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The transversal value: stable coalition structures in the workgroup formation game and the game of chairpersons

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The transversal value: stable coalition structures in the workgroup formation game and the game of chairpersons

This article investigates the existence of a stable coalition structure in coalition partition games dealing with a system of distinct representatives. The system of distinct representatives, or the transversal, is a coalition of agents whose members represent different groups. An agent gains utility if he/she belongs to a transversal. The transversal value of a cooperative game with coalition structure is introduced to solve the coalition formation problem. It is demonstrated that any cooperative game has a partition that is simultaneously Nash stable, permutation stable, and has a total payoff maximization for the transversal value. The transversal value expresses the players' payoff functions in the workgroup formation game and the game of chairpersons. A player's payoff in the workgroup formation game is a payment for the projects the player participated in. In the game of chairpersons, each player is interested in being their group's leader. It is demonstrated that in such games there exist punctually stable coalition structures.

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1 Introduction

1.1 Transversal problems

Many actors in the economy, management, sociology, and other spheres have qualifications. Agents develop in a certain direction to become specialists. To be able to implement some projects or assignments, however, representatives of different qualifications or groups have to form coalitions by collaborating. Each coalition member performs the job he/she specializes in. Then, this system of distinct representatives gets a certain utility. The system of distinct representatives, or the transversal, is a coalition of agents from different groups.

Suppose we have two groups: theorists and practitioners. Working together, a theorist and a practitioner can come up with an application for a theory. A coalition of a theorist and a practitioner is a transversal. A buyer and seller, and a sender and receiver are also examples of transversals. An agent gains utility only if he/she belongs to a transversal.

Formally, let N be a non-empty finite set of agents and $\pi = \{B_1, B_2, ..., B_l\}$ be a partition of the agents into disjoint non-empty groups. Then, the transversal of partition π is a coalition $K = \{t_1, t_2, ..., t_l\}$, where $t_j \in B_j \; \forall j \in \{1, 2, ..., l\}$. Consider two situations where transversals are important:

1. The formation of workgroups. Let N be the set of workers and l be the number of types of jobs to be fulfilled to implement one project. The set of workers is to be portioned into l groups so that one type of job is associated with each group. Then, any transversal of the resulting partition is a minimal coalition of workers able to implement the project.

2. The appointment of chairpersons. An institution needs to form l commissions, but the same person cannot be member of two commissions at a time. Chairpersons are appointed from among commission members. In this case, any coalition of chairpersons is a transversal of the partition.

Workgroup formation can result in any partition into l non-empty coalitions, i.e. $|\pi| = l$. In the following, workers form transversals and derive some utility thereby. Each worker is interested in getting as much profit from the eventual partition as possible. Before transversals are formed, workers can move between coalitions of the partition, meaning that a worker can shift from one type of job to another. In this case, the question arises of whether a stable coalition structure exists. A similar issue appears in the problem of chairpersons, because each agent is interested in chairing the respective commission.

The problem of forming working groups and the problem of appointing chairpersons differ from each other in the context, but both boil down to solving transversal problems.

1.2 A game-theoretic approach

The utility of a coalition in this paper is determined using cooperative game theory. The transversal value is introduced for cooperative games with coalition structure. The transversal value of player $i, i \in N$ for a given partition π is the sum of the values of partition π, the transversals of which player i is a member. Coalition values are determined by the characteristic function $v, v : 2^N \to \mathbb{R}$, which is independent of partition π. A feature of the transversal value of player i in partition π is that it does not depend on other players in the coalition that player i is a member of. This is a significant distinction from many other game-theoretic models.

The article introduces the workgroup formation game and the game of chairpersons. They are novel games of coalition partition, where players' payoffs are expressed through transversal values.

Players in the workgroup formation game are the firm owner and workers. Workers form a coalition partition which excludes the firm owner. The characteristic function v indicates the number of projects that can be implemented by the coalition $K, K \subseteq N$. When the partition is formed, workers form transversals to implement the projects. The firm owner gets a certain amount of money from each project implemented. The owner then pays the workers for the completed projects in which they participated. A Pareto-Nash-stable coalition structure is shown to exist in the workgroup formation game. This means that workers can be split into a fixed number of groups so that none of them will want to individually move to other groups and the owner's payoff will be maximized.

The characteristic function of the game of chairpersons represents *a priori* probabilities that a certain coalition is a coalition of chairpersons. Each player is interested in a partition in which his/her odds of becoming a chairperson are high. A player's payoff in the game of chairpersons is the probability of becoming the leader in his/her coalition, given that partition π is formed. We are thus dealing with a random variable that takes value 1 or 0. In this case, the expected value of such a random variable is the probability of being a leader for partition π. A Nash-stable coalition structure is shown to exist in the game of chairpersons.

Players' payoff functions in the workgroup formation game and the game of chairpersons are expressed in a special manner through the transversal value and its potential function. Hence, properties of the transversal value are applied to the players' payoff functions in these games.

1.3 Literature review

To the best of the author knowledge, the transversal meaning has not been investigated in the literature. The literature review below focuses on coalition formation and related problems.

The problem of forming coalitions is studied in many spheres of human activity which have specific features.

Economic actors often cooperate with each other to achieve their goals. A model for coalition formation in production was suggested in [1]. First, a collaborative agreement is signed between firms, and then a game in Cournot form takes place between the coalitions. It is demonstrated that there exists a market equilibrium. The formation of dynamic alliances of suppliers in a decentralized assembly system is proposed in [2].

In political science models, each agent is interested in belonging to the party that represents his/her interests and can provide benefits to society. Parties compete for power, therefore coalition formation is an important issue. [3] investigates the effect of lobbying coalitions on financial institutions. Relationships between agents, which can be described by using a graph, should be taken into account when dealing with coalition formation. For example, a new type of stability suggested in [4] takes intermediate payoffs and cautiousness into consideration. The existence of a profile of strategies with this type of stability was investigated. Models for bargaining in networks were suggested in [5, 6].

An interesting question in coalition formation studies is the existence of a stable partition. Types of stability are determined in different ways and depend on the problem statement.

Internal and external stability were defined in [7]. A coalition that is both externally and internally stable is called a von Neumann–Morgenstern (vNM) stable set. [8] expands the vNM concept to the case taking the history of the game into account. Myopia and farsightedness properties are introduced in [9] for players in games on graphs, where the existence of a vNM stable set was studied. The present paper demonstrates that any partition is externally stable with respect to the transversal value if the characteristic function values are non-negative. Following [10], we also assume a fixed number of coalitions in a coalition partition. Contrary to [10], this study shows that an expansion of a coalition is not beneficial for the players therein if a player's payoff is the transversal value.

Hedonic games are game-theoretic models of partition formation in which the payoff of a player depends on the coalition he/she belongs to. The existence of a stable partition in symmetric additively-separable games was studied in [11]. Core stability in a hedonic game with claims was suggested in [12]. Methods for analyzing the existence and uniqueness of the core in coalition partition games were proposed in [13]. The transversal value of a player, considered in this paper, does not depend on other players of the same coalition. This is a major distinction from hedonic game studies.

Players' payoffs can depend not only on their coalition but also on the partition in general. For example, [14] implies that the payoff of a player depends on the status of his/her coalition in the partition and the player's own position in the coalition. The effect of the player's status in the organization was studied in [15]. Players' payoffs in the game-theoretic models examined here also depend on the entire partition. Nash stability is analyzed here using the theory of potential functions. Potential functions for normal-form games were introduced in [16] and defined on a set of players' strategy profiles. The potential function in the present paper was determined on a set of coalition structures. This approach was used in [17] to prove the existence of a stable partition for the Aumann-Dréze value. In [18], potential functions were investigated to study the internal stability of coalitions. The present study employed the potential function methods from [19] and [17].

1.4 The contribution of the paper

The study produced the following results:

- We prove that any cooperative game has a coalition structure that is simultaneously Nash stable and permutation stable, and has a total payoff maximization (TPM) for the transversal value. If values of the characteristic function are non-negative, the coalition structure with these three types of stability is also externally stable (Theorem 1).

- Punctual stability is introduced for the coalition structures that maximize the potential function. This type of stability is studied for the transversal value (Theorem 2).

- A Pareto-Nash-stable coalition structure is proved to exist in the workgroup formation game (Theorem 3).

- The game of chairpersons is shown to be an ordinal potential game (Theorem 4).

- Punctual stability is studied for the game of chairpersons (Theorem 5).

In Theorem 1, the existence of a coalition structure that is simultaneously Nash stable and permutation stable is guaranteed by the fact that the potential functions for the different types of stability coincide. A similar result can be found in [19]. [19] examined games where a player's payoff depended on his/her coalition, whereas the transversal value did not have this property. Game-theoretic models from [19] and in the present paper do not follow from one another. Furthermore, there exists a coalition structure with two types of stability which is TPM for the transversal value. Where values of the characteristic function are non-negative, there is one more type of stability. No such result is found in [19].

The theory of potential functions was developed to prove the existence of a pure Nash equilibrium. What other properties can be found in an equilibrium that maximizes the potential function? This paper demonstrates that a partition that maximizes the potential function of the transversal value possesses punctual stability (Theorem 2). This stability means that any player benefits more from participating in the partition formation from the start rather than from joining the most personally profitable coalition afterwards. We elaborate on the application of the transversal value to problems of workgroup formation and the appointment of chairpersons. The corresponding coalition partition games are introduced and stable coalition structures are proved to exist (Theorems 3-5). To deal with the these problems, agents are advised to form a stable partition.

The article is structured as follows. The second section introduces the key notations and describes the types of stability. The third section gives the definition of the transversal value of a cooperative game with coalition structure. The main theorem about the existence of a stable coalition structure is proved, and punctual stability is introduced for potential games. Sections 4 and 5 describe the application of the transversal value to the problems of forming workgroups and appointing a chairperson.

2 Key notations and definitions

2.1 Coalition partition game

Let $N = \{1, 2, ..., n\}$ be a finite set of players and 2^N be the set composed of all subsets of the set N. The pair (N, v) is called a cooperative game where $v : 2^N \to \mathbb{R}, v(\emptyset) = 0$.

Denote by $\Pi(N)$ the set that consists of all partitions of the set N,

$$\Pi(N) = \left\{ \{B_1, B_2, ..., B_l\} \,\middle|\, \bigcup_{j=1}^{l} B_j = N, B_j \cap B_g = \emptyset, B_j \neq \emptyset, 1 \leq j < g \leq l \right\}.$$

Let $\pi \in \Pi(N)$. $B(i)$ is the element of coalition partition π in which player i is contained. A single-player coalition of π is coalition B, for which $B \in \pi$ and $|B| = 1$.

The pair (N, H) is a coalition partition game where N is the set of players, $H : \Pi(N) \to \mathbb{R}^n$. In game (N, H) each player $i, i \in N$ in any coalition partition $\pi, \pi \in \Pi(N)$ is assigned the number $H_i(\pi)$. This number is called the value or the payoff of player i in coalition structure π.

Let $M(\pi)$ be a set consisting of all the transversals of coalition structure π. Denote by $M_i(\pi), i \in N$ the set of the transversals of coalition structure π which contain player i.

2.2 Stability concepts

This subsection defines some stability types for a coalition partition game (N, H). Transversal-related problems usually assume that the number of groups in partition π is fixed and equal to l, i.e. $|\pi| = l$. The stability types below take this restriction into account.

Definition 1. *Coalition structure $\pi = \{B_1, B_2, ..., B_l\}$ is called externally stable in (N, H) if $\forall j \in \{1, 2, ..., l\}, \forall i \in B_j$ the following inequalities hold:*

$$H_i(\pi) \geq H_i(\{B_1 \setminus K, ..., B_{j-1} \setminus K, B_j \cup K, B_{j+1} \setminus K, ..., B_l \setminus K\}),$$

$$\forall K \subseteq N \setminus B_j \text{ and } |B_g \setminus K| \geq 1 \ \forall g \in \{1, 2, ..., l\} \setminus \{j\}.$$

Suppose π is an externally stable coalition structure. If coalition $B, B \in \pi$ is joined by players from other coalitions, then the payoff of each original player from B will not increase.

Next, we write the definition of a Nash-stable coalition partition. Let $A, B \in \pi, A \neq B$. Denote $\pi_{-A} = \pi \setminus \{A\}, \pi_{-A,B} = \pi \setminus \{A, B\}$. Introduce the sets $D_i(\pi), i \in N$ consisting of coalition structures:

$$D_i(\pi) = \{\pi\} \cup \{\{B(i) \setminus \{i\}, A \cup \{i\}, \pi_{-B(i),A}\} | B(i) \neq \{i\}, A \in \pi_{-B(i)}\}.$$

Definition 2. *Coalition structure π is called Nash-stable in (N, H) if $\forall i \in N$:*

$$H_i(\pi) - H_i(\rho) \geq 0 \ \forall \rho \in D_i(\pi).$$

Let π be a Nash-stable coalition structure. In this case, it is detrimental for each player to leave their coalition alone. The set $D_i(\pi)$ consists of π and the coalition structures that appear when player i moves to another coalition. In the following, the situations will be analyzed where the transversals of the coalition partition are important. The number of coalitions in such problems is fixed. The set $D_i(\pi)$ is therefore designed so as to prohibit individual transfers of players that will change the number of coalitions in the coalition partition. Thus, $\forall \pi \in \Pi(N), \ \forall \rho \in D_i(\pi) : |\pi| = |\rho|$.

In order to present the definition of the third type of stability, some notations need to be introduced. Denote

$$L(\pi) = \{\{i, j\} | B(i) \neq B(j) \ \forall i, j \in N\}.$$

Let $K \in L(\pi), K = \{i, j\}, \pi = \{B(i), B(j), ..., \pi_{-B(i),B(j)}\}$. Form the coalition structure $\rho = \rho(\pi, K)$ derived from π as follows

$$\rho = \rho(\pi, K) = \{B(i) \setminus \{i\} \cup \{j\}, B(j) \setminus \{j\} \cup \{i\}, \pi_{-B(i),B(j)}\}.$$

Coalition structure $\rho = \rho(\pi, K)$ is obtained from π in the following manner. Two players, i and j, are selected from different coalitions of coalition partition π. The two selected players form the coalition $K = \{i, j\}$. Then, players from K swap positions in partition π.

Definition 3. *Coalition structure π is a permutation-stable coalition structure in the game (N, H) if $\forall K \in L(\pi)$:*

$$\sum_{i \in K} H_i(\pi) - \sum_{i \in K} H_i(\rho) \geq 0, \ \textit{where } \rho = \rho(\pi, K).$$

Let π be a permutation-stable coalition structure. If two players from different groups of coalition structure π swap places this will not increase their total payoff.

As stated above, the number of coalitions in transversal-related problems is fixed. Hence, set $\Pi(N)$ can be mapped as a union of pairwise non-intersecting sets $X_1(N) \cup X_2(N) \cup ... \cup X_n(N)$, where

$$X_l(N) = \{\pi | \pi \in \Pi(N), |\pi| = l\}, l = 1, 2, ..., n.$$

Any partition from $X_l(N)$ contains l coalitions. If $l = 1$, then $X_1(N) = \{\{1, 2, ..., n\}\}$. If $l = n$, then $X_n(N) = \{\{1\}, \{2\}, ..., \{n\}\}$. Since sets $X_l, X_g, 1 \leq l < g \leq n$ do not intersect, the question of whether a stable coalition structure exists can be studied on sets $X_l(N), l = 2, ..., n-1$ rather than on the entire set of coalition partitions.

Definition 4. *Coalition structure π is a TPM coalition structure on the set $X_l(N), l = 2, ..., n-1$ in the game (N, H) if $\pi \in X_l(N)$ and*

$$\sum_{i \in N} H_i(\pi) - \sum_{i \in N} H_i(\rho) \geq 0 \ \forall \rho \in X_l(N).$$

We exclude the cases when $l = 1$ and $l = n$ in Definition 4 since $|X_1| = |X_n| = 1$. For TPM coalition structure π, the sum of payoffs of players is maximum.

We have defined some stability types that will be of interest to us. To analyze the existence of a Nash-stable and permutation-stable coalition structure, let us define the corresponding potential games.

Definition 5. *The game (N, H) is called a potential game if there is a function $P : \Pi(N) \to \mathbb{R}$ for which $\forall i \in N, \forall \pi \in \Pi(N)$ the following equality holds:*

$$H_i(\pi) - H_i(\rho) = P(\pi) - P(\rho) \ \forall \rho \in D_i(\pi).$$

Definition 6. *The game (N, H) is called an ordinal potential game if there is a function $P : \Pi(N) \to \mathbb{R}$ for which $\forall i \in N, \forall \pi \in \Pi(N)$ the following inequality holds:*

$$H_i(\pi) - H_i(\rho) > 0 \text{ iff } P(\pi) - P(\rho) > 0 \ \forall \rho \in D_i(\pi).$$

Definition 7. *The game (N, H) is called a permutable potential game if there is a function $R : \Pi(N) \to \mathbb{R}$, and $\forall \pi \in \Pi(N)$, $\forall K \in T(\pi)$ the following equality holds:*

$$\sum_{l \in K} H_l(\pi) - \sum_{l \in K} H_l(\rho) = R(\pi) - R(\rho), \text{ where } \rho = \rho(\pi, K).$$

Potential functions are usually defined on the set of player strategy profiles. In this paper, potential functions are defined on the set of coalition structures. The existence of a potential function from definitions 5, 6, or 7 guarantees that there exists a coalition partition with the respective type of stability. Let us assume that the coalition partition game meets definition 5, 6, or 7 and that we seek to find a stable coalition structure on the set $X_l(N), l = 2, ..., n-1$. Then, the coalition structure that maximizes the potential function on the set $X_l(N)$ is stable.

3 The transversal value and the stability of coalition structures

3.1 The main definition

This subsection introduces the transversal value for a cooperative game with coalition structure and describes some of its properties.

Definition 8. *The transversal value of player i in a cooperative game (N, v) with coalition structure π is*

$$T_i(N, v, \pi) = \sum_{K \in M_i(\pi)} v(K) \ \forall i \in N.$$

The number $T_i(N, v, \pi), i \in N$ is the sum of the values of the coalitions K that are transversals of partition π and $i \in K$.

Example 1. Let (N, v) be a cooperative game with coalition structure π, where $N = \{1, 2, ..., 7\}, \pi = \{\{1, 2, 3\}, \{4, 5\}, \{6, 7\}\}$. The transversal values of the players have the form:
if $i \in \{1, 2, 3\}$, then

$$T_i(N, v, \pi) = v(\{i, 4, 6\}) + v(\{i, 4, 7\}) + v(\{i, 5, 6\}) + v(\{i, 5, 7\});$$

if $i \in \{4, 5\}$, then

$$T_i(N, v, \pi) = v(\{1, i, 6\}) + v(\{2, i, 6\}) + v(\{3, i, 6\})$$
$$+v(\{1, i, 7\}) + v(\{2, i, 7\}) + v(\{3, i, 7\});$$

finally, if $i \in \{6, 7\}$, then

$$T_i(N, v, \pi) = v(\{1, 4, i\}) + v(\{1, 5, i\}) + v(\{2, 4, i\})$$
$$+v(\{2, 5, i\}) + v(\{3, 4, i\}) + v(\{3, 5, i\}).$$

Note that the transversal values of players from the same coalition are different, for example:

$$T_1(N, v, \pi) \neq T_2(N, v, \pi)$$
$$\Leftrightarrow v(\{1, 4, 6\}) + v(\{1, 4, 7\}) + v(\{1, 5, 6\}) + v(\{1, 5, 7\})$$
$$\neq v(\{2, 4, 6\}) + v(\{2, 4, 7\}) + v(\{2, 5, 6\}) + v(\{2, 5, 7\}).$$

Many classical values of cooperative games with coalition structure depend on the contributions of players to the coalition, i.e., on expressions of the form $v(K) - v(K \setminus \{i\}), i \in N, K \subseteq N$. Examples are the Owen value or the Aumann-Dréze value. The $v(K) - v(K \setminus \{i\})$ difference may not have a physical meaning when speaking of transversal-related problems. This happens because where K is a transversal of the coalition structure π, $K \setminus \{i\}$ is not a transversal for π. The new value of a cooperative game with coalition structure is constructed in such a way as to solve some transversal problems. Neither is the transversal value a special case of coalition partition games from [20].

The value of a cooperative game is called a division if the sum of players' payoffs is $v(N)$ and the payoff of player $i, i \in N$ is not less than $v(\{i\})$. The transversal value is not a division of the cooperative game. This is because if K is a transversal of partition π, then each player $i, i \in K$ gets the whole quantity $v(K)$ rather than a share of it. The transversal value is suitable for modeling the processes and phenomena in which the payoff of a coalition is not divided among players but goes to each player. Sections 4 and 5 describe situations in which the transversal value has a physical meaning.

Vastly popular in the literature are hedonic games [21]. Players in such games have preferences on a set of coalitions. Each player is interested in belonging to the most profitable coalition. Yet, the transversal value of player i is independent of other players in the same coalition. The transversal value of player i depends on players belonging to other coalitions than his/her own. Hence, the transversal value of a cooperative game is not equivalent to that of a hedonic game.

Observe that if $\pi = \{N\}$, then the set of transversals is empty. In this case, since $T_i(N, v, \pi) = v(\emptyset) = 0 \ \forall i \in N$, the transversal value of each player is 0.

3.2 Theorem of the existence of a stable coalition structure

This section investigates the existence of a stable coalition structure for the transversal value. The main result is the following theorem.

Theorem 1. *The following two statements are true:*

1. *In any cooperative game (N, v) on the set $X_l(N), l = 2, 3, ..., n-1$ there exists a coalition structure that is simultaneously Nash stable, permutation stable, and TPM for the transversal value.*

2. *If the values of the characteristic function are non-negative, the coalition structure with these three types of stability is also externally stable.*

The proof is in the appendix.

Theorem 1 is based on the existence of the potential function

$$P(\pi) = P(v, \pi) = \sum_{K \in M(\pi)} v(K). \tag{1}$$

Function $P(\pi)$ is a potential and a permutation-potential function at the same time.

Suppose that the values of the characteristic function are non-negative. We denote by $\pi_l^*, \pi_l^* \in X_l(N), l = 2, 3, ..., n-1$ a coalition structure that possesses four types of stability. In this case, to find π_l^* it suffices to solve the following discrete optimization problem,

$$\pi_l^* \in \underset{\pi \in X_l(N)}{\text{argmax}} \sum_{K \in M(\pi)} v(K).$$

Players' cooperation with respect to the transversal value can be analyzed by finding π_l^* for $l = 2, 3, ..., n-1$ and comparing the coalition partitions.

3.3 Punctual stability

This subsection investigates the following issue. Suppose players in a potential coalition partition game (N, H) have to break into non-intersecting coalitions. There are two variants of coalition structure formation. The first variant is finding the partition that maximizes the potential function. In this case, no player will want to move to another coalition alone because this will decrease his/her payoff. The second variant of coalition partition involves manipulations by player i. First, player i chooses not to participate in coalition structure formation. Next, players from the set $N \setminus \{i\}$ form a coalition structure that maximizes the potential function on the set $\Pi(N \setminus \{i\})$. When players from the set $N \setminus \{i\}$ have formed the coalition structure, player i shows up. Player i joins the coalition of the newly-formed partition in which his/her payoff will be the highest. The assumption is that a new player is free to join any coalition of the partition, whereas the transition of old players between coalitions is prohibited.

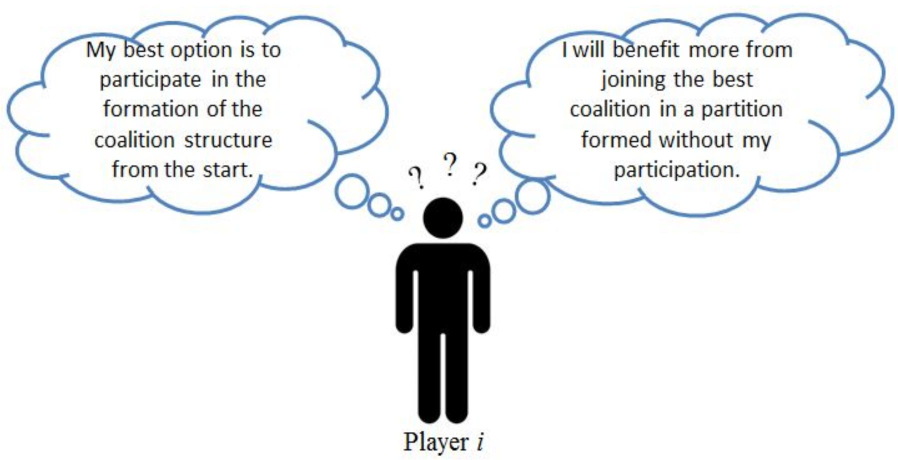

Fig. 1. Dilemma of player i.

In the second variant of coalition structure formation, player i can be said to be late since the coalition partition was formed without him/her. The question is the following: is it in the first or in the second scenario of coalition structure formation that the late player's payoff is the highest? Should one participate in the coalition structure formation from the start or is it more beneficial for player i to join the partition that has already been formed (see Fig. 1)? This section will answer this question for the transversal value.

The problem described above can be formalized as follows. Let $P(\pi)$ be the potential function of the coalition partition game (N, H). Denote

$$\pi^* \in \underset{\pi \in \Pi(N)}{\operatorname{argmax}} P(\pi), \quad \psi^*_{-i} \in \underset{\psi_{-i} \in \Pi(N \setminus \{i\})}{\operatorname{argmax}} P(\psi_{-i}),$$

$$Y(\psi^*_{-i}) = \Big\{ \{A \cup \{i\}, \psi^*_{-i,A}\} | \forall A \in \psi^*_{-i} \Big\}.$$

Coalition structure $\psi, \psi \in Y(\psi^*_{-i})$ is obtained from ψ^*_{-i} by joining the player i in some coalition. For example, if $\psi^*_{-1} = \{\{2\}, \{3\}, \{4, 5\}\}$, then

$$Y(\psi^*_{-1}) = \Big\{ \{\{1,2\}, \{3\}, \{4,5\}\}, \{\{2\}, \{1,3\}, \{4,5\}\}, \{\{2\}, \{3\}, \{1,4,5\}\} \Big\}.$$

Definition 9. *Let (N, H) be a potential coalition partition game. The coalition structure π^*, which maximizes the potential function of the game (N, H), is punctually stable if*

$$H_i(\pi^*) \geq H_i(\psi) \; \forall i \in N, \forall \psi \in Y(\psi^*_{-i}).$$

Punctual stability means that no player alone will benefit from manipulating with his/her presence. If a fixed player $i, i \in N$ chooses not to participate in partition formation but to join the most personally profitable coalition afterwards, his/her payoff will not increase as a result. Hence, any player wants to communicate their wish to participate in the coalition partition from the start.

Theorem 2. *Suppose that coalition structure π_l^* maximizes the potential function of the transversal value on the set $X_l(N), l = 2, 3, ..., n-1$ and does not contain single-player coalitions. In this case, π_l^* is punctually stable.*

The proof is in the appendix.

If the conditions of Theorem 2 are fulfilled, then it is not beneficial for any player to conceal their presence. Participating in coalition partition formation from the start is better than joining the most personally profitable coalition afterwards. We suppose the values of the characteristic function are non-negative, and a player's payoff is the transversal value. In this case, members of the coalition structure do not mind the arrival of new players. If a new player joins coalition B, the payoffs of players in B will not change, while the payoffs of players from other coalitions will increase.

The following conclusion can be drawn. Suppose the values of the characteristic function are non-negative, and there are no single-player coalitions in coalition structure π_l^* that maximize the potential function (1) on the set $X_l(N), l = 2, ..., n-1$. Partition π_l^* is then simultaneously Nash stable, permutation stable, TPM, externally stable, and punctually stable with respect to the transversal value. This result underlies the existence of a stable coalition structure in the workgroup formation game and the game of chairpersons.

4 The workgroup formation game

We consider a firm which implements some projects. Let $\{0, 1, 2, ..., n\}$ be a set of players where 0 is the owner of the firm, and $N = \{1, 2, ..., n\}$ is the set of working players. Projects are implemented by working players, and the owner of the firm pays the workers for their work.

It takes l fulfilled jobs to implement one project, $2 \leq l \leq n-1$. All jobs are different but equally difficult.

The firm owner gets paid α for the implementation of one project and pays each working player β for participating in one project. Working players get identical payments per project, proceeding from the condition that all jobs are equally difficult.

Let $\pi \in X_l(N)$. A unique job is bijected to each coalition in partition π. Players form transversals of partition π to implement projects and the number of transversals formed is assumed to be maximal. Each participant of a transversal implements the job corresponding to his/her coalition in partition π. The value $v(K), K \subseteq N, v(K) \geq 0, |K| = l$ is the number of projects coalition K can implement. Hence, player $i, i \in N$ participates in $\sum_{K \in M_i(\pi)} v(K)$ projects. Since player i receives a payment of β for participating in one project, his/her payoff is $\beta \cdot \sum_{K \in M_i(\pi)} v(K)$. The total number of completed projects will be $\sum_{K \in M(\pi)} v(K)$. The owner of the firm gets income from all the projects and transfers payoffs to players from N. Hence, the owner's net profit is

$$\alpha \cdot \sum_{K \in M(\pi)} v(K) - \beta \cdot \sum_{i \in N} \sum_{K \in M_i(\pi)} v(K) = (\alpha - \beta \cdot |\pi|) \cdot \sum_{K \in M(\pi)} v(K).$$

Since $|\pi| = l$, we assume that $\alpha - \beta \cdot l \geq 0$. We see from this inequality that the owner of the firm will not bear losses. The question is how the partition of working players should be formed? To answer this question, let us formalize the task as a coalition partition game.

Definition 10. *The workgroup formation game is a coalition partition game* $(N \cup \{0\}, H)$ *in which players' payoffs have the form*

$$H_0(\pi) = (\alpha - \beta \cdot |\pi|) \cdot \sum_{K \in M(\pi)} v(K),$$

$$H_i(\pi) = \beta \cdot \sum_{K \in M_i(\pi)} v(K), \forall i \in N,$$

where $\pi \in X_l(N)$.

Let $T(\pi)$ and $P(\pi)$ be the transversal value of the cooperative game (N, v) with coalition structure π and the potential function (1), respectively. The payoffs of players in the workgroup formation game can then be expressed as:

$$H_0(\pi) = (\alpha - \beta \cdot |\pi|) \cdot P(\pi), H_i(\pi) = \beta \cdot T_i(\pi) \; \forall i \in N.$$

Remember that the owner of the firm is not part of the coalition structure.

Definition 11. *Coalition structure* π *is called Pareto-Nash-stable in the workgroup formation game if the following inequalities hold:*

$$H_0(\pi) \geq H_0(\rho) \; \forall \rho \in X_l(N),$$

$$H_i(\pi) \geq H_i(\rho) \; \forall i \in N \; \forall \rho \in D_i(\pi).$$

The firm owner is interested in partition π, maximizing his/her payoff on the set $X_l(N)$. This is taken into account in Definition 11. A player will opt to move from one coalition to another if this will increase his/her payoff. Frequent migrations of players, however, can affect the work process and displease the team. If the partition is Pareto-Nash-stable, there will be no individual migrations between coalitions as this would reduce the moving player's payoff.

Theorem 3. *A Pareto-Nash-stable coalition structure on the set* $X_l(N), l = 2, ..., n-1$ *exists in the workgroup formation game.*

The proof is in the appendix.

The firm owner and the working players in the workgroup formation game can be advised to form a Pareto-Nash-stable partition, which always exists for any values of the characteristic function v. In this case, the owner's payoff will be maximal and the working players will not want to move to other groups alone.

If two working players from different groups in a Pareto-Nash-stable coalition structure agree to implement each other's jobs, their combined payoff will not increase. This comment is a corollary of Theorem 1, since a Pareto-Nash-stable coalition structure maximizes the potential function of the transversal value of the cooperative game.

Example 2. Let the set of working players in the workgroup formation game be $N = \{1, 2, 3, 4\}$. The number of projects coalition K can implement are given in Table 1. We assume that $\alpha = 4, \beta = 1$. Hence, the payoff of a working player in the workgroup formation game is the transversal value. Let us find the Pareto-Nash-stable coalition structure π_l^* for $l = 2$ and $l = 3$. The calculated payoffs of working players and the values of the potential function are given in Table 2.

Table 1: Values of v.

K	$\emptyset$	$\{1\}$	$\{2\}$	$\{3\}$	$\{4\}$	$\{1,2\}$	$\{1,3\}$	$\{1,4\}$
v	0	4	5	6	7	4	5	7
K	$\{2,3\}$	$\{2,4\}$	$\{3,4\}$	$\{1,2,3\}$	$\{1,2,4\}$	$\{1,3,4\}$	$\{2,3,4\}$	$\{1,2,3,4\}$
v	8	5	3	2	8	6	7	8

Table 2: Values of v.

π	$T_1(\pi)$	$T_2(\pi)$	$T_3(\pi)$	$T_4(\pi)$	$P(\pi)$
$\{\{1\},\{2\},\{3\},\{4\}\}$	8	8	8	8	8
$\{\{1,2\},\{3\},\{4\}\}$	6	7	13	13	13
$\{\{1,3\},\{2\},\{4\}\}$	8	15	7	15	15
$\{\{1,4\},\{2\},\{3\}\}$	2	9	9	7	9
$\{\{2,3\},\{1\},\{4\}\}$	14	8	6	14	14
$\{\{2,4\},\{1\},\{3\}\}$	8	2	8	6	8
$\{\{3,4\},\{1\},\{2\}\}$	10	10	2	8	10
$\{\{1\},\{2,3,4\}\}$	16	4	5	7	16
$\{\{2\},\{1,3,4\}\}$	4	17	8	5	17
$\{\{3\},\{1,2,4\}\}$	5	8	16	3	16
$\{\{4\},\{1,2,3\}\}$	7	5	3	15	15
$\{\{1,2\},\{3,4\}\}$	12	13	13	12	25
$\{\{1,3\},\{2,4\}\}$	11	12	11	10	22
$\{\{1,4\},\{2,3\}\}$	9	9	8	8	17
$\{\{1,2,3,4\}\}$	0	0	0	0	0

Let $l = 2$. According to the proof of Theorem 3, finding π_2^* requires finding the maximum of the potential function on the set $X_2(N)$. In this case, we get:

$$\pi_2^* = \{\{1, 2\}, \{3, 4\}\}.$$

The vector of working players' payoffs is $T(N, v, \pi_2^*) = (12, 13, 13, 12)$. The firm owner's profit is 50.

Let us now consider $l = 3$. The vector of players' payoffs is $T(N, v, \pi_3^*) = (8, 15, 7, 15)$. The firm owner's payoff is 15.

The cases with $l = 1$ or $l = 4$ are self-evident,

$$\pi_1^* = \{\{1, 2, 3, 4\}\}, \pi_4^* = \{\{1\}, \{2\}, \{3\}, \{4\}\}.$$

The transversal value of any player for π_1^* is equal to zero since the set of transversals is empty. If all players in the partition are solo players, then working players' payoffs are $T(N, v, \pi_4^*) = (8, 8, 8, 8)$ and the firm owner gets 0.

In this example, the firm owner's payoff is maximal among the detected stable partitions where $l = 2$. Hence, the owner in interested in taking up projects for which two types of jobs need to be implemented.

Consider the coalition partition $\pi = \{\{1, 2\}, \{3, 4\}\}$. The firm owner can organize the work as follows. All days are divided into even and odd. On even and odd days, projects perform transversals $\{1, 3\}, \{2, 4\}$ and $\{1, 4\}, \{2, 3\}$, respectively. The total number of projects in which player i participated in on an even and odd day is his transversal value. Since $v(\{1, 3\}) = v(\{2, 4\}) = 5$ and $v(\{1, 4\}) = 7, v(\{2, 3\}) = 8$, on even days, working players are less loaded with work. This means that there is no workload every day. Working players perform projects in various transversals, which leads to increased experience.

5 The game of chairpersons

5.1 Game formation and the existence of a stable coalition structure

Suppose $N = \{1, 2, ..., n\}$ are players who need to break up into l non-intersecting commissions. A chairperson is elected for each commission and each player wants to become one. Let $v(K)$ be the *a priori* probability that each member of coalition K is a chairperson, i.e.

$$v(K) \in (0; 1] \ \forall K \subseteq N, |K| = l, \sum_{K: K \subseteq N, |K| = l} v(K) = 1,$$

$$v(K) = 0 \ \forall K \subseteq N, |K| \neq l.$$

The coalition of chairpersons is the transversal of the players' partition. Denote by $\mathbb{P}\mathrm{r}(i|\pi)$ the probability that player i is the chairperson in his/her commission, provided that partition π has been formed. Player $i, i \in N$ is interested in a partition π for which the probability $\mathbb{P}\mathrm{r}(i|\pi)$ is the highest.

Example 3. Two commissions, A and B, have to be formed to supervise a project, $A \cup B = N = \{1, 2, 3, 4\}, A \cap B = \emptyset, A \neq \emptyset, B \neq \emptyset$. The following *a priori* probabilities are known:

$$v(\{1, 2\}) = 0.05, v(\{1, 3\}) = 0.1, v(\{1, 4\}) = 0.15,$$

$$v(\{2, 3\}) = 0.15, v(\{2, 4\}) = 0.25, v(\{3, 4\}) = 0.3.$$

The higher the probability $v(K), K \subseteq N, |K| = 2$, the better the members of the coalition K will cope with their responsibilities as chairpersons. The productivity of commissions A and B depends on the collaboration between their leaders. Since the *a priori* probability is the lowest for coalition $\{1,2\}$, the joint leadership skills of players 1 and 2 are poorer than in other pairs. The best collaborative leadership qualities are found in the coalition $\{3,4\}$.

Suppose there formed a partition of players into commissions $\pi = \{\{1\},\{2,3,4\}\}$. In this case, the probability that player $i, i \in N$ is a chairperson in his coalition is calculated as follows,

$$\mathbb{P}\mathrm{r}(1|\{\{1\},\{2,3,4\}\}) = \frac{v(\{1,2\}) + v(\{1,3\}) + v(\{1,4\})}{v(\{1,2\}) + v(\{1,3\}) + v(\{1,4\})} = 1,$$

$$\mathbb{P}\mathrm{r}(2|\{\{1\},\{2,3,4\}\}) = \frac{v(\{1,2\})}{v(\{1,2\}) + v(\{1,3\}) + v(\{1,4\})} = \frac{1}{6},$$

$$\mathbb{P}\mathrm{r}(3|\{\{1\},\{2,3,4\}\}) = \frac{v(\{1,3\})}{v(\{1,2\}) + v(\{1,3\}) + v(\{1,4\})} = \frac{1}{3},$$

$$\mathbb{P}\mathrm{r}(4|\{\{1\},\{2,3,4\}\}) = \frac{v(\{1,4\})}{v(\{1,2\}) + v(\{1,3\}) + v(\{1,4\})} = \frac{1}{2}.$$

Since the first player is the only member of the commission, he/she will be a chairperson with probability 1. It is more probable that the second chair will be taken by the fourth player. Player 2 has the lowest probability of becoming a chairperson. Suppose he/she moved to the commission of player 1. This move forms the partition $\{\{1,2\},\{3,4\}\}$. The conditional probabilities in this case will take the form:

$$\mathbb{P}\mathrm{r}(1|\{\{1,2\},\{3,4\}\}) = \frac{v(\{1,3\}) + v(\{1,4\})}{v(\{1,3\}) + v(\{1,4\}) + v(\{2,3\}) + v(\{2,4\})} = \frac{5}{13},$$

$$\mathbb{P}\mathrm{r}(2|\{\{1,2\},\{3,4\}\}) = \frac{v(\{2,3\}) + v(\{2,4\})}{v(\{1,3\}) + v(\{1,4\}) + v(\{2,3\}) + v(\{2,4\})} = \frac{7}{13},$$

$$\mathbb{P}\mathrm{r}(3|\{\{1,2\},\{3,4\}\}) = \frac{v(\{1,3\}) + v(\{2,3\})}{v(\{1,3\}) + v(\{1,4\}) + v(\{2,3\}) + v(\{2,4\})} = \frac{5}{13},$$

$$\mathbb{P}\mathrm{r}(4|\{\{1,2\},\{3,4\}\}) = \frac{v(\{1,4\}) + v(\{2,4\})}{v(\{1,3\}) + v(\{1,4\}) + v(\{2,3\}) + v(\{2,4\})} = \frac{7}{13}.$$

In the partition $\{\{1,2\},\{3,4\}\}$ it is the most probable that the coalition of chairpersons is $\{2,4\}$.

Let $i \in N, \pi \in \Pi(N)$. Consider a random variable

$$\xi_i(\pi) = \begin{cases} 1, & \mathbb{P}\mathrm{r}(i|\pi); \\ 0, & 1 - \mathbb{P}\mathrm{r}(i|\pi). \end{cases}$$

Player i is the chairperson of his coalition in partition π with a probability $\mathbb{P}\mathrm{r}(i|\pi)$. Then, $E\xi_i = \mathbb{P}\mathrm{r}(i|\pi)$.

How does one form a coalition partition considering that each player wants to be a chairperson? This question is investigated through the game of chairpersons introduced below.

Definition 12. *The game of chairpersons is a coalition partition game (N, H), in which the payoff of player i is the expected value of being the leader in his coalition of the partition π, i.e.*

$$H_i(\pi) = E\xi_i(\pi) = \frac{T_i(\pi)}{P(\pi)},$$

where $T_i(\pi) = T_i(N, v, \pi)$ and $P(\pi)$ is the transversal value and the potential function (1), respectively.

A curious feature of the game of chairpersons is that a player's payoff is the relation of the transversal value to its potential function. Since each player is interested in becoming a chairperson, the question arises of whether a stable coalition partition exists.

Theorem 4. *The game of chairpersons is an ordinal potential game with an ordinal potential function (1).*

The proof is in the appendix.

If every player wants to be the leader of his/her coalition in the game of chairpersons, then the players can be advised to form a partition that maximizes function (1). In this case, no player will want to move from one's own to another coalition alone.

Observe that the permutation stability has no physical meaning for the game of chairpersons. This happens because players cannot compensate each other's payoffs.

Our next concern is the punctuality of a stable partition.

5.2 Punctuality in the game of chairpersons

Suppose players in the game of chairpersons agree to form a partition that maximizes the potential function (1). In this case, a Nash-stable coalition partition is formed (see Theorem 4). However, player $i, i \in N$ chooses to conceal their presence or is late. The partition is then formed without them, and the *a priori* probabilities are re-calculated as follows:

$$v_{-i}(K) = \begin{cases} \frac{v(K)}{\sum\limits_{\substack{L: i \notin L \\ |L|=l}} v(L)}, & i \notin K; \\ 0, & i \in K. \end{cases} \quad \forall K \subseteq N, |K| = l.$$

Let us consider the characteristic function from Example 3. If player 1 has concealed his/her presence, the values of v_{-1} will take the form:

$$v_{-1}(\{2,3\}) = \frac{15}{70}, v_{-1}(\{2,4\}) = \frac{25}{70}, v_{-1}(\{3,4\}) = \frac{30}{70},$$
$$v_{-1}(\{1,2\}) = v_{-1}(\{1,3\}) = v_{-1}(\{1,4\}) = 0.$$

Players from the set $N \setminus \{i\}$ in the game of chairpersons form a partition that maximizes the potential function with the characteristic function v_{-i}. After the coalition structure is formed, player i appears, who joins the coalition most profitable to them. With the arrival of player i, the transversal of chairpersons is formed with regard to the values of the characteristic function v.

Since values of the characteristic function are dependent variables, we have to re-calculate them because of one player arriving late. Let us now formally write down the punctual stability specifically for the game of chairpersons. Let

$$\pi^* \in \underset{\pi \in X_l(N)}{\operatorname{argmax}}\, P(v, \pi), \quad \psi^*_{-i} \in \underset{\psi_{-i} \in X_l(N \setminus \{i\})}{\operatorname{argmax}}\, P(v_{-i}, \psi_{-i}),$$

$$Y(\psi^*_{-i}) = \Bigg\{ \{A \cup \{i\}, \psi^*_{-i,A}\} | \forall A \in \psi^*_{-i} \Bigg\}.$$

The coalition structure π^* is punctually stable in the game of chairpersons if the inequalities

$$H_i(\pi^*) \geq H_i(\psi) \ \forall \psi \in Y(\psi^*_{-i})$$

hold $\forall i \in N$.

Punctuality in the game of chairpersons is different from punctuality in Definition 9 in that the potential function depends on different characteristic functions.

Theorem 5. *Let π^* maximize the potential function $P(v, \pi)$ on set $X_l(N), l = 2, 3, ..., n-1$ and not contain single-player coalitions. Then, π^* is punctually stable in the game of chairpersons.*

The proof is in the appendix.

Punctuality is an important property in coalition formation. It often happens that before chairpersons are appointed, someone arrives late and joins a coalition that has already been formed. Theorem 5 states that there is no benefit for players in turning up late or concealing their presence.

5.3 The game of chairpersons and the weighted allocation rule

This subsection demonstrates that one classical coalition partition game is a special case of the game of chairpersons.

The game of chairperson assumes that the characteristic function v takes values from the interval $[0; 1]$. This is done to render a physical meaning to the players' payoff functions. If, however, the values of v are within the interval $(0; +\infty)$, this does not affect the outcome of Theorem 4.

Suppose w_i is the weight of player $i, w_i > 0 \ \forall i \in N$. We take a game of chairpersons with the characteristic function $v(K) = \prod_{j \in K} w_j$. In this case, players' payoffs can be simplified as follows:

$$H_i(\pi) = \frac{T_i(\pi)}{P(\pi)} = \frac{w_i \cdot \prod_{B \in \pi \setminus B(i)} \sum_{j \in B} w_j}{\prod_{B \in \pi} \sum_{j \in B} w_j} = \frac{w_i}{\sum_{j \in B(i)} w_j}.$$

Thus, the weighted allocation rule is a special case of the game of chairpersons. The ratio of a player's weight to the sum of weights of players belonging to the same coalition can be interpreted as an index. Suppose players have to break up into l coalitions and each player is interested in having the maximum index. In this case, they can be advised to form a partition that maximizes the ordinal potential function

$$P(\pi) = \sum_{K \in M(\pi)} v(K) = \prod_{B \in \pi} \sum_{j \in B} w_j.$$

If $\forall B \in \pi^* : |B| \neq 1$, then π^* is punctually stable.

5.4 Numerical example

Let $N = \{1, 2, ..., 5\}$ be the set of players in the game of chairpersons. The players have to break up into two groups. They agree to form a partition that maximizes the ordinal potential function (1). The values of the characteristic function v are given in Table 3. If player i is late, then players from the set $N \setminus \{i\}$ form the partition with regard to the values of the characteristic function $v_{-i}, i \in N$.

Table 3. Values of v and $v_{-i}, i \in N$.

K	v	v_{-1}	v_{-2}	v_{-3}	v_{-4}	v_{-5}
$\{1,2\}$	.0690	–	–	.1194	.1280	.1240
$\{1,3\}$	.0776	–	.1184	–	.1440	.1395
$\{1,4\}$	.1121	–	.1711	.1940	–	.2016
$\{1,5\}$	.0690	–	.1053	.1194	.1280	–
$\{2,3\}$	.0776	.1154	–	–	.1440	.1395
$\{2,4\}$	.0948	.1410	–	.1642	–	.1705
$\{2,5\}$	.1034	.1538	–	.1791	.1920	–
$\{3,4\}$	.1250	.1859	.1908	–	–	.2248
$\{3,5\}$	.1422	.2115	.2171	–	.2640	–
$\{4,5\}$	.1293	.1923	.1974	.2239	–	–

Since players form a partition consisting of two coalitions, the game of chairpersons can be matched to a weighted complete graph (N, E), where N is the set of vertices and $E = \{\{i, j\} | i, j \in N, i \neq j\}$ is the set of edges. The weight of edge $\{i, j\}$ is the value of the characteristic function $v(\{i, j\})$. Hence, the process of finding the optimal coalition structure can be reduced to the problem of finding a complete bipartite graph in the graph (N, E). Problems dealing with a search for an optimal sub-graph are usually NP complete, e.g. the clique problem.

In the game of chairpersons, players from the set N form the partition

$$\pi^* = \{\{1, 2, 3\}, \{4, 5\}\}.$$

The graph of the game of chairpersons and the graph corresponding to the partition π^* are shown in Fig. 2.

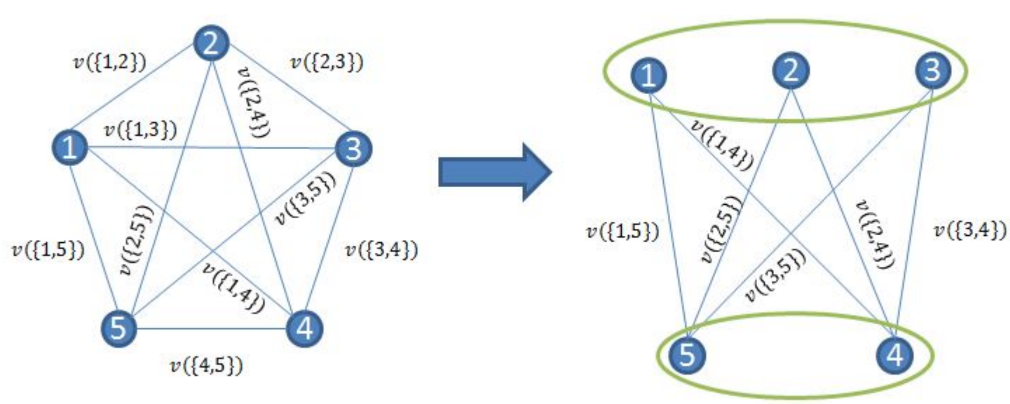

Fig. 2. Complete and bipartite graphs.

Suppose player $i, i \in N$ arrived late and the partition ψ^*_{-i} was formed without him/her. Next, player i joins the coalition he/she finds the most personally profitable, and the coalition structure $\psi, \psi \in Y(\psi^*_{-i})$ is formed. The payoffs of players and the optimal partitions are given in Table 4.

Table 4. Optimal coalition structures

i	ψ^*_{-i}	ψ	$H_i(\psi)$	$H_i(\pi^*)$
1	$\{\{2,3\},\{4,5\}\}$	$\{\{1,2,3\},\{4,5\}\}$	.2800	.2800
2	$\{\{1,5\},\{3,4\}\}$	$\{\{1,5\},\{2,3,4\}\}$	.2721	.3066
3	$\{\{1,5\},\{2,4\}\}$	$\{\{1,5\},\{2,3,4\}\}$	.3469	.4133
4	$\{\{1,5\},\{2,3\}\}$	$\{\{1,5\},\{2,3,4\}\}$	.3809	.5133
5	$\{\{1,3\},\{2,4\}\}$	$\{\{1,3,5\},\{2,4\}\}$	.3776	.4866

Players 2, 3, 4, 5 do not want to be late as this decreases their chances of becoming chairpersons. Player 1 lateness does not affect his/her payoff. This is explained by the fact that the coalition structure ψ_{-1} is a sub-partition of partition π^*.

The values of the characteristic function $v(K), 1 \in K$ are small relative to its other values. Player 1 can be said to be weaker than other players. Because of player $i, i \in N \setminus \{1\}$ was late, the punctual players have to form a partition which is not a sub-partition of π^*. That is why being late is detrimental for strong players in the game of chairpersons.

6 Conclusions

The transversal value of a cooperative game with coalition structure is a novel value that has been introduced to solve transversal problems. Its properties are studied and its applications to the working group formation problem and the appointment of chairpersons are shown.

The main conclusion is the following. If the values of the characteristic function are nonnegative and the partition that maximizes the potential function of the transversal value does not contain single-player coalitions, then such a partition is Nash stable, permutation stable, externally stable, TPM, and punctually stable. This is a new result in coalition formation theory.

The classical coalition formation problem means that a player gets a certain utility from the coalition he/she belongs to. A player's transversal value, however, does not depend on the players belonging to the same coalition. This assumption permits modeling problems related to transversals.

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Appendix

Proof of Theorem 1

The main idea of the proof of the first point is that function (1) is simultaneously potential and permutation-potential for the transversal value. Next, we show that the sum of players' payoffs is expressed through function (1) in a special manner. This proves the existence of a coalition structure with three types of stability.

For brevity, we denote $T_i(\pi) = T_i(N, v, \pi)\ \forall i \in N$. We assume hereafter that $|\pi| = l$, where $l = 2, 3, ..., n-1$.

Step 1. Let us demonstrate that the potential function

$$P(\pi) = \sum_{K \in M(\pi)} v(K).$$

exists for the transversal value.

Let

$$\pi \in X_l(N), \rho \in D_i(\pi), \pi = \{B(i), A, \pi_{-B(i),A}\}, \rho = \{B(i) \setminus \{i\}, A \cup \{i\}, \pi_{-B(i),A}\}.$$

We assume that $|B(i)| = 1$ is fulfilled in the coalition structure π. In this case, player i cannot move to another coalition, since the number of coalitions in the coalition partition is fixed and equals l. Then, $D_i(\pi) = \{\pi\}, \rho = \pi$. Therefore,

$$T_i(\pi) - T_i(\rho) = P(\pi) - P(\rho) = 0.$$

Now, let $|B(i)| \geq 2$. The following sequence of equalities is true,

$$P(\pi) - P(\rho) = \sum_{K \in M(\pi)} v(K) - \sum_{K \in M(\rho)} v(K)$$

$$= \sum_{K \in M_i(\pi) \cup (M(\pi) \setminus M_i(\pi))} v(K) - \sum_{K \in M_i(\rho) \cup (M(\rho) \setminus M_i(\rho))} v(K)$$

$$= \left(\sum_{K \in M_i(\pi)} v(K) + \sum_{K \in M(\pi) \setminus M_i(\pi))} v(K) \right) - \left(\sum_{K \in M_i(\rho)} v(K) + \sum_{K \in M(\rho) \setminus M_i(\rho))} v(K) \right).$$

The sets $M(\pi) \setminus M_i(\pi)$ and $M(\rho) \setminus M_i(\rho)$ consist of transversals of the partition π and do not contain player i. If player i moves to another coalition, the transversals that do not contain player i will not change. So, $M(\pi) \setminus M_i(\pi) = M(\rho) \setminus M_i(\rho)$. Hence, $\forall i \in N, \forall \pi \in X_l(N)$ the equality

$$P(\pi) - P(\rho) = \sum_{K \in M_i(\pi)} v(K) - \sum_{K \in M_i(\rho)} v(K) = T_i(\pi) - T_i(\rho)\ \forall \rho \in D_i(\pi)$$

holds. This means that a Nash-stable coalition structure on the set $X_l(N), l = 2, 3, ..., n-1$ can be found by simply finding the maximum of the potential function $P(\pi)$ on this set.

Step 2. Let us now demonstrate that the same function $P(\pi)$ is a permutation-potential function for the transversal value. Let $\pi \in X_l(N), \{i, j\} \in L(\pi)$,

$$\rho = \rho(\pi, \{i, j\}) = \{B(i) \setminus \{i\} \cup \{j\}, B(j) \setminus \{j\} \cup \{i\}, \pi_{-B(i),B(j)}\}.$$

We simplify the difference of the sums of payoffs of players i and j,

$$(T_i(\pi)+T_j(\pi))-(T_i(\rho)+T_j(\rho))$$

$$=\left(2\cdot\sum_{K\in M_i(\pi)\cap M_j(\pi)}v(K)+\sum_{K\in M_i(\pi),j\notin K}v(K)+\sum_{K\in M_j(\pi),i\notin K}v(K)\right)$$

$$-\left(2\cdot\sum_{K\in M_i(\rho)\cap M_j(\rho)}v(K)+\sum_{K\in M_i(\rho),j\notin K}v(K)+\sum_{K\in M_j(\rho),i\notin K}v(K)\right)$$

$$=\left(\sum_{K\in M_i(\pi),j\notin K}v(K)+\sum_{K\in M_j(\pi),i\notin K}v(K)\right)-\left(\sum_{K\in M_i(\rho),j\notin K}v(K)+\sum_{K\in M_j(\rho),i\notin K}v(K)\right).$$

Next, we simplify the difference of the potential functions,

$$P(\pi)-P(\rho)=\sum_{K\in M(\pi)}v(K)-\sum_{K\in M(\rho)}v(K)$$

$$=\left(\sum_{K\in M(\pi),i,j\in K}v(K)+\sum_{K\in M(\pi),i\in K,j\notin K}v(K)+\sum_{K\in M(\pi),i\notin K,j\in K}v(K)+\sum_{K\in M(\pi),i\notin K,j\notin K}v(K)\right)$$

$$-\left(\sum_{K\in M(\rho),i,j\in K}v(K)+\sum_{K\in M(\rho),i\in K,j\notin K}v(K)+\sum_{K\in M(\rho),i\notin K,j\in K}v(K)+\sum_{K\in M(\rho),i\notin K,j\notin K}v(K)\right).$$

Considering the properties of transversals, the following equalities hold,

$$\sum_{K\in M(\pi),i,j\in K}v(K)=\sum_{K\in M(\rho),i,j\in K}v(K),\quad \sum_{K\in M(\pi),i\notin K,j\notin K}v(K)=\sum_{K\in M(\rho),i\notin K,j\notin K}v(K),$$

$$\sum_{K\in M_i(\pi),j\notin K}v(K)=\sum_{K\in M(\pi),i\in K,j\notin K}v(K),\quad \sum_{K\in M_i(\rho),j\notin K}v(K)=\sum_{K\in M(\rho),i\in K,j\notin K}v(K).$$

The above equalities permit simplifying the difference of the potential functions even further,

$$P(\pi)-P(\rho)$$

$$=\left(\sum_{K\in M_i(\pi),j\notin K}v(K)+\sum_{K\in M_j(\pi),i\notin K}v(K)\right)-\left(\sum_{K\in M_i(\rho),j\notin K}v(K)+\sum_{K\in M_j(\rho),i\notin K}v(K)\right).$$

Thus, $\forall\pi\in X_l(N),\forall\{i,j\}\in L(\pi):$

$$(T_i(\pi)+T_j(\pi))-(T_i(\rho)+T_j(\rho))=P(\pi)-P(\rho),\rho=\rho(\pi,\{i,j\}).$$

Hence, the function $P(\pi)$ is a permutation-potential function for the transversal value. Since the potential functions for different types of stability coincide, the coalition partition that maximizes the potential function on the set $X_l(N), l = 2, 3, .., n-1$ possesses two types of stability.

Step 3. Let us demonstrate that the coalition structure that maximizes the potential function is TPM on the set $X_l(N), l = 2, 3, ..., n-1$. This can be done using the following property of transversals. If we write out all transversals of players from one coalition, we thereby write out all transversals of the coalition partition. This means that $\forall \pi \in X_l(N)$ the equality

$$\sum_{i \in B} T_i(\pi) = \sum_{i \in B} \sum_{K \in M_i(\pi)} v(K) = \sum_{K \in M(\pi)} v(K) = P(\pi) \ \forall B \in \pi$$

holds. In other words, the sum of the transversal values of players forming one coalition is the potential function. Then,

$$\sum_{i \in N} T_i(\pi) = \sum_{B \in \pi} \sum_{i \in B} T_i(\pi) = \sum_{B \in \pi} P(\pi) = |\pi| \cdot P(\pi).$$

On the set $X_l(N), l = 2, ..., n-1$ the value $|\pi|$ is constant. As a result, the coalition structure that maximizes $P(\pi)$ at the same time maximizes the sum of players' payoffs.

Thus, the coalition structure that maximizes the potential function on the set $X_l(N)$ is Nash stable, permutation stable, and TPM on this set. This proves the first point of the theorem.

Proof of the second point. External stability is defined so as to take into account players' transfers that cause no change in the number of coalitions in the partition. Let $\pi \in X_l(N), l = 2, ..., n-1, \pi = \{B_1, B_2, ..., B_l\}$. Without loss of generality, we consider the coalition B_1. Let ρ be the partition formed from π through the transfer of some players to the coalition B_1. The transversal value of any player from the coalition B_1 in the partition π consists of $|B_2| \cdot |B_3| \cdot ... \cdot |B_l|$ non-negative summands. If a player or a group of players from the set $N \setminus B_1$ moves to the coalition B_1, then some of the numbers $|B_2|, |B_3|, ..., |B_l|$ will decrease. Hence, $T_i(N, v, \pi) \geq T_i(N, v, \rho) \ \forall i \in B_1$. Thus, any coalition structure from the set $X_l(N), l = 2, ..., n-1$ is externally stable. So, the coalition structure that maximizes the potential function on the set $X_l(N)$ possesses four types of stability.

Proof of Theorem 2

For brevity, we denote $T_i(\pi) = T_i(N, v, \pi)$. We show that the equality

$$T_i(\pi) = P(\pi) - P(\pi_{-i})$$

$$\forall \pi \in X_l(N), P(\pi) = \sum_{K \in M(\pi)} v(K) \text{ and } |\pi_{-i}| = l \ \forall i \in N$$

is true for the transversal value. The set $M(\pi)$ consists of all the transversals of the partition π, and the set $M(\pi_{-i})$ consists of the transversals of the set π, which do not include the player i. Therefore, the following sequence of equalities is true

$$P(\pi) - P(\pi_{-i}) = \sum_{K \in M(\pi)} v(K) - \sum_{K \in M(\pi_{-i})} v(K)$$

$$= \sum_{K \in M(\pi) \setminus M(\pi_{-i})} v(K) = \sum_{K \in M_i(\pi)} v(K) = T_i(\pi).$$

Then, the transversal value of the player $i, i \in N$ can be written as follows,

$$T_i(\pi^*) = P(\pi^*) - P(\pi^*_{-i}),\ T_i(\psi) = P(\psi) - P(\psi^*_{-i}),$$

where

$$\pi^* \in \underset{\pi \in X_l(N)}{\operatorname{argmax}}\ P(\pi), \quad \psi^*_{-i} \in \underset{\psi_{-i} \in X_l(N \setminus \{i\})}{\operatorname{argmax}} P(\psi_{-i}), \psi \in Y(\psi^*_{-i}).$$

Then,

$$\begin{aligned} T_i(\pi^*) - T_i(\psi) &= P(\pi^*) - P(\pi^*_{-i}) - P(\psi) + P(\psi^*_{-i}) \\ &= (P(\pi^*) - P(\psi)) + \left(P(\psi^*_{-i}) - P(\pi^*_{-i})\right). \end{aligned}$$

The coalition structure π^* maximizes the potential function on the set $X_l(N)$. Since $\psi \in X_l(N)$, then in the equality written above, the first difference is non-negative. The coalition structure ψ^*_{-i} maximizes the potential function on the set $X_l(N \setminus \{i\})$. Since $\pi^*_{-i} \in X_l(N \setminus \{i\})$, then the second difference is also non-negative. Therefore, $T_i(\pi^*) \geq T_i(\pi)$ and π^* is punctually stable.

Proof of Theorem 3

Let π^* be a coalition structure that maximizes the payoff of the firm owner, that is

$$\pi^* \in \underset{\pi \in X_l(N)}{\operatorname{argmax}} \sum_{K \in M(\pi)} v(K).$$

Since $|\pi| = |\rho| = l\ \forall \rho \in X_l(N)$, then $\alpha - \beta \cdot |\pi| = \alpha - \beta \cdot |\rho|$. Hence, $H_0(\pi) \geq H_0(\rho)\ \forall \rho \in X_l(N)$.

Since π^* maximizes the potential function, then π^* is a Nash-stable partition with respect to the transversal value (Theorem 1). Therefore, π^* is a Pareto-Nash-stable coalition structure on the set of $X_l(N)$ in the workgroup formation game.

Proof of Theorem 4

We show that the game of chairpersons is an ordinal potential game with an ordinal potential function (1). Let

$$\pi \in X_l(N), \rho \in D_i(\pi), \pi = \{B(i), A, \pi_{-B(i),A}\}, \rho = \{B(i) \setminus \{i\}, A \cup \{i\}, \pi_{-B(i),A}\}.$$

In the proof of Theorem 2, it was shown that $\forall i \in N:$

$$T_i(\pi) = P(\pi) - P(\pi_{-i})\ \forall \pi \in X_l(N), |\pi_{-i}| = l.$$

Note that $\pi_{-i} = \rho_{-i}\ \forall \rho \in D_i(\pi)$. Then the following sequence of equalities is true,

$$H_i(\pi) - H_i(\rho) = \frac{T_i(\pi)}{P(\pi)} - \frac{T_i(\rho)}{P(\rho)} = \frac{P(\pi) - P(\pi_{-i})}{P(\pi)} - \frac{P(\rho) - P(\rho_{-i})}{P(\rho)}$$

$$= \frac{-P(\pi_{-i})}{P(\pi)} + \frac{P(\rho_{-i})}{P(\rho)} = \frac{P(\pi_{-i})}{P(\pi) \cdot P(\rho)} \cdot (P(\pi) - P(\rho)).$$

Since the characteristic function v takes positive values, then

$$\frac{P(\pi_{-i})}{P(\pi)\cdot P(\rho)} > 0.$$

Therefore, the sign of the difference $H_i(\pi) - H_i(\rho)$ coincides with the sign of the difference $P(\pi) - P(\rho)$, that is

$$H_i(\pi) - H_i(\rho) > 0 \text{ iff } P(\pi) - P(\rho) > 0 \ \forall i \in N, \forall \pi \in X_l(N), \forall \rho \in D_i(\pi).$$

By definition, the game of chairpersons is an ordinal potential game.

Proof of Theorem 5

We have $|\pi^*_{-i}| = l \ \forall i \in N$. Then, the difference of the payoff functions of player i can be transformed as follows,

$$H_i(v,\pi) - H_i(v,\psi) = \frac{T_i(v,\pi^*)}{P(v,\pi^*)} - \frac{T_i(v,\psi)}{P(v,\psi)} = \frac{P(v,\pi^*) - P(v,\pi^*_{-i})}{P(v,\pi^*)} - \frac{P(v,\psi) - P(v,\psi^*_{-i})}{P(v,\psi)}$$

$$= \frac{-P(v,\pi^*_{-i})}{P(v,\pi^*)} + \frac{P(v,\psi^*_{-i})}{P(v,\psi)} = \frac{P(v,\pi^*)\cdot P(v,\psi^*_{-i}) - P(v,\psi)\cdot P(v,\pi^*_{-i})}{P(v,\pi^*)\cdot P(v,\psi)}.$$

The values of the potential function are positive, that is, $P(v,\pi^*)\cdot P(v,\psi) > 0$. Since π^* maximizes the potential function on the set $X_l(N)$ and $\psi \in Y(\psi^*_{-i}) \subseteq X_l(N)$, then $P(v,\pi^*) \geq P(v,\psi)$. The following sequence of inequalities is true,

$$P(v_{-i},\psi^*_{-i}) \geq P(v_{-i},\pi^*_{-i})$$

$$\Leftrightarrow \sum_{K\in M(\psi^*_{-i})} \frac{v(K)}{\sum\limits_{\substack{L:i\notin L\\ |L|=l}} v(L)} \geq \sum_{K\in M(\pi^*_{-i})} \frac{v(K)}{\sum\limits_{\substack{L:i\notin L\\ |L|=l}} v(L)}$$

$$\Leftrightarrow \sum_{K\in M(\psi^*_{-i})} v(K) \geq \sum_{K\in M(\pi^*_{-i})} v(K) \Leftrightarrow P(v,\psi^*_{-i}) \geq P(v,\pi^*_{-i})$$

Thus, $P(v,\pi^*) \geq P(v,\psi) \ \forall \psi \in Y(\psi^*_{-i})$ and $P(v,\psi^*_{-i}) \geq P(v,\pi^*_{-i})$. Also note that the left and right sides of the inequalities are positive expressions. Hence, $P(v,\pi^*)\cdot P(v,\psi^*_{-i}) - P(v,\psi)\cdot P(v,\pi^*_{-i}) \geq 0$. Therefore, $H_i(\pi^*) \geq H_i(\psi) \ \forall i \in N$. So, π^* is punctually stable in the game of chairpersons.

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