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SENSITIVITY-BASED BOND PORTFOLIO IMMUNIZATION WITH NONPARAMETRIC TERM STRUCTURE MODELS

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Abstract

The traditional approach to bond portfolio immunization usually assumes that the possible future changes of the term structure of interest rates lie within a suitable parametric class of functions. The quantities of interest are the sensitivities of the portfolio value with respect to these parameters. Various kinds of term structure assumptions give rise to different bond portfolio immunization models—from the classical duration to key rate and parametric duration hedging. We propose a nonparametric version of this approach by introducing a suitable regularization—effectively imposing smoothness constraints on the term structure changes. This allows us to derive hedging equations without recourse to any specific parametric form. We test the proposed nonparametric immunization approach and find it performs slightly better than a traditional approach based on a popular Nelson-Siegel term structure parametric form.

JEL: G12, E43.

Keywords: immunization, bond portfolio, term structure of interest rates, nonparametric models.

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1 Introduction and Literature Review

Using a portfolio of bonds to immunize a financial obligation is a classical financial problem. Excellent review articles by Bierwag (2006) and Shah et al. (2020) gave detailed accounts of the history of the question and of possible approaches to it. De La Peña et al. (2021) discussed the use of this kind of immunization in the context of life insurance.

The literature differs in both the problem formulation and the assumptions made to arrive at a solution. Regarding problem formulation, there are two distinct branches of literature: worst-case immunization and sensitivity-based immunization. The worst-case branch initiated by Bierwag and Khang (1979) and Prisman (1986) considered a set of possible future scenarios and aimed at providing a max-min strategy to guarantee the best possible portfolio yield in the worst possible scenario. Fong and Vasicek (1984) derived a lower bound for the worst-case portfolio yield—maximizing this lower bound provided an immunization approach known as M^2 immunization. Balbás and Ibáñez (1998) and Barber and Copper (1998) considered other sets of possible scenarios—arriving at other bounds for the worst-case outcome. Balbás et al. (2002b), Balbás et al. (2002a) and Balbás and Romera (2007) further advanced this stream of research.

A very different and more mainstream branch is sensitivity-based immunization in which one considers a set of risk factors and immunizes a portfolio with respect to small changes in these factors. Immunization in this case might amount to simple delta-hedging, i.e. forming the portfolio so that the combined first-order sensitivities to small changes in all risk factors (also known as partial or parametric durations) should be zero. In more complicated cases immunization might involve second- or higher-order partial derivatives. This branch originated with Redington (1952) who considered the (flat) interest rate as the sole risk factor. The subsequent literature explored various risk factor specifications. Bierwag (1977) considered additive and multiplicative shocks to the term structure of interest rates as risk factors. Cooper (1977) assumed several specific parametric forms of the term structure and took these term structure parameters as factors. Willner (1996) used the now-classic term structure equation by Nelson and Siegel (1987). Boyle (1978), Ingersoll et al. (1978), Cox et al. (1979), and Beekman and Shiu (1988) among many others derived the term structure risk factors by postulating a stochastic differential equation governing the instantaneous short rate. Au and Thurston (1995) inferred durations as bond price sensitivities within a one-factor Heath-Jarrow-Morton model (Heath et al., 1992).

However, the risk factors in question do not have to originate from parametric assumptions. The duration-vector approach of Chambers et al. (1988) is equivalent to considering a Taylor expansion of an arbitrary term structure change—each term of the expansion introduces another risk factor in the model. Reitano (1990); Ho (1992) introduced the so-called key rate durations. Their approach is equivalent to assuming that the term structure is linearly interpolated from a set of key interest rates which are directly observed for a fixed set of key terms to maturity. Tark (1990) proposed estimating the most suitable risk factors via principal component analysis. D'ecclesia and Zenios (1994); Barber and Copper (1996); Hill and Vaysman (1998) developed and tested the corresponding immunization techniques. Lapshin (2019a) supposed that the term structure of interest rates is endogenously determined from observed prices of benchmark (government) coupon bonds via nonparametric least squares fitting with smoothing—thus, the observed prices of the benchmark bonds were the true risk factors in the model. Lapshin (2021) developed a similar model within the parametric term structure estimation approach of Nelson and Siegel (1987).

The remainder of the paper is organized as follows. Section 2 presents the general classification of existing immunization approaches and pinpoints the gap to be filled. Section 3 describes the model. Section 4 describes the data and the empirical analysis. Section 5 concludes.

2 The Framework

One of the two main modeling choices we consider is whether the term structure of interest rates is assumed to be exogenous and evolving over time or endogenous and determined by other variables which evolve and thus in turn provide for an evolving term structure estimate. Let P be the vector of observed bond prices and $r(t)$ be the term structure of interest rates for the term t to maturity. Endogenous term structure models can be schematically represented as

$$P \to r(\cdot),$$

where the arrow represents that the term structure $r(\cdot)$ is determined from bond prices P (which are assumed to evolve randomly).

Exogenous term structure models can be schematically represented as

$$r(\cdot) \to P,$$

where $r(\cdot)$ evolves randomly and determines P.

The second choice is whether the function $r(t)$ is assumed to be parametric or non-parametric. With respect to these two aspects, there are four possibilities:

1. Parametric exogenous term structure. This is the classical setup of parametric duration hedging dating back to Cooper (1977).
2. Parametric endogenous term structure. This is the setup of Lapshin (2021).
3. Nonparametric endogenous term structure. This is the setup of Lapshin (2019a).
4. Nonparametric exogenous term structure. To the best of our knowledge, this combination has not yet been implemented; this is where our paper comes in.

We now consider all four setups within the same framework and notation to ease comparison.

2.1 Parametric Exogenous Term Structure

This is the classical setup which we repeat here for the sake of completeness and to introduce the notation. Assume a parametric equation for the term structure $r(t)$:

$$r(t) = r_\theta(t), \tag{1}$$

where t is the term to maturity and θ is the vector of parameters. For example, a classical Nelson-Siegel parametric model has

$$r_\theta(t) = \beta_0 + \beta_1 \frac{1-e^{-\frac{t}{\tau}}}{t/\tau} + \beta_2 \left(\frac{1-e^{-\frac{t}{\tau}}}{t/\tau} - e^{-\frac{t}{\tau}} \right) \tag{2}$$

with $\theta = (\beta_0, \beta_1, \beta_2, \tau)$. We assume that the model parameters θ evolve randomly over time and that Eq. (1) always holds.

The bond prices P_k for $k = 1..K$ are assumed to be determined by the market from the term structure via discounted cash flows:

$$P_k = \sum_{i=1}^{N_k} F_{k,i} e^{-r(t_{k,i})t_{k,i}} = \mathrm{PV}(F_k, r), \tag{3}$$

where $F_{k,i}$ is the cash flow promised by the bond k at term $t_{k,i}$ to maturity.

In this classical setting, the only risk factors affecting the portfolio value are the parameters θ, thus the portfolio should be immunized by making its sensitivities to small changes in θ equal to zero. Let w_k be the amount of bond k in the immunized portfolio. The portfolio value V is thus

$$V(w, r) = -\mathrm{PV}(F_0, r) + w^T \mathrm{PV}(F, r), \tag{4}$$

where the index $k = 0$ denotes the original bond-like obligation to be immunized with all other bonds.

The sensitivities to small changes in θ are

$$\frac{\partial V}{\partial \theta} = -\frac{\partial \mathrm{PV}(F_0, r)}{\partial \theta} + w^T \frac{\partial \mathrm{PV}(F, r)}{\partial \theta} = 0,$$

which can be written as

$$QBw = QB_0, \tag{5}$$

where Q is the matrix of term structure sensitivities determined by the nature of the parametric term structure model used:

$$Q_{i,j} = \left(\frac{\partial r_\theta(t_i)}{\partial \theta_j} \right)^T$$

with B and B_0 being the matrices of instrument sensitivities determined by the nature of the financial instruments used:

$$B_{i,k} = \left(\frac{\partial \mathrm{PV}(F_i, r)}{\partial r(t_i)} \right)^T = -t_i F_{k,i} e^{-r(t_i)t_i},$$

where, for the sake of notation, we assumed that the cash flow times t_i are common for all instruments. If this is not the case, zero cash flows $F_{k,i} = 0$ can be introduced where necessary. The quantities $\frac{\partial PV(F,r)}{\partial \theta_n}$ are known as parametric or partial durations—price sensitivities with respect to changes in model parameters θ_n.

Note that even though B only depends on the current interest rates r, Q is likely to depend on the current estimate of model parameters θ. It is common to find them via nonlinear least squares:

$$\sum_{k=1}^{N}\left(\sum_{i=1}^{N} F_{k,i}e^{-r_\theta(t_i)t_i} - P_k\right)^2 \to \min_\theta, \tag{6}$$

where P_k are the observed bond prices. Gilli et al. (2010) showed that for the popular Nelson-Siegel term structure Eq. (2) this problem had relatively bad numerical properties and that in practice the solution was likely to be suboptimal.

Convexity and higher-order bond price sensitivities are also common in bond portfolio immunization. However they fall completely within this framework. As discussed by Chambers et al. (1988), bond convexity represents not only the second order sensitivity to parallel shifts but also the first order sensitivity to linear changes in the term structure. Analogously with higher order bond price sensitivities. Let $r(t) = \theta_1 + \theta_2 t + \theta_3 t^2 + \cdots + \theta_n t^{n-1}$ be the Taylor expansion of the term structure $r(t)$. Then

$$\frac{\partial \mathrm{PV}(F_k, r)}{\partial \theta_j} = -\sum_{i=1}^{N} t_i^j F_{k,i} e^{-r(t_i)t_i} = -\mathrm{PV}(F_k, r) D_{k,j},$$

where $D_{k,j}$ is the j-th order duration of bond k.

Unfortunately, this only allows immunizing with n bonds, where n is the number of term structure parameters. If there are more bonds available in the market (which is usually the case), modifications are in order. A common modification of Eq. (3) incorporates i.i.d. random pricing errors $\varepsilon \sim N(0, \Sigma_\varepsilon)$:

$$P_k = \mathrm{PV}(F_k, r) + \varepsilon_k. \tag{7}$$

Immunization is now understood as minimizing the variance of the portfolio price V:

$$\mathrm{Var}[V] = \mathrm{Var}[-\mathrm{PV}(F_0, r) - \varepsilon_0 + w^T(\mathrm{PV}(F, r) + \varepsilon)] \to \min_w.$$

Assuming the vector of parameter changes $\Delta\theta \sim N(0, \Sigma_\theta)$ is small and using the first order approximation, we get

$$\mathrm{Var}[V] = (Bw - B_0)^T Q^T \Sigma_\theta Q(Bw - B_0) + w^T \Sigma_\varepsilon w + \Sigma_\varepsilon,$$

which is a quadratic form in w minimized for

$$w = (B^T Q^T \Sigma_\theta Q B + \Sigma_\varepsilon)^{-1} B^T Q^T \Sigma_\theta Q B_0.$$

Assuming i.i.d. parameter changes $\Sigma_\theta = I$ and negligibly small price errors $\Sigma_\varepsilon = \alpha I$ for $\alpha \to 0$, with a bit of matrix algebra one can get that

$$w = B^T Q^T (QBB^T Q^T)^{-1} Q B_0, \tag{8}$$

which is exactly the least squares solution to Eq. (5), commonly used in practice when the number of bonds K is greater than the number of term structure parameters n. Nonzero choices for α will lead to regularized solutions satisfying Eq. (5) only approximately in order to achieve robustness.

However, there are other possible model setups.

2.2 Parametric Endogenous Term Structure

Here the logic is reversed—bond prices P are randomly evolving while the term structure $r(t)$ is determined by fitting the term structure parameters θ to observed bond prices P via least squares:

$$\sum_{k=1}^{N}\left(\sum_{i=1}^{N} F_{k,i} e^{-r_\theta(t_i)t_i} - P_k\right)^2 \to \min_{\theta}.$$

The obtained term structure $r_{\theta(P)}(t)$ can then be used to price either the entire portfolio:

$$V(w,P) = -\operatorname{PV}(F_0, r_{\theta(P)}) + w^T \operatorname{PV}(F, r_{\theta(P)})$$

or the original obligation only:

$$V(w,P) = -\operatorname{PV}(F_0, r_{\theta(P)}) + w^T P.$$

In both cases the risk factors are the observed bond prices P, so the immunization condition is

$$\frac{\partial V(w,P)}{\partial P} = 0,$$

which was shown by Lapshin (2021) to hold for

$$w = (B^T Q^T (QBB^T Q^T + A_\theta)^{-1} QB)^{-1} B^T Q^T (QBB^T Q^T + A_\theta)^{-1} QB_0$$

if the entire portfolio is priced via the estimated term structure or

$$w = B^T Q^T (QBB^T Q^T + A_\theta)^{-1} QB_0 \tag{9}$$

if only the original obligation is priced with the model. Here

$$A_\theta = \sum_{k=1}^{K} (\operatorname{PV}(F_k, r_{\theta(P)}) - P_k) \frac{\partial^2 \operatorname{PV}(F_k, r_\theta)}{\partial \theta^2}.$$

The formulas are a bit more complicated (although these are still closed form solutions). Lapshin (2021) examined this immunization setup and demonstrated it to be roughly equivalent to the classical immunization with exogenous term structure in both hedging portfolio composition and risk reduction—but only if the number of bonds in the market is large enough.

2.3 Nonparametric Endogenous Term Structure

In this setup bond prices are still the primary risk factors but the term structure is now estimated in a nonparametric fashion—via minimizing some sort of smoothness functional, e.g.

$$\sum_{k=1}^{K}\left(P_k - \sum_{i=1}^{N} F_{k,i} e^{-r(t_i)t_i}\right)^2 + \gamma \int_0^T [r^{(s)}(x)]^2\, dx \to \min_{r(\cdot)}, \tag{10}$$

where $r^{(s)}$ stands for the s-th derivative of the function $r(\cdot)$, γ is the regularization parameter and the term structure $r(\cdot)$ is sought within a suitable class of functions over $[0, T]$.

The hedging coefficients can still be derived from requiring sensitivities to the observed bond priced P_k to be zero (Lapshin, 2019a):

$$w = (B^T(BB^T + A_\theta + \gamma J^T J)^{-1}B)^{-1}B^T(BB^T + A_\theta + \gamma J^T J)^{-1}B_0, \tag{11}$$

where $J^T J$ is a nonnegative definite symmetrical matrix uniquely defined by the structure of the regularization functional and the set of bond cash flow terms t_i.

Lapshin (2019a) showed that these hedging weights naturally correspond to the classical hedging based on duration, duration and convexity, or higher-order sensitivities when the degree of smoothness $\gamma \to +\infty$. He also recovered the classical approach based on key rate durations as another particular case in a more restrictive theoretical setup.

Analogously, if the hedging instruments are assumed to be valued at their observed market prices P_k, the optimal hedging coefficients are given by a similar formula:

$$w = B^T(BB^T + A_\theta + \gamma J^T J)^{-1}B_0.$$

Note that even though the term structure is nonparametric (and thus infinite-dimensional), the number of risk factors is equal to the number of hedging instruments by construction—therefore the immunization problem always has a unique solution.

2.4 Nonparametric Exogenous Term Structure

To the best of our knowledge, this setup has not yet been studied in the literature. A nonparametric term structure function is an infinite-dimensional object—but sensitivity-based immunization requires all of its dimensions to be independent risk factors. One generally cannot hedge infinite number of risk factors with a finite number of instruments. Therefore, a suitable regularization is key to building a successful and useful model. This is the subject of the rest of this paper.

3 The Model

We assume that the term structure $r(\cdot)$, or rather its change $\Delta r(\cdot)$, is an infinite-dimensional risk factor randomly sampled from the space of sufficiently smooth functions. We adopt Gaussian process formalism to build a model. Let $\Delta r(t)$ be Gaussian with a known correlation function

(kernel)

$$\text{corr}[\Delta r(t), \Delta r(t')] = k(t,t')$$

subject to the usual positive semi-definiteness constraint

$$\int_0^T \int_0^T k(t,t') f(t) f(t') \, dt \, dt' \geq 0$$

for all $f \in L_2[0,T]$.

The choice of the kernel function k is tied to our assumptions about the smoothness of $r(\cdot)$ and $\Delta r(\cdot)$—whether we assume them to be continuous, differentiable, etc. The exact nature of these smoothness assumptions is not important now and will be discussed when choosing the kernel k in the next section. In what follows, we consider radial kernels, i.e. with the correlation depending only on the distance between the two points: $k(t,t') = k(|t-t'|)$. We do so mainly for simplicity, however nothing prevents one from using a more general kernel.

We now derive the optimal immunization weights from this assumption. The portfolio value V given the immunization weights w and the term structure r is

$$V(w,r) = -\text{PV}(F_0, r) + w^T \, \text{PV}(F, r),$$

which only depends on the values of $r(t_i)$ which are jointly normal:

$$r(t_i) \sim N(r^*, \Sigma_r),$$

where r_i^* is the current (unobserved) interest rate for the term t_i and $(\Sigma_r)_{i,j} = \sigma_i \sigma_j k(t_i, t_j)$ is the covariance matrix implied by kernel $k(\cdot,\cdot)$. Its variance is therefore given by

$$\text{Var}[V] = (Bw - B_0)^T \Sigma_r (Bw - B_0) + w^T \Sigma_\varepsilon w + \Sigma_\varepsilon, \tag{12}$$

which is minimized by

$$w = (B^T \Sigma_r B + \Sigma_\varepsilon)^{-1} B^T \Sigma_r B_0,$$

or, assuming i.i.d. ε_k and equal σ_i,

$$w = (B^T K B + \alpha I)^{-1} B^T K B_0, \tag{13}$$

where K is the correlation matrix derived from the kernel function $k(t_i, t_j)$ and α incorporates the ratio of variances of Δr and ε.

To finish the setup, we have to specify the kernel $k(t,t')$ and the signal-to-noise ratio α.

3.1 Choosing the Kernel

As discussed by Stein (1999), guessing a kernel based on scarce and noisy data is not a good idea. Therefore, we assume a Matérn kernel, which is widely used due to the presence of the parameter ν governing the degree of smoothness of random samples—from almost surely not differentiable to infinitely differentiable.[3] Due to the analytical simplicity, the following kernels are the most popular:

$$
\begin{aligned}
k_{1/2}(t,t') &= e^{-\frac{|t-t'|}{h}}, \\
k_{3/2}(t,t') &= \left(1+\frac{\sqrt{3}|t-t'|}{h}\right)e^{-\frac{\sqrt{3}|t-t'|}{h}}, \\
k_{5/2}(t,t') &= \left(1+\frac{\sqrt{5}|t-t'|}{h}+\frac{5|t-t'|^2}{3h^2}\right)e^{-\frac{\sqrt{5}|t-t'|}{h}}, \\
k_{\infty}(t,t') &= e^{-\frac{|t-t'|^2}{2h^2}},
\end{aligned}
$$

where h is the parameter governing the characteristic length of the changes which we expect to be of more practical importance than the choice of the kernel. For each of the four kernels we try h on a reasonable grid of values from 0.01 to 100 years and choose the best model.

3.2 Choosing the Signal-to-Noise Ratio

In contrast to the kernel, the signal-to-noise ratio α is close to observable, as we describe below, although we need to use either a holdout sample or a small, randomized subsample for the estimation of α to avoid overfitting.

As discussed by Rasmussen and Williams (2006), any chosen kernel can be viewed as imposing a certain prior probability distribution over the space of term structures $r(\cdot)$. Given this prior distribution, the observation model (7), and the observed values P, we can form the posterior distribution for both $r(\cdot)$ and Σ_ε via infinite-dimensional Bayesian inference as described by Lapshin (2019b). However, this approach is computationally intensive, therefore we propose an approximation as described below.

Instead of the Bayesian estimate of $r_P(\cdot)$ and Σ_ε, we can first find the *maximum a posteriori* (the Bayesian counterpart to the maximum likelihood) estimate of $r_P(\cdot)$ via a variant of (10):

$$
\sum_{k=1}^{K}\left(P_k-\sum_{i=1}^{N}F_{k,i}e^{-r(t_i)t_i}\right)^2+\gamma J_k\left[r(\cdot)\right], \tag{14}
$$

where the functional $J_k[\cdot]$ is determined by the chosen kernel. Smoothing splines arising as the solution to (10) are a particular case of this for a very special choice of the kernel (Rasmussen and Williams, 2006). However, it turns out that the next step is robust to different specifications of $J_k[\cdot]$, so we can use the solution to (10), which is much easier to compute, instead of the true *maximum a posteriori* estimate given by (14).

Now, given an estimate $r_P(\cdot)$, either *maximum a posteriori* or obtained otherwise, we calculate

[3] Smoothness characteristics of random samples generated by the Matérn kernel are discussed in more detail by Kanagawa et al. (2018).

the residuals

$$\varepsilon_k = \sum_{i=1}^{N} F_{k,i} e^{-r_P(t_i)t_i} - P_k,$$

which allows us to estimate Σ_ε or σ_ε since we assume $\mathrm{Var}[\varepsilon_k] = \sigma_\varepsilon^2$ to be equal for all k.

Now consider a portfolio consisting of a single observed bond k ($B_0 = B_k$) without hedging ($w = 0$). Then Eq. (12) becomes

$$\frac{1}{\sigma_\varepsilon^2} \mathrm{Var}[P_k] = \frac{1}{\alpha} B_k^T K B_k + 1,$$

which is linear in α^{-1}. We can now estimate α^{-1} from samples of P_k and B_k via simple linear regression.

This approximation seems rather crude, but we have found the overall results to be quite robust to variations in α up to the changes by a factor of 3–5. We thus need only to estimate the right order of magnitude for α, which we believe this algorithm suffices for.

4 The Empirical Analysis

Our dataset consists of end-of-day Spanish government bonds closing prices from 1996 to 2020 obtained from Bloomberg. All bonds have coupons; we drop the bonds for which the payment amounts were not known in advance. Figure 1 has a dot for a given date if a bond with a given maturity was traded on that day. Figure 2 presents a histogram of numbers of traded bonds—it ranged from 9 to 44 with typical values from 24 to 34. We use the first half of the data as the training set—to choose model parameters. The second half is used to estimate out-of-sample performance metrics.

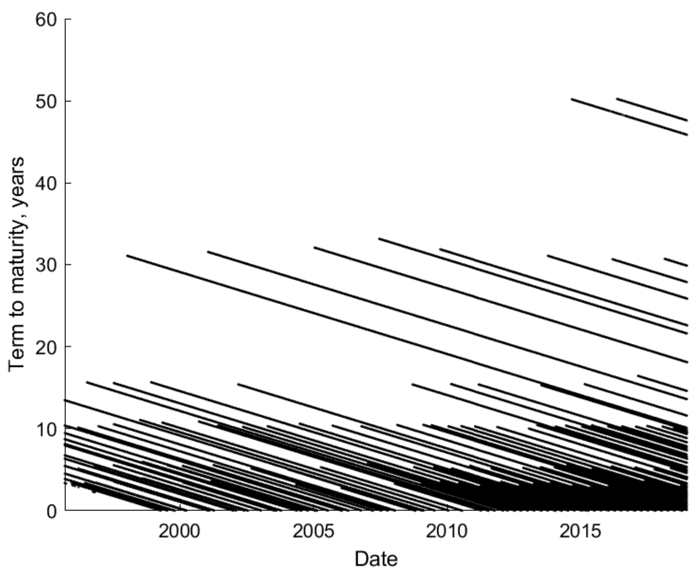

Figure 1: Maturities of all bonds in the dataset.

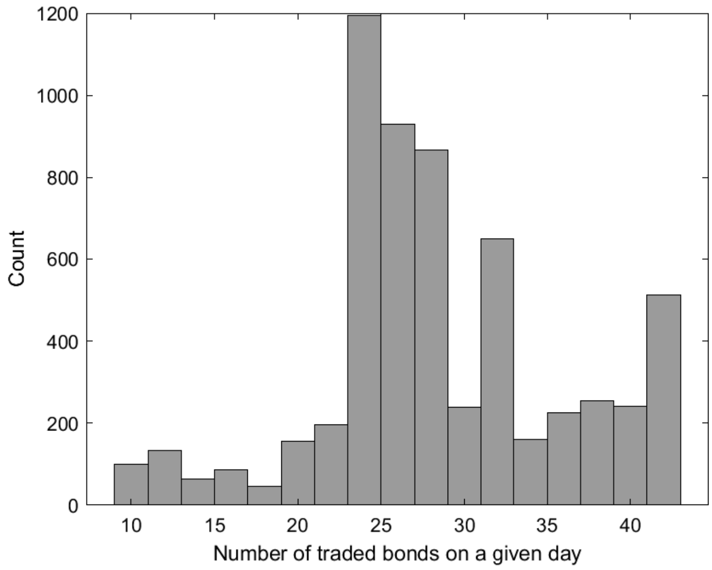

Figure 2: Number of bonds in the dataset.

We test the immunization performance via a randomized leave-out-one cross validation procedure. We consider a random trading day τ (in what follows we assume $\tau = 0$ for the sake of notation) and a random bond k traded on that day, which is neither the shortest nor the longest bond. This bond is chosen to model the obligation to be hedged—and is therefore assigned the index 0. We use all other bonds maturing after the immunization horizon to hedge this obligation according to Eq. (13).

We use 1 year as the immunization horizon as most financial reporting is done at least annually and do not consider interim portfolio rebalancing. All intermediate coupon payments are assumed to be reinvested until the end of the immunization period at the interest rate prevailing at the moment of receiving the payment.

If perfect immunization were possible, the immunized portfolio would be truly riskless and would therefore earn the risk-free rate. Therefore, if the initial portfolio consisted of a short position in 1 unit of bond 0 and the positions in other bonds described by the vector w, its projected price would be

$$V^H(w) = V^0(w)e^{r^0(H)H} + \sum_{i \| t_i < H} w^T F_{\cdot,i} e^{f^0(t_i,H)(H-t_i)},$$

where H is the immunization horizon, $r^0(\cdot)$ is the term structure estimated at time 0,

$$f(t_i, H) = r^0(H)H - r^0(t_i)t_i$$

is the forward rate for investing at time t_i until time H implied by the term structure $r^0(\cdot)$ observed at time 0, and the initial portfolio price $V^0(w)$ is given by

$$V^0(w) = -P_0 + w^T P.$$

The quality of immunization is measured by the mean absolute value of Δ, the deviation of the actual financial result from the value $V^H(w)$ one could hope to get:

$$\Delta = -P_0^H + w^T P^H + \sum_{i \| t_i < H} w^T F_{\cdot,i} e^{r^{t_i}(T-t_i)\cdot(T-t_i)} - V^H(w), \tag{15}$$

where P^H and P_0^H are the prices observed at the end of the immunization horizon and $r^{t_i}(x)$ is the interest rate estimated at time t_i for the term x to maturity, which is assumed to be the actual investment rate for the coupons received at time t_i.

The hedging error Δ is then calculated for $N = 10,000$ randomly chosen trading days to play the part of the day 0 in Eq. (15), each time with a randomly chosen bond substituted for the obligation to be hedged (bond 0 in Eq. (15)). Then we calculate the mean absolute hedging error for each hedging method over N simulations.

The hedging methods we consider and compare in our empirical exercise are as follows.

1. **No hedging**. We let $w = 0$, therefore we have $\Delta = -P_0^H + P_0 e^{r^0(H)H}$.

2. **Duration-based**. This is the usual approach based on reducing the duration of the immunized portfolio to zero. We choose the least squares portfolio among all portfolios satisfying the immunization equation.

3. **Duration-Convexity**. This approach sets both the duration and the convexity of the immunized portfolio to zero. We choose the least squares portfolio among all portfolios satisfying the immunization equations.

4. **Parametric Exogenous**. We use Nelson-Siegel parametric hedging as described by Eqs. (2) and (8) in Section 2.1.

5. **Parametric Endogenous**. Again, we use Nelson-Siegel parametric hedging, now described by Eq. (9) in Section 2.2.

6. **Nonparametric Endogenous**. We use the spline approach as described by Eqs. (10) and (11) in Section 2.3. We try both $s = 1$ (piecewise linear splines) and $s = 2$ (cubic splines).

7. **Nonparametric Exogenous**. This is the approach we describe in Section 3. Optimal parameters were estimated on the training sample and found to be $\alpha \approx 7$, $\varepsilon \approx 0.0043$, $\nu = 3/2$, $h \approx 0.25$ years.

Table 1 presents the results of our numerical experiment—the mean absolute hedging errors and their standard deviations. We have found the relative performance on the train and test samples to be very similar, therefore we report only the latter.

Note that even though Table 1 suggests that the nonparametric endogenous approach with linear splines is the clear winner, there are many factors influencing the outcome of such comparison experiment. Lapshin (2022) shows that some modelling choices implicitly or explicitly made during such experiment might significantly influence the results. Therefore, we cautiously interpret the results as indicative of the performance of the proposed method being comparable with the alternatives, thus justifying further empirical studies.

Hedging method	MAE, basis points
No hedging	497 (6.7)
Duration	149 (1.5)
Duration and convexity	93.2 (0.9)
Parametric Exogenous (Nelson-Siegel)	40.4 (0.4)
Parametric Endogenous (Nelson-Siegel)	49.0 (1.2)
Nonparametric Endogenous (linear)	44.0 (0.5)
Nonparametric Endogenous (cubic)	29.6 (0.3)
Nonparametric Exogenous (new)	39.9 (0.4)

Table 1: Mean absolute hedging errors for various methods. Estimated standard deviations in parentheses.

5 Conclusion

Nonparametric bond portfolio immunization without assuming any particular form of the term structure equation is completely feasible—gaussian processes provide all the necessary machinery, however some parameters still need to be estimated though. With reasonable parameter tuning, nonparametric immunization performs slightly better than parametric in an out-of-time cross-validation exercise—and significantly better than the traditional approach based on duration matching and duration-convexity matching.

A technical advantage of the nonparametric approach is even though it requires a term structure estimate for calculations (the matrix B depends on the term structure), this estimate does not have to come from any specific term structure estimation method. This could be viewed as an advantage over the parametric immunization approach, where we need to estimate the current term structure equation parameters in order to carry out the hedging. We can thus use any term structure estimation method, either parametric or nonparametric. Moreover, the nonparametric hedging portfolio turns out to be quite robust to perturbations in the estimated term structure. This is in sharp contrast to the parametric approach, which is quite sensitive to the solutions of the nonlinear least squares problem in Eq. (6) being suboptimal.

There is not enough evidence to recommend switching from the existing parametric approach to the proposed nonparametric one. However, it is a viable candidate if an immunization procedure is only being developed. We can also recommend the proposed approach for consideration in practical situations where parametric term structure estimates are unavailable or unreliable.

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