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Some problems in noncommutative Hodge theory

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One of the features of algebraic geometry over the field of complex numbers is existence of a pure Hodge structure on de Rham cohomology of smooth proper varieties. Noncommutative algebraic geometry studies k-dg-categories, aka noncommutative schemes. A fruitful idea is in the realm of dg-categories one can still define many properties and invariants of schemes, such as smoothness, properness, (direct sums of) Dolbeault cohomology groups and de Rham cohomology, – the correspondence between the commutative and noncommutative worlds being the functor assigning to a scheme X/k the dg-category of perfect complexes $\mathrm{Perf}(X)$. The last two invariants are presented by Hochschild homology and periodic cyclic homology and they come with the Hodge-de Rham spectral sequence, which degenerates for smooth proper dg-categories when k is a field of characteristic 0 ([Kal], see also [M] and [KKM]). In [KKP], the authors consider noncommutative schemes over $\mathbb{C}$ and suggest to look for a counterpart for Hodge structures, which are already partially presented thanks to the aforementioned degeneration of the spectral sequence. One of the missing parts is a natural rational structure, i.e. a functor $F : dgCat_{\mathbb{C}} \longrightarrow Mod_{\mathbb{Q}}$ and a natural transformation $F \to \mathrm{HP}(\cdot/\mathbb{C})$, such that, for a smooth proper dg-category T, the induced morphism $F(T) \otimes_{\mathbb{Q}} \mathbb{C} \to \mathrm{HP}(T/\mathbb{C})$ is an equivalence (i.e. a quasi-isomorphism).

In our thesis, we consider the topological K-theory of dg-categories functor $\mathrm{K}^{\mathrm{top}}$, defined by A. Blanc as a promising candidate for the role of integral structure ([Bla]). We prove a statement to which we later will refer as *lattice property* of a dg-category T; namely we show that topological K-theory provides an integral structure in HP in the sense as below in several cases, collected in the following theorem (cf. Corollary **??**, Theorem **??**, Corollary **??**, Theorem **??**).

Theorem 0.1. *Let $LC \subset dgCat_{\mathbb{C}}$ be the full subcategory of dg-categories on which the natural transformation of functors $\mathrm{K}^{\mathrm{top}}(T) \otimes \mathbb{C} \to \mathrm{HP}(T/\mathbb{C})$ is an equivalence (which we call dg-categories satisfying lattice property). Then LC contains the following classes of dg-categories:*

a) $T = \mathrm{Perf}(B)$ where B is a connected proper dg-algebra;

b) $T = \mathrm{Perf}(B)$ where B is a connected dg-algebra, such that $\mathrm{H}_0 B$ is a nilpotent extension of a commutative $\mathbb{C}$-algebra of finite type;

*c) $T = \mathrm{Loc}(M, \mathbb{C})$ where M is a connected locally contractible space with some condition on its fundamental group (see Theorem **??**);*

d) $T = \mathrm{Perf}(\mathfrak{X})$ where $\mathfrak{X}$ is a derived $\mathbb{C}$-scheme, such that its classical part is a separated scheme of finite type. LC satisfies 2-out-of-3 property with respect to exact triples of dg-categories and is closed under Morita-equivalences and taking retracts;

e) finite-dimensional smooth $\mathbb{C}$-dg-algebras.

In section 4.7, [Bla], the author considered finite-dimensional classical algebras and used a variant of $\mathrm{K}^{\mathrm{top}}$, called pseudo-connective topological K-theory, to provide periodic cyclic homology of such algebras with a rational structure. Since these algebras lie in the class a) of Theorem 0.1, it follows that better-behaving topological K-theory works just as well, which can also be seen directly (see Proposition **??**). The class e) provides another generalization of the case of finite-dimensional algebras; it directly follows from Orlov's result (Theorem 2.19, [Orl]), while for other classes we use Theorem 0.2 below.

The cases covered by Theorem 0.1 include non-smooth and also non-proper dg-categories (unlike, e.g. Conjecture 4.25 of [Bla]). Indeed, in the commutative case, even though one has to require the variety to be smooth and proper for Hodge-de Rham spectral sequence to collapse, topological K-theory provides an integral structure inside HP for every quasi-projective variety, so in the noncommutative case we also expect the lattice property to be satisfied under very mild assumptions.

Topological K-theory of dg-categories of dg-categories is defined using a finer invariant called semi-topological K-theory K^{st} (we remind the definitions of both in the section 3). The main technical ingredient is the topological realization functor, which generalizes the procedure of analytification from $\mathbb{C}$-varieties to arbitrary (spectrum-valued) invariants of schemes. To prove Theorem 0.1, we study the behaviour of the realization functor, focusing on the case of Hochschild-type invariants. It allows us to establish the following result, crucial for proving the main theorem.

Theorem 0.2. *Let $v : B \to A$ be a nilpotent extension of connective $\mathbb{C}$-dg-algebras. Then the induced map $\mathrm{K}^{\mathrm{st}}(B) \to \mathrm{K}^{\mathrm{st}}(A)$ is an equivalence.*

After derived nil-invariance is established, proving most of the cases does not require much work, but for the case of local systems on M we need to understand topological K-theory of group algebras. This we can do only under some assumptions on the group, which corresponds to putting assumptions on the fundamental group of M. Concretely, we ask the group to satisfy the Burghelea conjecture and the rational Farrell-Jones conjecture, which are both established for a large class of groups, – and under these assumptions we prove the lattice conjecture. We also suggest a new approach to constructing a counterexample to the Farrell-Jones conjecture.

Structure of the paper. The first section is devoted to considering two realizations functors, which allow one to extend the functor of taking the space of complex points with analytic topology from schemes to spectral presheaves. We show that these two functors coincide, which allows us later to use properties of both.

The realization formalism was used in [Bla] to define semi-topological K-theory, which after inverting the Bott element becomes topological K-theory. In the second section, we recall the necessary definitions and statements from [Bla].

Semi-topological K-theory of a dg-category T is built from algebraic K-theories of different base-changes of T. And, while algebraic K-theory itself is a very complicated invariant, to some extent, it can be approximated by Hochshild and (variants of) cyclic homology. In the third section, we consider realizations of Hochschild-type invariants. In particular, we show that the realizations of HH and HC vanish.

In the fourth section, we recall the definition of derived nilpotent invariance and prove the Theorem 0.2 using the computations from the previous section.

The last section is devoted to considering consequences of Theorem 0.2. In particular, we prove Theorem 0.1 and sketch some other possible applications of our ideas.

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