behaviour obtained from exact solution [69]. The universal power law form of cumulants of KPZ interface height of arbitrary order was conjectured in [28, 73] basing on the analysis of dimensions supplied with the scaling invariance arguments. Specifically the model dependent dimensionful factors were predicted, which come with the power laws, while the universal dimensionless constants are to be determined from exact solutions. The program of determining the universal constants was first realized within the exact calculation of the second current cumulant [69, 41] and of the cumulants of arbitrary order [33, 72] in TASEP and ASEP. For the EW universality class a few first current cumulants were also calculated exactlty in [34], the mentioned solution of SSEP. The asymptotic scaling form of the cumulants was also verified at several other models [67, 76, 77, 78]. Here, using the method of TQ-equation we reproduce the earlier result [67] on the scaling form of the diffusion coefficient in the q-boson ZRP. We obtain expected power law of $1/\sqrt{N}$ in Theorem 2.2 which agrees with the hypothesis of the KPZ class universality.

In addition the scaling form of the current cumulants and the scaling functions that describes the crossover between the KPZ and EW universality classes are also expected to be universal. The first attempts to determine the form of the scaling functions were also realized within TASEP [33] and ASEP [41], respectively. We derive the scaling function interpolating between the EW and KPZ classes, which confirms the universality of the expression conjectured from the ASEP solution [41]. This result is given by Theorem 2.3.

### Thesis results

In the thesis we study the families of the finite dimensional quotient algebras of $\mathbb{C}[B_3]$ by a $p$-th order generic monic polynomial relation on the elementary braids and the statistics of particle current for the $q$-boson zero range process.

- In the diagonal basis for the first elementary braid generator we construct a series of representations of dimension $\leq 6$ and find the conditions under which they are irreducible. For the considered quotient algebras we formulate semisimplicity criteria, and if those criteria are satisfied we give a complete classification of irreducible representations.
- We have calculated exact expressions for the first two current cumulants in the $q$-boson ZRP on a ring in the form of contour integrals. In the large system size limit we find that at generic values of parameter $q$ controlling interparticle interaction the second cumulant reproduces the scaling expected for the models in the Kardar-Parisi-Zhang universality class. Another result is the universal scaling function describing the crossover between the Kardar-Parisi-Zhang and Edwards-Wilkinson universality classes.

Below we formulate the results of the thesis in more details. Note that introduction in the thesis almost coincides with the Summary, although there are more references and links to the main part of the Thesis. We hope it will help a reader to navigate.

# 1 Representations of finite-dimensional quotients of $\mathbb{C}[B_3]$

Next, we provide a summary of the first chapter of the thesis, presenting the finite-dimensional quotient algebras and formulating the results.

## 1.1 Braid group $B_3$ and its quotient algebras $Q_X$

In the first section we introduce the finite dimensional quotient algebras of $\mathbb{C}[B_3]$.

**Definition 1.1.** *Braid group $B_3$ on three strings is an abstract associate group, generated by a pair of elementary braids – $g_1$ and $g_2$ – satisfying the braid relation*

$$g_1 g_2 g_1 = g_2 g_1 g_2. \tag{2}$$

Alternatively it can be given in terms of generators

$$a = g_1 g_2, \quad b = g_1 g_2 g_1, \tag{3}$$

and relations

$$a^3 = b^2 = c\,, \tag{4}$$

where $c = (g_1 g_2)^3 = (g_1 g_2 g_1)^2$ is a central element of $B_3$ which generates the center $\mathbb{Z}(B_3)$ [80]. Thus, the quotient group $B_3/\mathbb{Z}(B_3) = \langle a, b | \, a^3 = b^2 = 1 \rangle$ is the free product $\mathbb{Z}_3 * \mathbb{Z}_2$ of two cyclic groups, which is known to be isomorphic to $PSL(2, \mathbb{Z})$.

**Definition 1.2.** *For a finite set $X$ of pairwise different nonzero complex numbers:*

$$X = \{x_1, x_2, \ldots, x_n\}, \qquad x_i \in \mathbb{C} \setminus \{0\}, \;\; x_i \neq x_j \; \forall i \neq j. \tag{5}$$

*and a polynomial*

$$P_X(g) := \prod_{i=1}^{n=|X|} (g - x_i 1), \qquad \textit{where } g \in \{g_1, g_2\}. \tag{6}$$

*the quotient algebra of the braid group on 3-strings is*

$$Q_X := \mathbb{C}[B_3]/\langle P_X(g) \rangle \tag{7}$$

**Remark 1.1.** *It is enough to impose a polynomial condition in the first braid only. Since all elementary braids are conjugate to each other, therefore, conditions on them are identical.*

As it was already mentioned in the introduction the quotient algebras $Q_X$ are finite dimensional if and only if $|X| = n < 6$. With a particular choice of polynomials $P_X(g) = g^n - 1$ they are the group algebras of the quotient groups $B_3/\langle g^n \rangle$. By the Tits deformation argument, in the generic situation the algebras $Q_X$ with $|X| \leq 5$ are isomorpic to $\mathbb{C}[B_3/\langle g^{|X|} \rangle]$ and, hence, in a generic situation semisimple.

### 1.1.1 Spectra of the central element and of generators

Our aim is to construct enough irreducible representations for quotient algebras $Q_X$ and using the Artin-Wedderburn prove that we obtain the complete classification of the irreducible representations. It turns out that their dimensions do not exceed 6. We show that in these irreducible representations the spectra of the central element $c$ (2) and of generators $a$ and $b$ (3) are, up to a discrete factor, defined by the eigenvalues $x_i$ of the elementary braids.

Let $V$ be a finite dimensional linear space, with $\dim V = d$ and let

$$\rho_{X,V} : Q_X \to \mathrm{End}(V)$$

be an irreducible representation of $Q_X$.

**Statement 1.1.** *The characteristic polynomial of elementary braids $g_1, g_2$ in representation $\rho_{X,V}(g)$ has the form*

$$\Pi_\rho(g) := \prod_{i=1}^{n=|X|} (g - x_i)^{m_i}, \qquad \textit{where } \; m_i \in \mathbb{N}^+ \textit{ such that } \sum_{i=1}^{n} m_i = d. \tag{8}$$

Since $c$ is central, we apply Schur's lemma and we have that $c$ acts in the irreducible representation $\rho_{X,V}$ as a scalar operator. We denote

$$A := \rho_{X,V}(a), \quad B := \rho_{X,V}(b), \quad \rho_{X,V}(c) := C_\rho \, \mathrm{Id}_V. \tag{9}$$

The following proposition describes explicitly the spectrum of operators $A$ and $B$ in low dimensional representations.

**Proposition 1.1.** *Let $\rho_{X,V} : Q_X \to \mathrm{End}(V)$ be a family of irreducible representations of algebras $Q_X$ (7) such that*
*a) their characters are continuous functions of parameters $x_i \in X$;*
*b) the characteristic and minimal polynomials of the matrices $\rho_{X,V}(g_1)$ and $\rho_{X,V}(g_1)$ are given by $\Pi_\rho$ (8) and $P_X$ (6), respectively.*

*Let $A$, $B$, $C_\rho$ be as defined in (9). Denote $\nu := e^{2\pi \mathrm{i}/3}$, and introduce notation $e_k(X)$ for $k$-th elementary symmetric polynomial in the set of variables $X = \{x_i\}_{i=1,\dots,n}$.*

*Then for $n = |X| \leq 5$ and $d = \dim V \leq 6$ the coefficient $C_\rho$ and eigenvalues of operators $A$ and $B$ can take the following values.*

$$\begin{aligned}
&\textit{If } d=n=2, \textit{then} && C_\rho = -e_2(X)^3, \\
& && \mathit{Spec}\, A = -e_2(X)\cdot\{\nu,\nu^{-1}\}, \quad \mathit{Spec}\, B = \mathrm{i}\, e_2(X)^{\frac{3}{2}}\cdot\{1,-1\}; && (10)\\
&\textit{If } d=n=3, \textit{then} && C_\rho = e_3(X)^2, \\
& && \mathit{Spec}\, A = e_3(X)^{\frac{2}{3}}\cdot\{1,\nu,\nu^{-1}\}, \quad \mathit{Spec}\, B = e_3(X)\cdot\{1,-1^{\sharp 2}\}, && (11)\\
& && \textit{where } m^{\sharp n} \textit{ denotes an eigenvalue } m \textit{ with multiplicity } n\,; \\
&\textit{If } d=n=4, \textit{then} && \textit{for any root } h(X) := \sqrt[2]{e_4(X)}\textit{:} \quad C_\rho = h(X)^3, \\
& && \mathit{Spec}\, A = h(X)\cdot\{1^{\sharp 2},\nu,\nu^{-1}\}, \quad \mathit{Spec}\, B = h(X)^{\frac{3}{2}}\cdot\{1^{\sharp 2},-1^{\sharp 2}\}; && (12)\\
&\textit{If } d=n=5, \textit{then} && \textit{for any root } f(X) := \sqrt[5]{e_5(X)}\textit{:} \quad C_\rho = f(X)^6, \\
& && \mathit{Spec}\, A = f(X)^2\cdot\{1,\nu^{\sharp 2},(\nu^{-1})^{\sharp 2}\}, \quad \mathit{Spec}\, B = f(X)^3\cdot\{1^{\sharp 3},-1^{\sharp 2}\}; && (13)
\end{aligned}$$

$$\begin{aligned}
&\textit{If } d=6,\ n=5,\ m_i=2,\ 1\leq i\leq 5, \textit{then} \quad C_\rho = -x_i e_5(X), \\
&\quad \mathit{Spec}\, A = -\sqrt[3]{x_i e_5(X)}\cdot\{1^{\sharp 2},\nu^{\sharp 2},(\nu^{-1})^{\sharp 2}\}, \quad \mathit{Spec}\, B = \mathrm{i}\sqrt[2]{x_i e_5(X)}\cdot\{1^{\sharp 3},-1^{\sharp 3}\}. \qquad (14)
\end{aligned}$$

Therefore, the spectra of elements $a$ and $b$ in all irreducible representations are, up to a discrete factor, defined by the eigenvalues $x_i$ of the elementary braids. This proposition becomes a key ingredient that makes the search of all irreducible representations possible.

### 1.1.2 Irreducible representations of quotient algebras $Q_X$

We use the data given in the Proposition 1.1 for an explicit construction of the irreducible representations in $Q_X$ of dimention up to 6. Choosing the basis of eigenvectors of $g_1$ we present them.

**Proposition 1.2.** *The algebras $Q_X$ in cases $|X| \leq 5$ have the following representations of dimensions* $\dim V \leq 6$.

*If* $\underline{|X| = \dim V = 1}$, *there exists a unique representation*

$$\rho_X^{(1)}(g_1) = \rho_X^{(1)}(g_2) = x_1. \qquad (15)$$

*If* $\underline{|X| = \dim V = 2}$, *there exists a unique representation*

$$\rho_X^{(2)}(g_1) = diag\{x_1, x_2\}, \qquad \rho_X^{(2)}(g_2) = \frac{1}{x_1 - x_2}\begin{pmatrix} -x_2^2 & -x_1x_2 \\ x_1^2 - x_1x_2 + x_2^2 & x_1^2 \end{pmatrix}. \qquad (16)$$

*If* $\underline{|X|=\dim V=3,}$ *there exists a unique representation*

$$\rho_X^{(3)}(g_1)=diag\{x_1,x_2,x_3\},\quad \rho_X^{(3)}(g_2)=\begin{pmatrix} \frac{x_2x_3(x_2+x_3)}{\Delta_1(X)} & \frac{x_3(x_1^2+x_2x_3)}{\Delta_1(X)} & \frac{x_2(x_1^2+x_2x_3)}{\Delta_1(X)} \\ \frac{x_3(x_2^2+x_1x_3)}{\Delta_2(X)} & \frac{x_1x_3(x_1+x_3)}{\Delta_2(X)} & \frac{x_1(x_2^2+x_1x_3)}{\Delta_2(X)} \\ \frac{x_2(x_3^2+x_1x_2)}{\Delta_3(X)} & \frac{x_1(x_3^2+x_1x_2)}{\Delta_3(X)} & \frac{x_1x_2(x_1+x_2)}{\Delta_3(X)} \end{pmatrix}, \tag{17}$$

*where we introduced notation*

$$\Delta_i(X):=\prod_{j=1,\ j\neq i}^{|X|}(x_j-x_i). \tag{18}$$

*If* $\underline{|X|=\dim V=4,}$ *there exist two inequivalent representations depending on a choice of the square root* $h=\sqrt{e_4(X)}$*:*

$$\begin{aligned} \rho_{h,X}^{(4)}(g_1) &= diag\{x_1,x_2,x_3,x_4\}, \\ \rho_{h,X}^{(4)}(g_2) &= \begin{pmatrix} \frac{\alpha_1}{\Delta_1(X)} & \frac{\beta_1\,\gamma_3\,\gamma_4}{\Delta_1(X)} & \frac{\beta_1\,\gamma_2\,\gamma_4}{\Delta_1(X)} & \frac{\beta_1\,\gamma_2\,\gamma_3}{\Delta_1(X)} \\ \frac{\beta_2}{\Delta_2(X)} & \frac{\alpha_2}{\Delta_2(X)} & \frac{\beta_2\,\gamma_2}{\Delta_2(X)} & \frac{\beta_2\,\gamma_2}{\Delta_2(X)} \\ \frac{\beta_3}{\Delta_3(X)} & \frac{\beta_3\,\gamma_3}{\Delta_3(X)} & \frac{\alpha_3}{\Delta_3(X)} & \frac{\beta_3\,\gamma_3}{\Delta_3(X)} \\ \frac{\beta_4}{\Delta_4(X)} & \frac{\beta_4\,\gamma_4}{\Delta_4(X)} & \frac{\beta_4\,\gamma_4}{\Delta_4(X)} & \frac{\alpha_4}{\Delta_4(X)} \end{pmatrix}. \end{aligned} \tag{19}$$

$$\begin{aligned} \text{Here}\quad \alpha_i(h,X) &:= e_3(X^{\setminus i})\,e_1(X^{\setminus i})\ -\ h\,e_2(X^{\setminus i}),\quad X^{\setminus i}:=X\setminus\{x_i\}, \\ \beta_i(h,X) &:= e_4(X)/x_i^2\ -\ h,\quad i=1,2,3,4, \\ \gamma_a(h,X) &:= x_1x_a\ +\ x_bx_c\ -\ h,\quad a,b,c\in\{x_2,x_3,x_4\}\ \textit{are pairwise distinct.} \end{aligned} \tag{20}$$

*If* $\underline{|X|=\dim V=5,}$ *there exist five inequivalent representations corresponding to different values of the root* $f(X):=\sqrt[5]{e_5(X)}$*:*

$$\rho_{f,X}^{(5)}(g_1)\ =\ diag\{x_1,x_2,x_3,x_4,x_5\},\quad \rho_{f,X}^{(5)}(g_2)\ =\ ||m_{ij}||_{1\le i,j\le 5}, \tag{21}$$

$$m_{ii}(f,X)\ :=\ \frac{e_4(X^{\setminus i})\,e_1(X^{\setminus i})+f(X)\,x_i\,e_3(X^{\setminus i})+f(X)\prod_{k=1,\,k\neq i}^{5}(f(X)+x_k)}{\Delta_i(X)}, \tag{22}$$

$$m_{ij}(f,X)\ :=\ \frac{(x_i^2+f(X)\,x_i+f(X)^2)\prod_{k=1,\,k\neq i,j}^{5}(f(X)^2+x_ix_k)}{f(X)\,x_i\,x_j\,\Delta_i(X)},\quad \forall i\neq j. \tag{23}$$

*If* $\underline{|X|=5,\ \dim V=6,}$ *there exist five inequivalent representations* $\rho_{i,X}^{(6)}$, $i=1,\dots,5$, *corresponding to all admissible values* $C_\rho=-x_ie_5(X)$ *of the central element c. formulae for* $\rho_{5,X}^{(6)}$ *are given in table 1. formulae for the other representations can be obtained by the transposition of the eigenvalues* $x_5$ *and* $x_i$*:* $\rho_{i,X}^{(6)}=\sigma_{i5}\circ\rho_{5,X}^{(6)}$, $i=1\dots4$.

**Remark 1.2.** *As it is noticed above a representation of* $Q_X$ *stays also a representation of* $Q_{X'}$ *if* $X\subset X'$.

Table 1: 6-dimensional representation of $Q_X$, $|X|= 5$.

| | |
|---|---|
| $\rho^{(6)}_{5,X}(g_1) \;=\; \mathrm{diag}\{x_1, x_2, x_3, x_4, x_5, x_5\}\,,$ | $\rho^{(6)}_{5,X}(g_2) \;=\; \lVert g_{ij}\rVert_{1\le i,j\le 6},$ |
| $G := \lVert g_{ij}\rVert_{1\le i,j\le 4}:$ | $g_{ii} = \frac{e_4(X^{\setminus i})e_1(X^{\setminus i}) - x_i x_5 e_3(X^{\setminus i})}{\Delta_i(X)},\; X^{\setminus i} := X \setminus \{x_i\},\; {\scriptstyle i=1,\dots,4};$<br>$g_{1a} \;=\; \frac{p_a\, q_b\, q_c}{x_1^2 \Delta_a(X)}, \quad g_{a1} \;=\; \frac{p_1}{x_a^2 \Delta_1(X)}, \quad g_{ab} \;=\; \frac{q_a\, p_b}{x_a^2 \Delta_b(X)},$<br>where indices $a, b, c \in \{2, 3, 4\}$ are pairwise distinct, and<br>$q_a(X) := x_1 x_a + x_b x_c\,, \quad p_i(X) := e_5(X) - x_i^3 x_5^2\,;$ |
| $G_{31} := \begin{pmatrix} g_{51} & g_{52} \\ g_{61} & g_{62} \end{pmatrix}:$ | $\mathrm{diag}\{\frac{1}{\Delta_1(X)}, \frac{1}{\Delta_2(X)}\};$ |
| $G_{32} := \begin{pmatrix} g_{53} & g_{54} \\ g_{63} & g_{64} \end{pmatrix}:$ | $\begin{pmatrix} q_4\, r & q_3\,(\sigma_{34}\circ r) \\ (\sigma_{12}\circ r) & (\sigma_{12}\sigma_{34}\circ r) \end{pmatrix}$, where $r(X) := \frac{x_3}{x_1(x_2-x_1)\Delta_3(X^{\setminus 2})},$<br>and $\sigma_{ij}\circ f(\dots x_i \dots x_j \dots) := f(\dots x_j \dots x_i \dots)$ for all $f(X)$ |
| $G_{33} := \begin{pmatrix} g_{55} & g_{56} \\ g_{65} & g_{66} \end{pmatrix}:$ | $\begin{pmatrix} u & q_3\, q_4\, v \\ (\sigma_{12}\circ v) & (\sigma_{12}\circ u) \end{pmatrix}$, where $v(X) := \frac{p_2(X)}{x_1 x_5 (x_2-x_1)\Delta_5(X^{\setminus 2})},$<br>and $u(X) := \frac{x_1 x_2 (x_3+x_4)(x_3 x_4 - x_1 x_5) + x_3 x_4 (x_2-x_1)(x_1^2 + x_2 x_5)}{(x_2-x_1)\Delta_5(X^{\setminus 2})};$ |
| $G_{23} := \begin{pmatrix} g_{35} & g_{36} \\ g_{45} & g_{46} \end{pmatrix}:$ | $\frac{1}{x_5 \Delta_5(X)} \begin{pmatrix} \frac{w}{x_3^2} & \frac{q_3\,(\sigma_{12}\circ w)}{x_3^2} \\ \frac{(\sigma_{34}\circ w)}{x_4^2} & \frac{q_4\,(\sigma_{12}\sigma_{34}\circ w)}{x_4^2} \end{pmatrix},$<br>$w(X) := p_1(X)\big(x_1 x_2 x_3 x_4 \{x_1 x_3 + x_5(x_2+x_4)\} - x_5^3\{x_1 x_3 (x_2+x_4) + x_5 x_2 x_4\}\big);$ |
| $G_{13} := \begin{pmatrix} g_{15} & g_{16} \\ g_{25} & g_{26} \end{pmatrix}:$ | $\frac{1}{\Delta_5(X)} \begin{pmatrix} \frac{z}{x_1} & \frac{q_3\, q_4\,(\sigma_{12}\sigma_{23}\circ w)}{x_1^2 x_5} \\ \frac{(\sigma_{23}\circ w)}{x_2^2 x_5} & \frac{(\sigma_{12}\circ z)}{x_2} \end{pmatrix},$<br>$z(X) := (e_1 e_3 - x_1^2 e_2)(x_1 e_1 e_3 - e_2 x_5^3) x_1 x_5 +$<br>$e_3(x_1 - x_5)\big(x_1^2(e_1 - x_1)\{e_3(x_1-x_5) - e_1 x_5^3\} + (x_1 e_2 - e_3)\{x_1 e_2 + (x_1 - x_5)x_5^2\} x_5\big),$<br>where $e_i$ are elementary symmetric polynomials in variables $x_2, x_3, x_4$. |

### 1.1.3 Semisimplicity criteria for algebras $Q_X$

In this subsection we present reducibility conditions for the obtained representations and formulate semisimplisity criteria for algebras $Q_X$.

**Proposition 1.3.** *For the algebras $Q_X$ (7) defined by the set of data $X$ (5) the representations $\rho^{(d)}_{\dots}$, $d \le 5$, described in Proposition 1.2 are irreducible if and only if the following* *conditions on their parameters are satisfied.*

$$\text{For } \rho_X^{(2)}, \quad |X|=2 \quad I_{ij}^{(2)} := x_i^2 - x_i x_j + x_j^2 \neq 0, \quad \text{where indices } {\scriptstyle i,j\in\{1,2\}} \text{ are distinct;} \tag{24}$$

$$\text{For } \rho_X^{(3)}, \quad |X|=3 \quad I_{ijk}^{(3)} := x_i^2 + x_j x_k \neq 0, \quad \text{where } {\scriptstyle i,j,k\in\{1,2,3\}} \text{ are pairwise distinct;} \tag{25}$$

$$\text{For } \rho_{h,X}^{(4)}, \quad |X|=4 \quad I_{h,i}^{(4)} := x_i^2 - h \neq 0,\ J_{h,ijkl}^{(4)} := x_i x_j + x_k x_l - h \neq 0, \quad \text{where } {\scriptstyle i,j,k,l\in\{1,2,3,4\}} \text{ are pairwise distinct;} \tag{26}$$

$$\text{For } \rho_{f,X}^{(5)}, \quad |X|=5 \quad I_{f,i}^{(5)} := x_i^2 + x_i f + f^2 \neq 0,\ J_{f,ij}^{(5)} := x_i x_j + f^2 \neq 0, \quad \text{where } {\scriptstyle i,j\in\{1,2,3,4,5\}} \text{ are pairwise distinct;} \tag{27}$$

*Otherwise, they are reducible but indecomposable.*

*For representations $\rho_{s,X}^{(6)}$, $s = 1, \ldots, 5$, also given in proposition 1.2 we present less detailed statement, which describes conditions under which all of them are irreducible.*

$$\text{If} |X|=5,\ \rho_{s,X}^{(6)}, 1 \leq s \leq 5 \text{ are irreducible if}$$
$$I_i^{(6)} := e_5(X) + x_i^5 \neq 0,\ J_{ij}^{(6)} := e_5(X) - x_i^3 x_j^2 \neq 0,\ K_{i,jklm}^{(6)} := x_j x_k + x_l x_m \neq 0, \tag{28}$$

*where ${\scriptstyle i,j,k,l,m\in\{1,2,3,4,5\}}$ are pairwise distinct. Otherwise, among them there are reducible but indecomposable representations.*

As a direct consequence of Proposition 1.2 and Proposition 1.3 we formulate our main result.

**Theorem 1.1.** *For $|X|\leq 5$ the algebra $Q_X$ (7) defined by a set of data $X$ (5) is semisimple if and only if one of the following conditions holds.*

$$|X|=2: \quad I_{12}^{(2)} \neq 0; \tag{29}$$

$$|X|=3: \quad \{I_{ij}^{(2)}, I_{ijk}^{(3)}\} \cap \{0\} = \emptyset \ \text{ for all pairwise distinct indices } i,j,k \in \{1,2,3\}; \tag{30}$$

$$|X|=4: \quad \{I_{ij}^{(2)}, I_{ijk}^{(3)}, I_{h,i}^{(4)}, J_{h,ijkl}^{(4)}\} \cap \{0\} = \emptyset \tag{31}$$
$$\text{for any } h \text{ such that}: h^2 = e_4(X), \ \text{ and for all pairwise distinct indices } i,j,k,l \in \{1,2,3,4\};$$

$$|X|=5: \quad \{I_{ij}^{(2)}, I_{ijk}^{(3)}, I_{h,i}^{(4)}, J_{h,ijkl}^{(4)}, I_{f,i}^{(5)}, J_{f,ij}^{(5)}, I_i^{(6)}, J_{ij}^{(6)}, K_{i,jklm}^{(6)}\} \cap \{0\} = \emptyset \tag{32}$$
$$\text{for any } h \text{ such that}: f^5 = e_5(X), \quad \forall h: h^2 = e_4(X^{\setminus i}),$$
$$\text{and for all pairwise distinct indices } i,j,k,l,m \in \{1,2,3,4,5\}.$$

*In the semisimple case all irreducible representations of these algebras are described in Proposition 1.2.*

# 2 $q$-boson zero range process

In the second chapter of the thesis we introduce the $q$-boson zero range process and the particle flow as an observable of our interest, sketch necessary information about

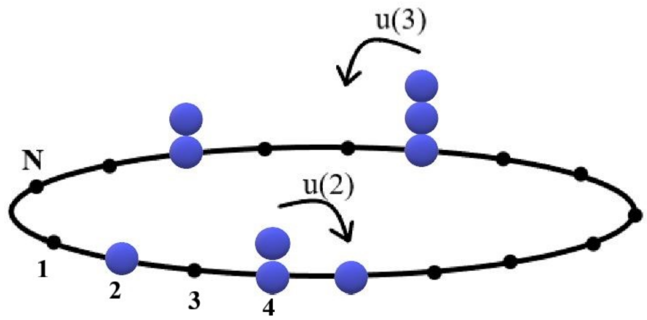


Figure 1: $q$-boson zero range process on a ring with $N$ sites and $p = 9$ particles.

the stationary state and current cumulants and state one of the main the results, exact expression for the diffusion coefficient. Then, we analyze the asymptotic and scaling limit of the obtained exact formulae testing the scaling hypotheses formulated for the models in KPZ and EW universality classes.

## 2.1 The model and its observables

ZRP is a stochastic interacting particle system. We define it on a periodic one dimensional lattice with $N$ sites (sites $i$ and $N+i$ are identical) and $p$ particles. Each lattice site can be occupied by an integer number of particles $n_i \geq 0$. A particle configuration is specified by the set of occupation numbers $\boldsymbol{n} = \{n_1, \dots n_N\}$. The total number of configurations is $C^p_{N+p-1}$.

We consider a continuous time Markov process on the set of particle configurations. Each site has its own Poissonian alarm clock, which rings with rate $u(n_i)$. When the clock rings a particle from site $i$ jumps to the neighbouring site $i+1$ (we imply that $u(0) = 0$). See figure 1.

Let $P_t(\boldsymbol{n})$ be the probability for the system to be in configuration $\boldsymbol{n}$ at time $t$. The probability solves the forward Kolmogorov equation

$$\partial_t P_t(\boldsymbol{n}) = \mathcal{L} P_t(\boldsymbol{n}). \tag{33}$$

Here $\mathcal{L}$ is the operator, whose action on probability is defined by

$$\mathcal{L} P_t(\boldsymbol{n}) = \sum_{\boldsymbol{n}'} (u(\boldsymbol{n}' \to \boldsymbol{n}) P_t(\boldsymbol{n}') - u(\boldsymbol{n} \to \boldsymbol{n}') P_t(\boldsymbol{n})),$$

where the rate $u(\boldsymbol{n}' \to \boldsymbol{n})$ of transition from configuration $\boldsymbol{n}'$ to $\boldsymbol{n}$ is equal to $u(n'_i)$ if the configuration $\boldsymbol{n}$ is obtained from $\boldsymbol{n}'$ by a single jump of a particle from site $i$ the to the site $i+1$ and zero otherwise. In the following we will deal with the particular choice of the rates

$$u(n) = [n]_q = \frac{1-q^n}{1-q},$$

which was shown to be the one necessary for the Bethe ansatz integrability [67]. These rates are positive when $q > -1$. This is the range we consider below.

Having a solution of the master equation corresponding to particular initial conditions one can compute the expectation of any function of configuration at given time. Often one would also like to study the statistics of additive functionals on trajectories of the process, by which we mean a quantity $Y_t$ changing its value by a fixed amount $\delta Y_{\boldsymbol{n}'\to\boldsymbol{n}}$ every time the system jumps from $\boldsymbol{n}'$ to $\boldsymbol{n}$. To this end, one considers the joint probability $P_t(\boldsymbol{n},Y)$ for the configuration to be $\boldsymbol{n}$ and the value of $Y_t$ to be $Y$ at time $t$. Its generating function

$$G_t(\boldsymbol{n},\gamma)=\sum_{Y=0}^{\infty}P_t(\boldsymbol{n},Y)e^{\gamma Y}$$

is a solution of the non-stochastic deformation of (33)

$$\partial_t G_t(\boldsymbol{n},\gamma)=\mathcal{L}_\gamma G_t(\boldsymbol{n},\gamma),$$

where the matrix of the deformed operator $\mathcal{L}_\gamma$ is obtained from that of $\mathcal{L}$ by multiplying every off-diagonal element corresponding to transition from $\boldsymbol{n}'$ to $\boldsymbol{n}$ by $e^{\gamma\delta Y_{\boldsymbol{n}'\to\boldsymbol{n}}}$. We consider a particular example of $Y_t$, the total distance traveled by all particles by time $t$. In this case the increase of $Y_t$ due to jump of a single particle is always $\delta Y_{\boldsymbol{n}'\to\boldsymbol{n}}=1$, so that the action of $\mathcal{L}_\gamma$ is as follows.

$$\mathcal{L}_\gamma G_t(\boldsymbol{n},\boldsymbol{\gamma})=\sum_{\boldsymbol{n}'}(e^{\gamma}u(\boldsymbol{n}'\to\boldsymbol{n},\gamma)G_t(\boldsymbol{n}',\gamma)-u(\boldsymbol{n}\to\boldsymbol{n}')G_t(\boldsymbol{n},\gamma)) \tag{34}$$

The moment generating function of the random variable $Y_t$ is given in terms of $G_t(\boldsymbol{n},\gamma)$.

$$\mathbb{E}e^{\gamma Y_t}=\sum_{\boldsymbol{n}}G_t(\boldsymbol{n},\gamma)$$

The utility of $G_t(\boldsymbol{n},\gamma)$ reveals itself in an observation that in the long time limit its behaviour is dominated by the largest eigenvalue $\lambda(\gamma)$ of matrix $\mathcal{L}_\gamma$,

$$\lambda(\gamma)=\lim_{t\to\infty}\frac{\ln\mathbb{E}e^{\gamma Y_t}}{t}=\sum_{n=1}^{\infty}c_n\frac{\gamma^n}{n!},$$

i.e the function $\lambda(\gamma)$ plays the role of the generating function of scaled cumulants

$$c_n=\lim_{t\to\infty}\frac{\langle Y_t^n\rangle_c}{t}$$

of $Y_t$, where we use notation $\langle\xi^n\rangle_c$ for $n-$th cumulant of the random variable $\xi$. In particular, the first two scaled cumulants, which we deal with below, have a simple physical meaning. The first one

$$J=J(N,p):=c_1=\lambda'(0)$$

is the expected number of particle jumps in the system per unit time, aka mean integral particle current, obtained by time-averaging of the expected total number of jumps made by all particles by time $t$ growing to infinity. The second scaled cumulant is the group diffusion coefficient

$$\Delta=\Delta(N,p):=c_2=\lambda''(0)$$

associated with the joint motion of all particles.

## 2.2 Stationary state and scaled current cumulants

The peculiarity of ZRP is the factorized form of the stationary probability distribution [64], which makes the analysis of the stationary state particularly simple. This is to say that the probability of finding the system in a configuration $\boldsymbol{n}$ is given by a product of one-site weights

$$\boldsymbol{P}_{st}(\boldsymbol{n}) = \frac{\prod_{i=1}^{N} f(n_i)}{Z(N,p)}, \tag{35}$$

where the one-site weight is given by

$$f(m) = \begin{cases} \prod_{j=1}^{m} \frac{1}{u(j)}, & m > 0 \\ 1, & m = 0 \end{cases}, \tag{36}$$

and

$$Z(N,p) = \sum_{\{\boldsymbol{n}: n_1+\cdots+n_N = p\}} \prod_{i=1}^{N} f(n_i). \tag{37}$$

is the normalization factor referred to as the (canonical) partition function. The partition function can be given an integral representation with the use of the generating function of one-site weights

$$F(z) = \sum_{n=0}^{\infty} f(n) z^n.$$

In the case of q-boson ZRP the series $F(z)$ is convergent for $z$ in the disk $|z| < 1/(1-q)$ when $|q| \leq 1$ and for $z$ in the whole complex plain when $|q| > 1$ to infinite products, which give two different q-exponential functions.

$$F(z) = \begin{cases} e_q(z) := (z(1-q);q)_\infty^{-1}, & |q| < 1, \\ E_{1/q}(z) := (z(q^{-1}-1);q^{-1})_\infty, & |q| > 1, \end{cases} \tag{38}$$

where $(z;q)_\infty = \prod_{i=0}^{\infty}(1-zq^i)$. The canonical partition function $Z(N,p)$ has a contour integral representation

$$Z(N,p) = \oint \frac{F^N(z)}{z^{p+1}} \frac{dz}{2\pi i}, \tag{39}$$

and appear further. For example, the integrated particle current calculated from the stationary state analysis is

**Proposition 2.1.**

$$J = \mathbb{E}\left(\sum_{i=1}^{N} u(n_i)\right) = N \frac{Z(N,p-1)}{Z(N,p)}. \tag{40}$$

The second scaled cumulant, aka diffusion coefficient, can not be obtained from the simple stationary state analysis, being the simplest observable that implicitly contains unequal time correlations. To find this quantity we need to address the full dynamical problem. Here we give the final expression, which is one of the main results of the thesis.

**Theorem 2.1.** *The group diffusion coefficient* $\Delta$ *has the following representation*

$$\begin{aligned}\Delta &= pJ + \frac{2N^2}{Z(N,p)^2}\oint \frac{dy}{2\pi i}\frac{F^N(y)}{y^p}\oint\limits_{|y|<|t|}\frac{dt}{2\pi i}\frac{F^N(t)}{t^p}\frac{\phi(y)}{t-y} \\ &+ \frac{2N^2}{Z(N,p)^2}\sum_{i=1}^{\infty}\oint \frac{dt}{2\pi i}\frac{F^N(t)}{t^p}\oint\limits_{|y|<|t|}\frac{dy}{2\pi i}\frac{F^N(y)}{y^p}\frac{q^{\pm i}\phi(yq^{\pm i})+\phi(y)}{t-yq^{\pm i}},\end{aligned} \tag{41}$$

*where plus and minus signs in the powers of* $q$ *correspond to* $|q|<1$ *and* $q>1$ *respectively and the integration contours are two nested simple counterclockwise loops around the origin, which do not contain any other poles. Also we defined the function*

$$\phi(z) = \frac{J}{p}(\ln F(z))' - 1. \tag{42}$$

*where we use notation* $(\ln F(z))' = \partial_z(\ln F(z))$ *for derivative of function* $\ln F(z)$*.*

The formula (41) is valid for $-1 < q \neq 1$.

Of course of physical interest is the behaviour of the cumulants in the thermodynamic limit, in which the notion of universality becomes relevant.

## 2.3 Interface growth and KPZ-EW universality

Exploiting the relation of q-boson ZRP with an interface growth model we discuss the asymptotic limit of the announced exact formulae in context of the KPZ-EW universality. We formulate the scaling hypotheses which can be extracted from the KPZ equation. Then we match this picture with the asymptotic results obtained for the interface associated with the q-boson ZRP.

The $q$-boson zero range process on the periodic lattice can be mapped onto a growing interface on a cylinder $\mathbb{R}\times[0,N]$. For $x\in[0,N]$ and time $t$ we define a piece-wise constant height function $h(x,t)$, which experiences a jump

$$h(x+0,t) - h(x-0,t) = n_x(t) \tag{43}$$

at each integer coordinate $x = 1,\dots N$ and is constant otherwise. Periodicity of the particle system implies helicoidal boundary conditions for the height (see Fig.2)

$$h(x+N,t) = \rho N + h(x,t), \tag{44}$$

where the particle density $\rho = p/N$ plays the role of the mean tilt of the interface. We are interested in the late-time behaviour of the large system, implying that that the limit $t\to\infty$ is taken first. The statistics of the interface height in this limit is dictated by the KPZ-EW universality.

### 2.3.1 KPZ equation and scaling hypotheses

The aim we pursue in the discussion below is to describe a few conjectures made on the basis of the heuristic analysis of the KPZ equation [27], [28], as well as the exact solution of ASEP [41].

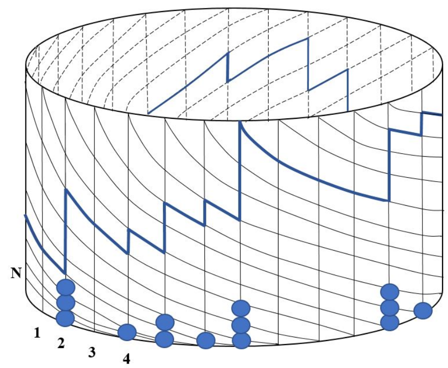


Figure 2: The mapping between configuration $\boldsymbol{n} = (0,3,0,1,2,1,3,0,0,0,3,1,0,\ldots,0)$ and an interface.

Let us consider the KPZ equation for the interface height $h(x,t)$

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial x^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial x}\right)^2 + \eta(x,t), \tag{45}$$

$\eta(x,t)$ is a Gaussian white noise with zero mean and covariance

$$\mathbb{E}\eta(x,t),\eta(x',t') = D\delta(x-x')\delta(t-t'), \tag{46}$$

where $\nu, \lambda$ and $D$ are three parameters of the model. The particular case $\lambda = 0$ is referred to as EW equation. We choose helicoidal boundary conditions $h(x+N,t) = h(x,t) + \rho N$ with the tilt $\rho$ for it to agree with the mapping (43). We are interested in the statistics of interface height function in the large time limit. In this limit the statistics should not depend on initial conditions. Consider the quantity characterizing the fluctuations of the interface height on the cylinder, its dispersion,

$$W^2(N,t) = \left\langle h^2(x,t)\right\rangle_c, \tag{47}$$

with a particular case of the flat initial conditions, which ensure that this quantity do not depend on the coordinate $x$ at any time.

In the thesis we reproduce the arguments of dimensional analysis about the behaviour of the fluctuations in the asymptotic limits, make scaling argument about the interpolation of this quantity between size independent small regime and the diffusive late regime and use the Family-Viseck-like ansatz [83]. These arguments leads us to the following scaling hypotheses for KPZ and EW universality classes.

**Scaling hypothesis 1.** *The late-time behaviour of the fluctuations $W^2(N,t)$ of the interface height in the large system (limit $t \to \infty$ is taken first) in the KPZ class universality is*

$$W^2(N,t) \simeq \kappa_{KPZ}(D/2\nu)^{\frac{3}{2}}|\lambda|N^{-\frac{1}{2}}t, \quad t \gg N^{3/2},\ N \to \infty, \tag{48}$$

*and in the EW class universality*

$$W^2(N,t) \simeq \kappa_{EW}\frac{Dt}{N}, \quad t \gg N^2,\ N \to \infty, \tag{49}$$

*where $\kappa_{KPZ}$ and $\kappa_{EW}$ are the universal dimensionless constants specific for given universality class.*

These constants can be obtained from exact solutions. The latter

$$\kappa_{EW} = 1 \tag{50}$$

follow directly from the EW equation, while the former

$$\kappa_{KPZ} = \frac{\sqrt{\pi}}{4} \tag{51}$$

was first obtained from the exact solution of the TASEP [33] and conjectured to be universal for the whole KPZ universality class.

Note that there are two different scaling functions used in the scaling arguments for EW and KPZ universality classes. The crossover between their late time asymptotics is given in terms of yet another scaling function.

**Scaling hypothesis 2.** *The late-time evolution of height dispersion in the diffusive scale* $t/N^2 \to \infty$ *under the scaling* $\lambda \asymp 1/\sqrt{N}$, $N \to \infty$ *is described by the crossover function so that*

$$\lim_{t\to\infty} \frac{W^2(N,t)}{t} = \frac{D}{N}\mathcal{F}(g,\infty), \quad g = \frac{\lambda^2 DN}{\nu^3} \tag{52}$$

*is given in terms of* $\mathcal{F}(g,\infty)$, *which is conjecturally the universal crossover function. The candidate for this function*

$$\mathcal{F}(g,\infty) = \frac{\sqrt{g}}{2\sqrt{2}} \int_0^{+\infty} \frac{y^2 e^{-y^2}}{\tanh((\sqrt{g}/\sqrt{32})y)} dy. \tag{53}$$

*was first obtained in [41] as a scaling limit of the exact diffusion constant at the weak asymmetry.*

### 2.3.2 Dimensionful invariants and asymptotic results.

We explain how to identify the model parameters with those of the KPZ equation and then test the formulated conjectures against the asymptotic results obtained from the q-boson ZRP.

The way of identification of the model dependent constants in the KPZ universality class was proposed in [28, 73]. It is based on the observation that the parameters $A = D/2\nu$ and $\lambda$ are stable with respect to scale transformation

$$x \to bx, t \to b^z t, h \to b^\zeta h, \tag{54}$$

with $z = 3/2$ and $\zeta = 1/2$, which together with the corresponding transformation of $D, \lambda$ and $\nu$ leaves the KPZ equation invariant. It was then conjectured that the dimensionful model-dependent constants within the universal functions must appear as a combination of these two parameters. It is indeed the case in (48), which suggests

$$W^2(N,t) \simeq \kappa_{KPZ} A^{\frac{3}{2}} |\lambda| N^{-\frac{1}{2}} t, \tag{55}$$

and is conjecturally true for all the systems of KPZ class.

**Statement 2.1.** *For an associated interface corresponding to the q-boson ZRP the non-linearity coefficient is*

$$\lambda = \frac{z^*}{h_2}\left(\frac{1}{|h_2|} - \frac{h_3}{h_2^2}\right) \tag{56}$$

*given in terms of derivatives* $h_k = (z\partial_z)^k h(z)|_{z=z^*}$ *of the function* $h(z) = \ln F(z) - \rho \ln(z)$ *calculated at the critical point* $z^*$ *which is the smallest positive solution of the equation* $z^*(\ln F(z^*))' = \rho$*. The constant* $A$ *is*

$$A = h_2. \tag{57}$$

This statement is a result of identification procedure that includes saddle point approximation of countour integrals.

The first part of the asymptotic analysis which we undertake in the thesis is aimed to the calculation of the diffusion coefficient in the thermodynamic limit.

**Theorem 2.2.** *Diffusion coefficient in the large system size limit*

$$\lim_{N\to\infty} \frac{\Delta}{N^{\frac{3}{2}}} = \frac{\sqrt{\pi}}{8\sqrt{h_2}}\left(\frac{\phi_1 h_3}{|h_2|} - \phi_2\right), \tag{58}$$

*where* $\phi_k = (z\partial_z)^k \phi(z)|_{z=z^*}$ *with* $\phi(z)$ *defined in* (42)*.*

The second part of the asymptotic analysis is devoted to the crossover regime. It corresponds to the scaling, in which the dimensionless variable $g = \lambda^2 D N \nu^{-3}$ from (52) stays finite as $N \to \infty$. This can be realized by taking

$$q = e^{-\frac{\alpha}{\sqrt{N}}}.$$

Then calculation yields

$$\lambda \simeq -\frac{\alpha}{\sqrt{N}}.$$

Thus $\alpha$ varying from zero to infinity brings the system from EW to KPZ universality class. Calculating the dimensional parameters in both classes of universality, we test the second scaling hypothesis.

**Theorem 2.3.**

$$\lim_{N\to\infty} \frac{\Delta}{N} = \rho\, \mathcal{F}(g, \infty), \tag{59}$$

*which matches with the conjectured expression (52) with the universal scaling function* $\mathcal{F}(g,\infty)$ *from (53).*

**Statement 2.2.** *The scaling hypothesis 1 and 2 are true for the associated interface corresponding to the q-boson ZRP.*

## Applications of the results

We indicate several areas in which the results of this thesis are relevant

- the theory of knots,
- representation theory,
- quantum groups theory,
- stochastic integrable processes.

## Publications

The main results of the thesis are presented in two papers:

1. Pyatov P., Trofimova A., *Representations of finite-dimensional quotient algebras of the 3-string braid group*, Moscow Math. J., 2021, **21**, 427-442.

2. A. Trofimova, A. Povolotsky, *Current statistics in the q-boson zero range process*,J. Phys. A: Math. Theor. 2020, **53**, 283003-1–283003-35.

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