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Global dynamics of regular homeomorphisms and topological flows on manifolds

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The study of multidimensional dynamical systems has its own specifics, due to the fact that many methods for studying such systems are based on the approximation of smooth invariant subsets of the system by piecewise linear objects. For example, a topological classification of Morse-Smale diffeomorphisms on a multidimensional sphere [9] was obtained by considering their topological analogues, Morse-Smale homeomorphisms, for which an analog of Smale's theorem [38, theorem 2.3] was proved (detailed proof Smale's theorems can be found, for example, in the monograph [14]).

The transition to the topological category is related to the possible existence of several smooth structures on the same manifold, starting from dimension 4. Initially, this was discovered by J. Milnor in the form of exotic 7-spheres [29]. Moreover, the closures of invariant manifolds of periodic points of a smooth system are often not even topological submanifolds, due to which the dynamics on such subsets is no longer smooth. Numerous examples of such subsets were induced by the work of D. Pixton [34] (see, for example, [42], [25]), as well as by the study of systems with surface dynamics (see, for example, [8], [11]).

Starting from dimension four, so-called exotic manifolds appear that do not allow smooth structures; manifolds that do not allow triangulation, and other features that prevent the use of the technique of studying smooth manifolds. A nonsmooth topological manifold was first demonstrated by Kervaire M. [20] in dimension 10. Thanks to S. Donaldson and M. Friedman [6] it became clear that many simply connected compact topological 4-manifolds do not admit smooth structures. There are examples of topological manifolds on which it is impossible to introduce a smooth structure, but at the same time there are topological flows on them. Such a situation arises, for example, in cases when non-smoothable manifolds admit the existence of a continuous Morse function, which (see, for example, [21]) generates a continuous flow there. A good illustration of this situation is the projective-like manifolds of dimensions 4, 8 and 16, including non-smooth ones, on which J. Ils and N. Cooper [7] constructed topological Morse functions with exactly three critical points.

In connection with the above, the idea of describing the properties of dynamical systems or functions on multidimensional manifolds exclusively in topological terms is quite natural. Moreover, in the absence of a differentiable structure for the phase space, these properties are interpreted as a topological tracing paper of the behavior of a smooth object. Often, topological dynamical systems and continuous functions retain the properties of their smooth counterparts and remain closely related to the topology of the ambient manifold. For example, the concept of a continuous Morse function was introduced back in 1959 in [30], at the same time the validity of Morse's inequalities was proved for it. However, the question of the existence of a continuous Morse function on an arbitrary topological manifold is still an open question. Similarly to its smooth counterpart, a continuous Morse-Bott function is defined, which also retains a close connection with the topology of the carrier space.

The classical definition of a hyperbolic set of a smooth dynamical system uses the decomposition of the tangent bundle into a direct sum of subspaces on which the differential acts in a special way (compresses, stretches). Dynamical systems with a hyperbolic chain-recurrent set consisting of a finite number of orbits with transversally intersecting invariant manifolds are widely known as Morse-Smale systems, and are so named because S. Smale proved the analogs of Morse inequalities [39] for such flows. Topological analogues of Morse-Smale systems with continuous and discrete time were introduced in [27], [10]. Moreover, the question of the existence of such systems on an arbitrary manifold is also open, in contrast to smooth analogues.

In this dissertation work, the concept of regular homeomorphisms and topological flows on topological manifolds is introduced. Regular topological dynamical systems are defined as dynamical systems whose chain-recurrent set is topologically hyperbolic and consists of a finite number of fixed points and periodic orbits. For such systems, the dissertation provides an exhaustive description of the behavior of invariant manifolds of chain components, both from the point of view of asymptotics and from the point of view of the topology of their embedding in the carrier manifold.

Also in the dissertation it is proved that for a regular flow without periodic orbits, given on a topological manifold of any dimension, there exists a (continuous) Morse energy function. The result obtained is an ideological continuation of the work of S. Smale [40], in which he established the existence of a smooth energy Morse function for any gradient-like flow on a manifold, and a partial solution of the Morse problem on the existence of continuous Morse functions on any topological manifolds. Namely, a topological manifold admits a continuous Morse function if and only if it admits a regular topological flow without periodic orbits. This result was obtained in the present work within the framework of constructing a continuous Morse-Bott energy function for an arbitrary continuous regular flow on a topological manifold, and is an analogue of the theorem K. Meyer [28], who in 1968 constructed the Morse-Bott energy function for an arbitrary Morse-Smale flow on a smooth closed n-manifold (see also the review [15] on the construction of energy functions for structurally stable systems).

The global properties of regular homeomorphisms, made it possible to obtain a complete topological classification of some classes of regular homeomorphisms that have classical smooth analogs, studied in the works of E. A. Leontovich, A. G. Mayer, M. M. Peixoto [23], [33], [24]. Namely, in the language of a three-color graph with periodic substitution, a complete topological invariant of gradient-like homeomorphisms of surfaces is described. At the same time, an exhaustive description of the set of admissible graphs is obtained and the implementation problem is solved. An efficient algorithm is also found (the running time of the algorithm has a polynomial dependence on the number of input data) for distinguishing isomorphism classes of allowable three-color graphs. The n-dimensional Cartesian products of regular homeomorphisms of the circle are also classified.

Within the framework of this dissertation work, methods have been developed for studying the dynamics of regular topological dynamical systems, as well as approaches to solving the problem of their classification and constructing energy functions for them. The dissertation consists of an introduction, four chapters, a conclusion and a list of references.

In **Chapter 1** formulated the main results of the work and provided information on approbation of the results of the study.

In **Chapter 2** introduced the notion of a regular dynamical system. Namely, in Section 2.1 regular homeomorphisms f on a topological n-manifold M^n are introduced.

Recall that an *ε-chain of length $m \in \mathbb{N}$* connecting a point x to a point y for a homeomorphism f is called a finite set of points $x = x_0, \dots, x_m = y$ such that $d(f(x_{i-1}), x_i) < \varepsilon$ for $1 \leqslant i \leqslant m$ (see Fig. 1).

An *ε-chain of length T* connecting a point x to a point y for the flow f^t is called a finite set of points $x = x_0, \dots, x_n = y$ which corresponds to a set of times $t_1, \dots, t_n$ such that $d(f^{t_i}(x_{i-1}), x_i) < \varepsilon$, $t_i \geqslant 1$ for $1 \leqslant i \leqslant n$ and $t_1 + \dots + t_n = T$.

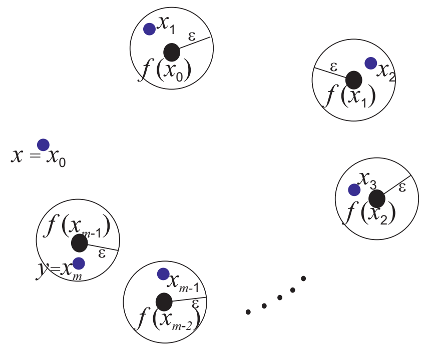

Fig. 1: ε-chain of length $m \in \mathbb{N}$

A point $x \in M^n$ is called *chain recurrent* for a homeomorphism f (a flow f^t) if for any $\varepsilon > 0$ there exists m (T) depending on from $\varepsilon > 0$, and an ε-chain of length m (T) connecting the point x to itself. The set of all chain-recurrent points is called the *chain-recurrent set* and is denoted by $\mathcal{R}_f$ $(\mathcal{R}_{f^t})$. On a chain-recurrent set, one can introduce an equivalence relation by the following rule: $x \sim y$ if for any $\varepsilon > 0$ there are ε-paths connecting x to y and y to x. Then the chain-recurrent set is divided into equivalence classes called *chain components*.

Definition 1. *A fixed point p of a homeomorphism $f : M^n \to M^n$ is called topologically hyperbolic if its neighborhood $U_p \subset M^n$ exists, the numbers $\lambda_p \in \{0, 1, ..., n\}$, μ_p, $\nu_p \in \in \{-1, +1\}$ and homeomorphism $h_p : U_p \to \mathbb{R}^n$ conjugating the homeomorphism $f|_{U_p \cap f^{-1}(U_p)}$ with the linear diffeomorphism $a_{\lambda_p,\mu_p,\nu_p} : \mathbb{R}^n \to \mathbb{R}^n$, given formula*

$$a_{\lambda_p,\mu_p,\nu_p}(x_1, ..., x_{\lambda_p}, x_{\lambda_p+1}, ..., x_n) = (\mu \cdot 2x_1, 2x_2, ..., 2x_{\lambda_p}, \nu \cdot 2^{-1}x_{\lambda_p+1}, 2^{-1}x_{\lambda_p+2}, ..., 2^{-1}x_n).$$

The number λ_p will be called the *Morse index* of the hyperbolic point p. The index points n and 0 will be called *source* and *sink*, respectively; any point p such that $\lambda_p \in \{1, \cdots, n-1\}$ will be called a *saddle point* (see Fig. 2). The topological hyperbolicity of a periodic point p of period $per(p)$ is determined by the hyperbolicity of the point p as a fixed point of the homeomorphism $f^{per(p)}$.

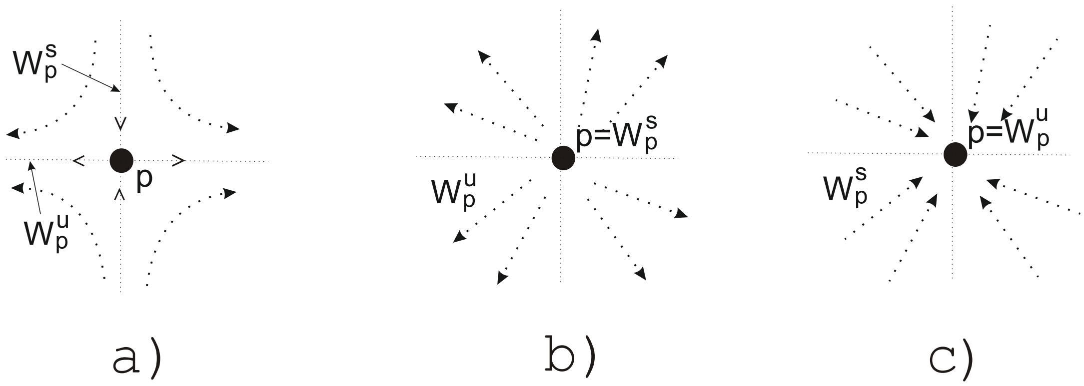

Fig. 2: Dynamics in the vicinity of a topologically hyperbolic fixed point: a) saddle point, b) source point, c) sink point

Definition 2. *A homeomorphism $f : M^n \to M^n$ is called regular if its chain-recurrent set is finite (hence, it consists of a finite number of periodic orbits) and topologically hyperbolic.*

Denote by G the class of regular homeomorphisms.

For a topologically hyperbolic fixed point p of a homeomorphism f of the set

$$W_p^s = \bigcup_{k\in\mathbb{Z}} f^k(h_p^{-1}(E^s_{\lambda_p})),\ W_p^u = \bigcup_{k\in\mathbb{Z}} f^k(h_p^{-1}(E^u_{\lambda_p})),$$

where $E^s_{\lambda_p} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_1 = \cdots = x_{\lambda_p} = 0\}$, $E^u_{\lambda_p} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_{\lambda_p+1} = = \cdots = x_n = 0\}$, we will call *stable and unstable invariant manifolds of the point* p. Invariant manifolds $W_p^s(f)$, $W_p^u(f)$ of the periodic point p with respect to the homeomorphism f coincide with invariant manifolds $W_p^s(f^{per(p)})$, $W_p^u(f^{per(p)})$ of a fixed point p relative to $f^{per(p)}$. For a periodic orbit $\mathcal{O}$ of a regular homeomorphism $f \in G$, we set

$$W^s_{\mathcal{O}} = \bigcup_{p\in\mathcal{O}} W_p^s,\ W^u_{\mathcal{O}} = \bigcup_{p\in\mathcal{O}} W_p^u,\ \ \lambda_{\mathcal{O}} = \lambda_p.$$

For the class G of regular homeomorphisms, are established the basic dynamical properties in Sec. 2.1. In particular, we prove the existence and uniqueness of stable and unstable varieties of periodic points, and the absence of cycles for any homeomorphism $f \in G$ in the sense of the following definition.

On the set of periodic orbits of the homeomorphism $f \in G$, we introduce the S. Smale relation by the condition

$$\mathcal{O}_i \prec \mathcal{O}_j \iff W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset.$$

A *k-cycle* ($k \geqslant 1$) is a set of pairwise distinct periodic orbits $\mathcal{O}_1, \mathcal{O}_2, \cdots, \mathcal{O}_k$ satisfying the condition $\mathcal{O}_1 \prec \mathcal{O}_2 \prec \cdots \prec \mathcal{O}_k \prec \mathcal{O}_1$.

Since the homeomorphisms of the class G have no cycles (see Statement 2.2), Smale's relation on the set of periodic orbits of a regular homeomorphism extends by transitivity to a partial order relation, and, therefore, by Szpilrajn's Theorem [41], can be extended to the set of all periodic orbits to a full order relation. In what follows, we assume that the orbits of the homeomorphism $f \in G$ are numbered, consistent with some fixed order:

$$\mathcal{O}_1 \prec \dots \prec \mathcal{O}_k.$$

The complete ordering of the orbits of a regular homeomorphism allows us to describe its global dynamics as follows.

Theorem 1. ([36]*[1], theorem 1). *Let $f \in G$. Then*

(1) $M^n = \bigcup\limits_{i=1}^{k} W^u_{\mathcal{O}_i} = \bigcup\limits_{i=1}^{k} W^s_{\mathcal{O}_i}$;

(2) each connected component of the manifold $W^u_{\mathcal{O}_i}$ ($W^s_{\mathcal{O}_i}$) is a topological submanifold of M^n, homeomorphic $\mathbb{R}^{\lambda_{\mathcal{O}_i}}$ ($\mathbb{R}^{n-\lambda_{\mathcal{O}_i}}$);

(3) $cl(W^u_{\mathcal{O}_i}) \setminus W^u_{\mathcal{O}_i} \subset \bigcup\limits_{j=1}^{i-1} W^u_{\mathcal{O}_j}$ $\quad$ $(cl(W^s_{\mathcal{O}_i}) \setminus W^s_{\mathcal{O}_i} \subset \bigcup\limits_{j=i+1}^{k} W^s_{\mathcal{O}_j})$.

In Section 2.2 introduced the notion of a regular topological flow.

Recall that a *topological flow* on a manifold M^n is a family of homeomorphisms $f^t : M^n \to M^n$ that depends continuously on $t \in \mathbb{R}$ and has group properties:

1) $f^0(x) = x$ for any point $x \in M^n$;

2) $f^t(f^s(x)) = f^{t+s}(x)$ for any $s, t \in \mathbb{R}$, $x \in M$.

The *trajectory or orbit* of a point $x \in M^n$ with respect to the flow f^t is the set $\mathcal{O}_x = \{f^t(x), t \in \mathbb{R}\}$.

Definition 3. *A fixed point p of a topological flow f^t is said to be topologically hyperbolic if there exists its neighborhood $U_p \subset M^n$, the number $\lambda_p \in \{0, 1, ..., n\}$, and the homeomorphism $h_p : U_p \to \mathbb{R}^n$ conjugating the flow $f^t|_{U_p \cap (f^t)^{-1}(U_p)}$ with the linear flow $a^t_{\lambda_p} : \mathbb{R}^n \to \mathbb{R}^n$, given formula*

$$a^t_{\lambda_p}(x_1, ..., x_{\lambda_p}, x_{\lambda_p+1}, ..., x_n) = (2^t x_1, ..., 2^t x_{\lambda_p}, 2^{-t} x_{\lambda_p+1}, ..., 2^{-t} x_n).$$

Definition 4. *A periodic orbit ℓ of period T_ℓ of a topological flow f^t is said to be topologically hyperbolic if there exists its neighborhood $U_\ell \subset M^n$, numbers $\lambda_\ell \in \{0, 1, ..., n-1\}$, μ_ℓ, $\nu_\ell \in \{-1, +1\}$ and the homeomorphism $h_\ell : U_\ell \to \mathbb{R}^{n-1} \times \mathbb{S}^1$ for $\mu_\ell \nu_\ell = 1$ ($h_\ell : U_\ell \to \mathbb{R}^{n-1} \widetilde{\times} \mathbb{S}^1$ for $\mu_\ell \nu_\ell = -1$)[2] conjugating flow $f^t|_{U_p \cap (f^t)^{-1}(U_p)}$ with superstructure $b^t_{\lambda_\ell, \mu_\ell, \nu_\ell, T_\ell}$ over a linear diffeomorphism $a_{\lambda_\ell, \mu_\ell, \nu_\ell} : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$, given formula*

$$a_{\lambda_\ell, \mu_\ell, \nu_\ell}(x_1, ..., x_{\lambda_\ell}, x_{\lambda_\ell+1}, ..., x_{n-1}) = (\mu_\ell \cdot 2x_1, 2x_2, ..., 2x_{\lambda_\ell}, \nu_\ell \cdot 2^{-1} \cdot x_{\lambda_\ell+1}, 2^{-1} \cdot x_{\lambda_\ell+2}, ..., 2^{-1} \cdot x_{n-1}).$$

[1]Here and below, a star marks works in which one of the co-authors is a dissertation candidate and the results of which are presented in this dissertation.

[2]The notation $\mathbb{R}^{\lambda_\ell} \widetilde{\times} \mathbb{S}^1$ means the skew product of $\mathbb{R}^{\lambda_\ell}$ by $\mathbb{S}^1$.

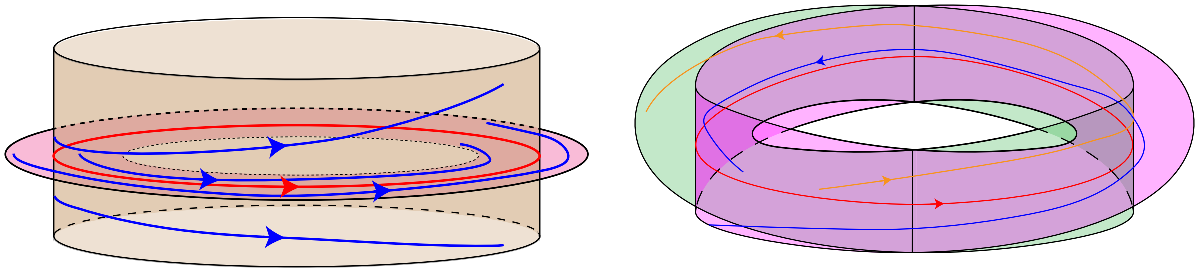

Fig. 3: Dynamics in a neighborhood of a topologically hyperbolic saddle periodic orbit ℓ of a flow on a three-dimensional manifold: (a) $\lambda_\ell = 1$, $\mu_\ell = \nu_\ell = +1$, (b) $\lambda_\ell = 1$, $\mu_\ell = \nu_\ell = -1$

Definition 5. *A topological flow $f^t : M^n \to M^n$ is called regular if its chain-recurrent set consists of a finite number of topologically hyperbolic periodic orbits and fixed points.*

Denote by G^t the class of regular flows on a closed n-manifold M^n. The dynamics of flows of the class G^t is close in its properties to the dynamics of Morse-Smale flows. Namely, if on the set of chain components of the flow $f^t \in G^t$ we introduce the S. Smale relation by the condition

$$\mathcal{O}_i \prec \mathcal{O}_j \iff W^s_{\mathcal{O}_i} \cap W^u_{\mathcal{O}_j} \neq \emptyset,$$

then we can prove the absence of cycles and extend the Smale relation to a relation of full order. In what follows, we assume that the orbits of the flow f^t are numbered in accordance with some fixed order: $\mathcal{O}_1 \prec \cdots \prec \mathcal{O}_k$.

Similarly to theorem 1, the following properties of regular flows are established.

Theorem 2 ([35]*, theorem 1).

Let $f \in G^t$. Then

1. $M^n = \bigcup\limits_{i=1}^{k} W^u_{\mathcal{O}_i} = \bigcup\limits_{i=1}^{k} W^s_{\mathcal{O}_i}$;
2. *unstable (stable) manifold W^u_p (W^s_p) fixed point $\mathcal{O}_i = p$ is a topological submanifold of M^n, homeomorphic to $\mathbb{R}^{\lambda_p}$ ($\mathbb{R}^{n-\lambda_p}$).*
3. *unstable (stable) manifold W^u_ℓ (W^s_ℓ) periodic orbit $\mathcal{O}_i = \ell$ is a topological submanifold of M^n, homeomorphic $\mathbb{R}^{\lambda_\ell} \times \mathbb{S}^1$ ($\mathbb{R}^{n-\lambda_\ell-1} \times \mathbb{S}^1$) for $\mu_\ell = +1$ u $\mathbb{R}^{\lambda_\ell} \widetilde{\times} \mathbb{S}^1$ ($\mathbb{R}^{n-\lambda_\ell-1} \widetilde{\times} \mathbb{S}^1$) for $\mu_\ell = -1$.*
4. $cl(W^u_{\mathcal{O}_i}) \setminus W^u_{\mathcal{O}_i} \subset \bigcup\limits_{j=1}^{i-1} W^u_{\mathcal{O}_j}$ $(cl(W^s_{\mathcal{O}_i}) \setminus W^s_{\mathcal{O}_i} \subset \bigcup\limits_{j=1}^{i-1} W^s_{\mathcal{O}_j})$.

A complete presentation of the results of this chapter is published in the papers [36]* and [35]*.

Chapter 3 proves the existence of a continuous energy function for any regular flow. The results of Chapter 3 are the ideological continuation of the works of S. Smale [40] and K. Meyer [28] on the existence of the Morse energy function for gradient-like flows and the Morse-Bott energy function for Morse-Smale flows, respectively.

Recall that the *Lyapunov function* for a dynamical system is a continuous function that decreases along orbits outside a chain-recurrent set and is a constant on each chain component[3]. By virtue of Conley's results [5], such a function exists for any (including continuous) dynamical system, and the fact of existence itself is called the «fundamental theorem of dynamical systems» (see, [37] Chapter IX, Theorem 1.1). It follows from the definition of a topologically hyperbolic point that any Lyapunov function $\varphi : M^n \to \mathbb{R}$ for a regular flow f^t has critical points on a chain-recurrent set in the sense of the following definition.

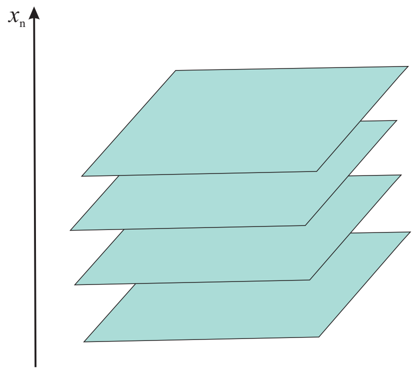

Fig. 4: Level lines of the φ function in a neighborhood of a regular point

Let $\varphi : M \to \mathbb{R}$ be a continuous function. A point $p \in M$ is called a *regular point* of a function φ if there exists a local map $(V_p, \phi_p : y \in V_p \mapsto (x_1(y), \ldots, x_n(y)) \in \mathbb{R}^n$ such that

$$\varphi(y) = \varphi(p) + x_n(y).$$

Otherwise, p is called a *critical point*. Denote by Cr_φ the set of critical points of the function φ. It is natural to expect that the property of strict decrease of the Lyapunov function outside a chain-recurrent set leads to the absence of critical points there. However, this is not true for arbitrary dynamical systems. Therefore, the Lyapunov function whose set of critical points coincides with the chain-recurrent set of the dynamical system is called the *energy function.*

Since the chain-recurrent set of a regular flow is finite and hyperbolic, it is natural to assume that they have an energy function with nondegenerate critical points. Recall that a point $p \in Cr_\varphi$ is called a *non-degenerate critical point of index* $\lambda_p \in \{0, \cdots, n\}$ if there exists a local chart (V_p, ϕ_p) such that

$$\varphi(y) = \varphi(p) - \sum_{i=1}^{\lambda_p} x_i^2(y) + \sum_{i=\lambda_p+1}^{n} x_i^2(y).$$

A function φ is called a *continuous Morse function* if the set Cr_φ consists of non-degenerate critical points.

[3]Outside the chain-recurrent set, the points on the trajectories are ordered by time.

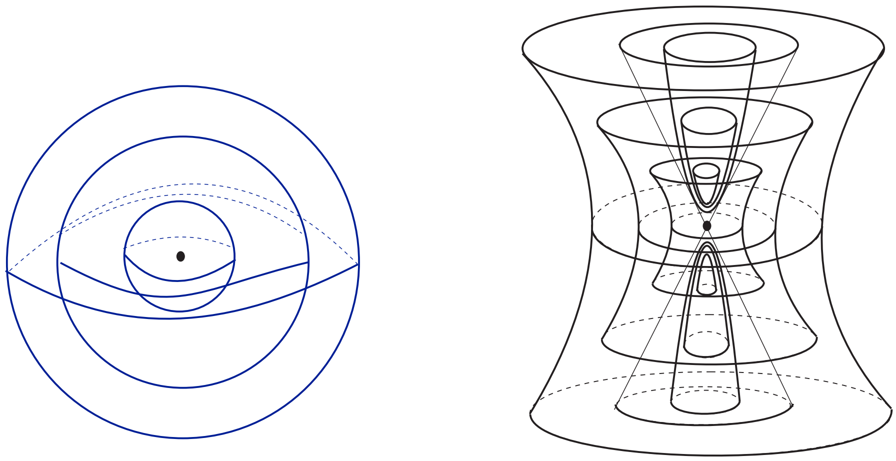

Fig. 5: Level lines of the function φ in a neighborhood of a non-degenerate critical point

A connected topological submanifold $C \subset Cr_\varphi$ of dimension $k \in \{1, \dots, n-1\}$ of M^n is called a *non-degenerate critical k-manifold of index* $\lambda_p \in \{0, \cdots, n-k\}$ if at any point $p \in C$ there exists a local map (V_p, ϕ_p) such that $\phi_p(V_p \cap C) \subset \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_1 = \cdots = x_{n-k} = 0\}$ and

$$\varphi(y) = \varphi(p) - \sum_{i=1}^{\lambda_p} x_i^2(y) + \sum_{i=\lambda_p+1}^{n-k} x_i^2(y).$$

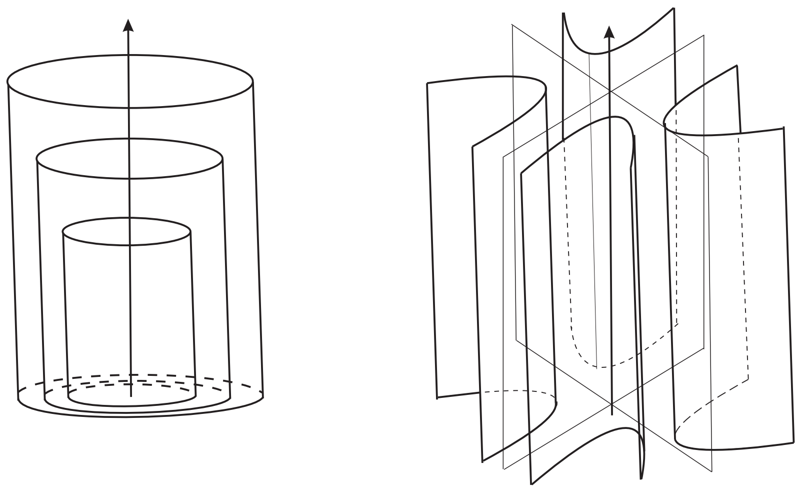

Fig. 6: Level lines of the function φ in a neighborhood of a non-degenerate critical manifold

A function φ is called a *continuous Morse-Bott function* if any connected component of the set Cr_φ is either a non-degenerate critical point or belongs to a non-degenerate critical submanifold.

Statement 3.1 ([26]*, theorem). *Any regular topological flow $f^t : M^n \to M^n$ without periodic orbits has a continuous energy Morse function*[4].

The concept of a continuous Morse function was introduced by Morse back in 1959 in

[4]For $n = 2$ the result follows from [43]*.

[30], at the same time the validity of the Morse inequalities was proved for it, and later (in [21]) a number of properties similar to the properties of the smooth Morse function . However, the question of the existence of a continuous Morse function on an arbitrary topological manifold is still an open question. Since the continuous Morse function generates a topological gradient-like flow on the manifold [21], then Statement 3.1 is a partial solution of the Morse problem: a topological manifold admits a continuous Morse function if and only if it admits a topological flow with a finite hyperbolic chain-recurrent set.

Statement 3.1 follows directly from a more general result in Chapter 3.

Theorem 3. ([35]*, theorem 2). *Any regular flow $f^t \in G^t$ has a continuous energy Morse-Bott function whose critical points are either non-degenerate or form non-degenerate one-dimensional manifolds.*

The existence of an energy function fundamentally distinguishes flows from cascades. For the latter, an obstacle to the construction of the energy function is the possible presence of wild saddle separatrices discovered by D. Pixton [34] in 1977 in dimension three. Examples of regular flows with wild separatrices are also known; such flows are constructed, for example, in recent works by V. Medvedev and E. Zhuzhoma [27]. However, it follows from the results of this paper that for regular flows, the wildness of the separatrices is not an obstacle to the existence of the Morse energy function.

A complete presentation of the results of this chapter is published in the papers [35]*, [26]*, [43]*.

In Chapter 4 a topological classification of some meaningful classes of regular homeomorphisms is obtained. Namely, in section 4.1. we introduce gradient-like homeomorphisms, regular homeomorphisms $f : M^2 \to M^2$, whose invariant manifolds of different saddle points do not intersect. Denote by Q the class of such homeomorphisms.

The dynamics of a gradient-like surface diffeomorphism is closely related to the dynamics of a gradient-like flow, since it differs from it by multiplying by a periodic transformation (see, for example, [4], [14]). The dynamics of gradient-like flows has historically been studied by selecting *cells* – regions with the same asymptotic behavior of trajectories [33], [1], [23], [2], [3].

In Chapter 4 it is proved that the cells of gradient-like homeomorphisms have the same types as the cells of Leontovich-Mayer-Peixoto [23], [33]. Further, these cells are subdivided into triangular regions with uniform dynamic behavior, similar to Oshemkov-Sharko atoms [31]. Each homeomorphism $f \in Q$ is associated with a three-color graph T_f (see Fig.7), whose vertex set is isomorphic to the set of triangular regions, and whose edge set is isomorphic to the set of boundaries of these regions. The graph is equipped with the automorphism P_f induced by the dynamics of the homeomorphism on the cells. The following theorem is proved.

Theorem 4. ([12]*, Theorem 1). *Homeomorphisms f, f' from the class Q are topologically conjugate if and only if the graphs $(T_f; P_f)$, $(T_{f'}; P_{f'})$ are isomorphic .*

To solve the implementation problem, a set of admissible three-color graphs (T, P) is

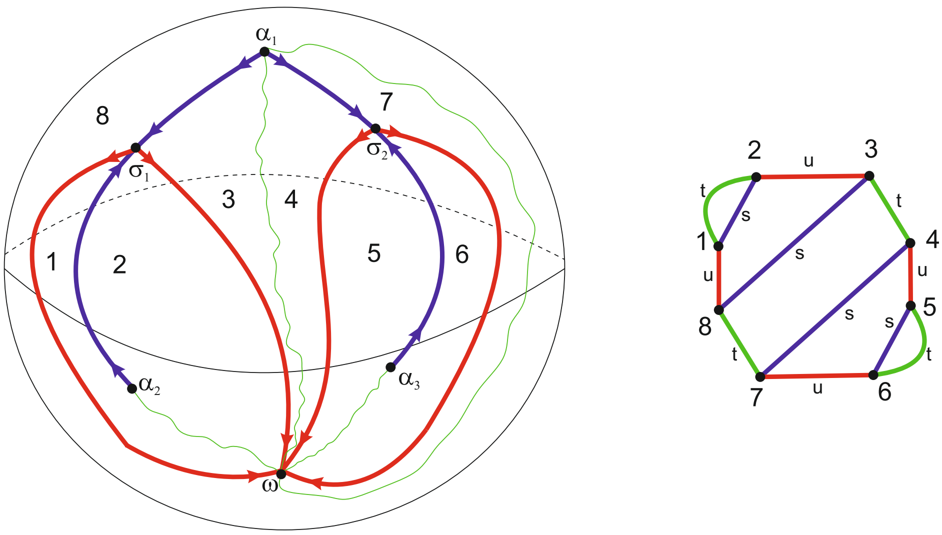

Fig. 7: Homeomorphism of the class Q and the corresponding three-color graph

singled out, for each of which a procedure for implementing a gradient-like homeomorphism on it is described.

Theorem 5. ([12]*, Proposition 3). *For any admissible graph* (T, P)*, there exists a homeomorphism* $f : M^2 \to M^2$ *from the class* Q *whose graph* (T_f, P_f) *is isomorphic to the graph* (T, P).

An efficient algorithm for distinguishing isomorphism classes of admissible three-color graphs is also found.

Theorem 6. ([13]*, theorem 7). *An isomorphism of two three-color n-vertex graphs* T_f*,* $T_{f'}$ *homeomorphisms* $f, f' \in Q$ *can be recognized in time*

$$O(n^3 log(n)).$$

Moreover, the orientability and genus of the supporting surface M^2 *can be determined in linear time from the number of vertices of the three-color graph* T_f.

Chapter 4 also considers the class $\mathcal{H}_n$ of ϕ homeomorphisms that are Cartesian products of n regular circle homeomorphisms

$$\phi = \phi_1 \times \cdots \times \phi_n,\ \phi_i : S^1 \to S^1.$$

Regular homeomorphisms of the circle are a topological generalization of rough transformations of the circle, which were exhaustively studied by A. G. Mayer in [24]. Thus, orientation-preserving transformations are a composition of regular homeomorphisms with fixed points and rotations through a rational angle, while transformations that change are compositions of regular homeomorphisms and orientation-changing involutions.

Homeomorphisms of the considered class $\mathcal{H}_n$ are regular homeomorphisms of the n-dimensional torus $\mathbb{T}^n$. One of the main results of Chapter 3 is finding necessary and sufficient conditions for topological conjugacy of homeomorphisms $\phi,\ \phi' \in \mathcal{H}_n$.

Theorem 7. ([16]*, Theorem 1). *Homeomorphisms* $\phi = \phi_1 \times \dots \times \phi_n$, $\phi' = \phi'_1 \times \dots \times \phi'_n \in \mathcal{H}_n$ *are topologically conjugate if and only if when there is a substitution* $\eta = \begin{pmatrix} 1 & 2 & \dots & n \\ \eta_1 & \eta_2 & \dots & \eta_n \end{pmatrix}$ *on the index set* $\{1, 2, \dots, n\}$, $\eta(i) = \eta_i$, *such that the homeomorphisms* ϕ_i *and* ϕ'_{η_i} *are topologically conjugate for* $i = 1, \dots, n$.[5]

Note that if we consider n-multiple Cartesian products of rotations by a rational number on the circle, as was done in [22], then the period of their periodic points is a complete invariant under the topological conjugation of such homeomorphisms. The case of the topological classification of products of regular homeomorphisms of the circle differs essentially from the results presented in the above paper.

A full presentation of the results of this chapter is published in the papers [18]*, [16]*, [12]*, [13]*.

Conclusion. This dissertation is devoted to the study of the dynamics of regular homeomorphisms and topological flows, as well as the topological classification and construction of energy functions for such systems. All the results obtained in the dissertation are new and the author of the dissertation owns the proofs of all the main results of the work submitted for defense.

- A class of regular dynamical systems is introduced, the dynamics of regular homeomorphisms (theorem 1) and topological flows (theorem 2) is studied, including

  - representation of the ambient manifold as a union of invariant manifolds of fixed points and periodic orbits;

  - description of the topology of embedding invariant manifolds of fixed points and periodic orbits in the ambient manifold;

  - description of the asymptotic behavior of invariant manifolds of fixed points and periodic orbits.

- For regular topological flows without periodic orbits, a constructive proof of the existence of a continuous energy Morse function is obtained (statement 3.1).

- For arbitrary regular topological flows, we prove the existence of a continuous energy Morse-Bott function whose critical points are either non-degenerate or form non-degenerate one-dimensional manifolds. (theorem 3).

- A complete topological classification is obtained for the following meaningful classes of regular homeomorphisms

  - gradient-like homeomorphisms of surfaces, including the construction of a combinatorial invariant, which is a three-color graph with periodic substitution and the proof of the topological conjugacy criterion by means of graph isomorphism

[5] For $n = 2$ the result follows from [18]*.

(theorem 4); selection of a class of admissible three-color graphs with periodic substitutions with the help of which the implementation problem is solved (theorem 5); finding an efficient algorithm for distinguishing isomorphism classes of admissible three-color graphs (theorem 6).

- n-multiple Cartesian products of regular circle homeomorphisms (theorem 7).

The research results are contained in 6 articles published in publications included in the Scopus and Web of Science:

1. V. Z. Grines, S. Kh. Kapkaeva, O. V. Pochinka A three-colour graph as a complete topological invariant for gradient-like diffeomorphisms of surfaces // Sb. Math. - 2014. - Vol. 205, No.10. - P. 1387–1412.
2. V. Z. Grines, D. S. Malyshev, O. V. Pochinka, S. Kh. Zinina Efficient Algorithms for the Recognition of Topologically Conjugate Gradient-like Diffeomorhisms // Regular and Chaotic Dynamics. - 2016. - Vol. 21, No. 2. - P. 189–203.
3. O. V. Pochinka, S. Kh. Zinina A Morse Energy Function for Topological Flows with Finite Hyperbolic Chain Recurrent Sets // Math. Notes. - Vol. 107, No. 2. - P. 313–321
4. T. V. Medvedev, O. V. Pochinka, S. Kh. Zinina On existence of Morse energy function for topological flow // Advances in Mathematics. - 2021. - Vol. 378. - 107518.
5. O. V. Pochinka, S. Kh. Zinina Construction of the Morse–Bott Energy Function for Regular Topological Flows // Regular and Chaotic Dynamics. - 2021. - Vol. 26, No. 4. - P. 350–369.
6. I. V. Golikova, S. Kh. Zinina Topological conjugacy of n-multiple Cartesian products of circle rough transformations // Izvestiya Vysshikh Uchebnykh Zavedenii. Applied Nonlinear Dynamics - 2021. - Vol. 29, No. 6. - P. 851–862.

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