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Newton polytopes, sparse resultants and enumeration of singularities

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Introduction

This work addresses the following two problems, arising in elimination theory in the context of Newton polytopes. The first one is to compute the signs of the leading coefficients of the sparse mixed resultant. The second problem concerns the singular points of a plane projection of a complete intersection curve given by a generic system of polynomials with given support.

In a broader sense, this work is devoted to developing new methods of enumeration of singularities which arise naturally in the context of Newton polytopes and sparse polynomials.

As an application of the obtained methods we describe the Newton polytope of the Morse discriminant – the closure of the set of all non-Morse polynomials in the space of polynomials of a given degree.

In subsections below, we provide a more detailed overview of the abovementioned problems and the application.

Leading coefficients of the sparse resultant

The classical resultant was initially studied by Sylvester (1853), and later extended to the case of a system of n homogeneous polynomials in n variables by Cayley (1948) and Macaulay (1902). In the 1990s, the advances in several fields, such as symbolic algebra and multivariate hypergeometric functions, revived the interest in resultants. Sparse resultants were introduced and studied by Gelfand, Kapranov, Zelevinsky, and Sturmfels (see e.g. [9]). In particular, in [17], Sturmfels gives an explicit combinatorial construction of the Newton polytope of the sparse resultant, and proves that the leading coefficient of the resultant with respect to an arbitrary monomial order is equal to ± 1. However, the signs of such coefficients have been computed explicitly only for some special cases so far, although the general answer might be useful for the purposes of real algebraic geometry.

In our work, we construct the *2-mixed volume* (Definition 3.14), which is an analogue of the classical mixed volume of convex lattice polytopes taking values in $\mathbb{F}_2$. Besides that, we express the signs of the leading coefficients of the sparse resultant in terms of the 2-mixed volume of certain tuples of polytopes (Theorem 3.19).

The 2-mixed volume is a symmetric and multilinear function of lattice polytopes (Proposition 3.15). However, its convex-geometric nature remains unclear, because we cannot define it as a polarization of any kind of an additive measure, or characterize it by any kind of its monotonicity properties.

Our explicit formula for the 2-mixed volume employs the so-called 2-determinant, that is, the unique nonzero multilinear function of $n+1$ vectors in the n-dimensional vector space over the field $\mathbb{F}_2$ which ranges in $\mathbb{F}_2$, remains invariant under all linear transformations, and equals zero whenever the rank of the $n+1$ vectors is less than n. This function implicitly appeared in the context of the class field theory for multidimensional local fields by Parshin and Kato (see e.g. Remark 1 in Section 3.1 of [15], which is probably the first occurence of the 2-determinant in the literature). Later this notion was explicitly introduced in full generality by A.Khovanskii in [10] for the purpose of his multidimensional version of the Vieta formula (i.e. the computation of the product in the group $(\mathbb{C} \setminus \{0\})^n$ of all the roots for a system of n polynomial equations with sufficiently generic Newton polytopes).

The algebro-geometric part Chapter 3 of our work includes an extension of related results by A. Khovanskii. In particular, our notion of the 2-mixed volume is the result of our effort to provide an invariant interpretation of the sign in Khovanskii's multivariate version of the Vieta formula, and to relax the genericity assumptions on the Newton polytopes in this formula. The convex-geometric part employs the techniques of tropical geometry to prove the existence of the 2-mixed volume (Theorem 3.13).

Singular points of a plane projection of a complete intersection curve

One of the main tools in the study of singularities of maps is the theory of Thom polynomials. They express the fundamental classes of the multisingularity strata for a generic map of arbitrary compact smooth manifolds in terms of their characteristic classes. This theory is however not applicable to a natural class of generic maps, namely the maps between varieties that are defined by generic Laurent polynomials with given Newton polytopes. Such varieties are not compact, and the maps are not proper. At the same time, their toric compactifications associated with the Newton polytopes do not satisfy the genericity conditions necessary for classical Thom polynomials to be applicable (for details see e.g. Example 1.1 and Remark 1.2 in [6]). Therefore, working with multisingularity strata of the abovementioned class of maps requires alternative methods. We will now give a short overview of developments in this direction.

The discriminant (i.e. $\mathcal{A}_1$ stratum) of a projection of a generic hypersurface was described in [16]. If H is a hypersurface given by a generic polynomial $f(x_1, \dots, x_d, y)$ and π is the projection forgetting the last coordinate, then the Newton polytope of the polynomial defining the abovementioned $\mathcal{A}_1$ stratum is equal to the fiber polytope $\mathcal{Q}_\pi(f) \subset \mathbb{R}^d$ of the Newton polytope $\mathcal{N}(f)$.

The image (i.e. $\mathcal{A}_0$ stratum) of a projection of a generic complete intersection was studied by A. Esterov and A. Khovanskii in [4]. They proved that the image under an epimorphism $\pi\colon (\mathbb{C}\setminus\{0\})^n \to (\mathbb{C}\setminus\{0\})^{n-k}$ of a complete intersection $\{f_1 = \ldots = f_{k+1}\} \subset (\mathbb{C}\setminus\{0\})^n$ defined by generic polynomials with given Newton polytopes $\mathcal{N}(f_i) = \Delta_i$ is a hypersurface $\{g = 0\} \subset (\mathbb{C} \setminus \{0\})^{n-k}$, whose Newton polytope $\mathcal{N}(g) \subset \mathbb{R}^{n-k}$ is equal to the so-called *mixed fiber polytope* $\mathrm{MP}_\pi(\Delta_1, \dots, \Delta_{k+1})$ of the polytopes $\Delta_1, \dots, \Delta_{k+1}$.

An approach to studying the strata of higher codimension, e.g for $\mathcal{A}_2$ (cusps) and $2\mathcal{A}_1$ (double points), is suggested in [6]. However, this approach only works under additional assumptions on the Newton polytopes and employs operations with tropical fans of dimension that is too high for practical applications (namely, those dimensions are of the same order as the number of monomials inside the given Newton polytopes).

For low dimensional maps or projections, the above mentioned strata are 0-dimensional, and the problem of computing their cardinalities in terms of the given Newton polytopes arises naturally. For example, this problem was solved in [7] for a mapping $\mathbb{C}^2 \to \mathbb{C}^2$ whose components are generic polynomials of given degrees. See [6] for an overview of some other literature on problems of this kind.

To the best of our knowledge, our work is the first one where such a problem is solved for polynomials with arbitrary Newton polytopes.

Let us state the main result of Chapter 4.

Theorem 1. *Let Δ be a lattice polytope in $\mathbb{Z}^n \oplus \mathbb{Z}^2$. If, for generic polynomials $f_1, \dots, f_{n+1}$ supported at Δ, the image of the complete intersection curve*

$$\widetilde{\mathcal{C}} = \{f_1 = \ldots = f_{n+1} = 0\} \subset (\mathbb{C} \setminus \{0\})^n \times (\mathbb{C} \setminus \{0\})^2$$

under the projection $\pi\colon (\mathbb{C} \setminus \{0\})^n \times (\mathbb{C} \setminus \{0\})^2 \to (\mathbb{C} \setminus \{0\})^2$ is a reduced nodal curve, then the number of its nodes is given by formula (5) in Theorem 4.9.

Remark 2. The classification of the Newton polytopes for which the abovementioned projection is not a nodal curve, is a non-trivial problem (see Example 4.2) that is not addressed in this thesis. Instead, we will prove a certain generalization of Theorem 1 which is applicable to all lattice polytopes Δ (see Theorem 4.9).

For some support sets $A \subset \mathbb{Z}^{n+2}$, it is quite easy to show that the projection of a complete intersection given by generic polynomials supported at A has only nodes as singularities. For instance, this is the case for $A = dT \cap \mathbb{Z}^3$, where $d \in \mathbb{Z}_{>0}$, and $T \subset \mathbb{R}^3$ is the standard simplex.

Example 3 (Counting the nodes of the projection of a complete intersection curve defined by generic equations of given degree)**.** Let us apply (5) to $A = dT \cap \mathbb{Z}^3$, where $d \in \mathbb{Z}_{>0}$, and $T \subset \mathbb{R}^3$ is the standard simplex. In the notation of this formula, we then have $n = 1$, the area of the polygon P is equal to d^4, the term d^2 comes from the area of the horizontal facet of dT, and the non-horizontal facets do not contribute, since for every $\Gamma \in \mathcal{F}(dT) \setminus \mathcal{H}(dT)$, we have $\mathrm{ind}_v(\Gamma \cap A) = 1$. Thus, the answer is

$$Đ = \frac{d^4 - 2d^3 + d^2}{2} = \frac{d^2(d-1)^2}{2}.$$

Remark 4. The same answer for the number of nodes of the curve $\mathcal{C}$, as in Example 3 can be obtained by the double-point formula (see Theorem 9.3 in [8]). Passing to a suitable toric compactification, we reduce the computation to finding the number of double points for the corresponding map $F\colon X \to Y$, where $X \subset \mathbb{CP}^3$ is the closure of the curve $\widetilde{C}$ and $Y = \mathbb{CP}^2$. Applying the double-point formula

$$|2\mathcal{A}_1| = \frac{1}{2}\Big((F_*1)^2 - F_*c_1(F^*T_Y / T_X)\Big) \in H^4(Y, \mathbb{C})$$

to the case considered, we obtain

$$|2\mathcal{A}_1| = \frac{1}{2}(d^4 - (2d^3 - d^2)) = \frac{d^2(d-1)^2}{2}.$$

This approach, however, does not work in general, because it is directly applicable to counting self-intersections of the image only for a map whose multisingularities are stable, and our projection of the compactified curve may have arbitrarily complicated singularities and multisingularities at infinity.

The problem addressed in Chapter 4 of our work might also be of interest with regard to the study of algebraic knots, which is motivated by Viro's work [18] about the rigid isotopy invariant called encomplexed writhe, as well as the works of Mikhalkin and Orevkov (see e.g. [14], [13], [12]). Namely, it is quite natural to estimate the complexity of an algebraic knot/link (i.e., the minimal crossing number) in terms of the algebraic complexity of its defining equations (i.e., the size of their Newton polytopes).

Theorem 1 gives an upper bound for the number of self-intersections of a projection of a real complete intersection curve onto a coordinate plane. It is easy to show that this upper bound is sharp for the case of real complete intersection links given by a pair of polynomials of given degree. Much harder problems, such as obtaining a sharp upper bound for arbitrary plane projections (not only onto coordinate planes), or topological classification of complete intersection links given by polynomials with arbitrary Newton polytopes, still remain unsolved.

The Newton polytopes of objects such as the image or the discriminant M of the projection of a complete intersection are known (see [5], [4], [16]). Therefore a natural first step in the study of the singularities of M would be passing to the corresponding toric compactification $\overline{M}$. If $\overline{M}$ did not have any additional singularities, then the problem of describing the simplest singularity strata of M, such as $\mathcal{A}_2$ and $2\mathcal{A}_1$, could be solved using classical methods. Unfortunately, the compactification $\overline{M}$ does in general have singularities at infinity. Moreover, these singularities are significantly more complicated than the ones studied. Dealing with them turns out to be the most challenging part in this class of problems. Our approach is to realize M as the base space of a certain covering, such that the covering space has significantly less complicated singularities at infinity, namely, so-called *forking–path singularities.*

The Newton polytope of the Morse Discriminant

For a set $A \subset \mathbb{Z}$, by $\mathbb{C}^A$ we denote the space of Laurent polynomials with support in A :

$$\mathbb{C}^A = \{\sum_{p \in A} \alpha_p x^p \mid \alpha_p \in \mathbb{C}\}.$$

Definition 5. The *caustic* in the space $\mathbb{C}^A$ is the set of all Laurent polynomials $f \in \mathbb{C}^A$ such that the map $f\colon (\mathbb{C} \setminus \{0\}) \to \mathbb{C}$ has a degenerate critical point.

Definition 6. The *Maxwell stratum* in $\mathbb{C}^A$ is the set of all Laurent polynomials $f \in \mathbb{C}^A$ such that the map $f\colon (\mathbb{C} \setminus \{0\}) \to \mathbb{C}$ has a pair of coinciding critical values taken at distinct points.

Definition 7. A polynomial $f \in \mathbb{C}^A$ is called *Morse,* if it belongs neither to the caustic, nor to the Maxwell stratum.

Definition 8. The *Morse discriminant* is the closure of the set of all non-Morse polynomials $f \in \mathbb{C}^A$. It is given by the polynomial $h_m^2 h_c$, where h_m and h_c are polynomials defining the Maxwell stratum and the caustic, respectively, if these two sets are hypersurfaces. Otherwise we set the corresponding defining polynomial to 1.

The problem of describing the Newton polytope and other closely related invariants of the Morse discriminant was studied by various authors. The degree of the Morse discriminant for general degree d univariate polynomials was computed in [11]. The tropical fan of the variety of univariate degree d polynomials having two multiple roots was studied in [3] and in [6] (in a more general setting). The maximal cones of the tropical fan, that were computed in these works, under the projection along a line spanned by a constant monomial, define the directions of all the edges of the Morse polytope. However, due to non-trivial intersections of the images of the cones under this projection, the results obtained in [3] and in [6] cannot be directly used to enumerate the edges and vertices of the Morse polytope.

In Chapter 5, using the methods and results obtained in Chapters 3 and 4, we will compute an explicit formula for the support function of the Newton polytope of the Morse discriminant in the space of univariate polynomials of given degree, which yields a combinatorial description of all the vertices of this polytope.

3 Signs of the leading coefficients of the resultant

This chapter is organized as follows. Section 3.1 is devoted to the notion of the *2-mixed volume.* First, we recall the definition and the basic properties of the *2-determinant*, and use it to define the so-called *2-intersection number* of tropical hypersurfaces, which in fact depends only on the Newton polytopes of the hypersurfaces, provided that those Newton polytopes satisfy a certain genericity condition. The latter allows to construct a well-defined function of lattice polytopes — the so-called *2-mixed volume.*

Section 3.2 concerns the multivariate Vieta's formula which expresses the product of roots for a polynomial system of equations in terms of the 2-mixed volume of its Newton polytopes.

In section 3.3 we compute the signs of the leading coefficients of the resultant reducing this problem to finding the product of roots for a certain system of equations (see Theorem 3.19).

3.1 The 2-mixed volume

Definition 3.1. We define $\det_2$ to be the function of $n+1$ vectors in an n-dimensional linear space over $\mathbb{F}_2$, that takes values in $\mathbb{F}_2$ and satisfies the following properties:

- $\det_2(k_1, \ldots, k_{n+1})$ is equal to zero, if the rank of the collection of vectors $k_1, \ldots, k_{n+1}$ is smaller than n;
- $\det_2(k_1, \ldots, k_{n+1})$ is equal to $\lambda^1 + \ldots + \lambda^{n+1} + 1$, if the vectors $k_1, \ldots, k_{n+1}$ are related by the unique relation $\lambda^1 k_1 + \ldots + \lambda^{n+1} k_{n+1} = 0$.

Lemma 3.2. *The function* $\det_2$

1. *is* $\mathrm{GL}_n(\mathbb{F}_2)$*-invariant, i.e. for any linear transformation* $A \in \mathrm{GL}_n(\mathbb{F}_2)$ *the equality* $\det_2(k_1, \ldots, k_{n+1}) = \det_2(Ak_1, \ldots, Ak_{n+1})$ *holds;*

2. *is multilinear.*

Theorem 3.3. *[10] There exists a unique nonzero function* $\det_2$ *which satisfies the properties listed in Lemma 3.2.*

Theorem 3.4. *[10] In coordinates the function* $\det_2$ *can be expressed by the formula*

$$\det_2(k_1, \ldots, k_{n+1}) = \sum_{j>i} \Delta_{ij},$$

where Δ_{ij} *is the determinant of the* $n \times n$ *matrix whose first* $n-1$ *columns represent the sequence of vectors* $k_1, \ldots k_{n+1}$ *from which the vectors with the indices* i *and* j *are deleted, and the last column is the coordinate-wise product of the vectors* k_i *and* k_j*.*

Definition 3.5. Let $H_1, \ldots, H_n$ be tropical hypersurfaces. We say that $H_1, \ldots, H_n$ intersect *transversely* (denote by $H_1 \pitchfork \ldots \pitchfork H_n$), if $|H_1 \cap H_2 \cap \ldots \cap H_n| < \infty$ and all the points $x \in H_1 \cap H_2 \cap \ldots \cap H_n$ are smooth for every H_i.

Definition 3.6. Let $H_1 \pitchfork \ldots \pitchfork H_n$ be a transverse tuple of tropical hypersurfaces. The *intersection number* $\iota(H_1, \ldots, H_n) \in \mathbb{Z}$ is the sum

$$\iota(H_1, \ldots, H_n) \stackrel{\text{def}}{=} \sum_{x \in H_1 \cap H_2 \cap \ldots \cap H_n} \det(\mathcal{N}_x(H_1), \ldots \mathcal{N}_x(H_n)). \tag{1}$$

It is well known that the intersection number of tropical hypersurfaces depends only on their Newton polytopes (and coincides with the mixed volume of the Newton polytopes). This fact is often referred to as the tropical Bernstein–Kushnirenko formula. We shall need the following $\mathbb{F}_2$-verison of the intersection number.

Definition 3.7. Consider an arbirtary point $\zeta \in \mathbb{Z}^n$. Let $H_1 \pitchfork \ldots \pitchfork H_n$ be a transverse tuple of tropical hypersurfaces. We define the *2-intersection number* $\iota_2(H_1, \ldots, H_n; \zeta) \in \mathbb{F}_2$ as follows:

$$\iota_2(H_1, \ldots, H_n; \zeta) \stackrel{\text{def}}{=} \sum_{x \in H_1 \cap H_2 \cap \ldots \cap H_n} \det_2(\mathcal{N}_x(H_1), \ldots \mathcal{N}_x(H_n), \zeta). \tag{2}$$

Unfortunately, in general, the 2-intersection number does depend on the tropical hypersurfaces, and not only on their Newton polytopes. However, this dependence disappears if the Newton polytopes themselves are in general position in a sense that we describe below.

Definition 3.8. Let $P \subset \mathbb{R}^n$ be a polytope or a finite set. We define the *support face* of a covector $v \in (\mathbb{R}^n)^*$ to be the maximal subset of P on which $v\,|_P$ attains its maximum. We shall denote this face by P^v.

Definition 3.9. A finite set $P \subset \mathbb{Z}^n$ is called a 2*-vertex*, if for any pair of points $p_1, p_2 \in P$, $p_1 \equiv p_2 \pmod 2$ (i.e., the corresponding coordinates of the points p_1, p_2 are of the same parity). A lattice polytope is called a 2*-vertex*, if the set of its vertices is a 2*-vertex*.

Definition 3.10. Let $P_1, \ldots, P_n$ be convex lattice polytopes in $\mathbb{R}^n$ or finite sets in $\mathbb{Z}^n$, and ζ be a point in $\mathbb{Z}^n$. The tuple $P_1, \ldots, P_n$ is said to be *2-developed with respect to* ζ if, for any covector $v \in (\mathbb{Z}^n)^*$ such that $v(\zeta) \not\equiv 0 \mod 2$, there exists $i \in \{1, \ldots, n\}$ such that the support face P_i^v is a 2-vertex.

Definition 3.11. A tuple $P = (P_1, \ldots, P_n)$ of convex lattice polytopes is said to be ζ*–prickly*, if for any covector $v \in (\mathbb{R}^*)^n$ such that $v(\zeta) \neq 0$, there exists $i \in \{1, \ldots, n\}$ such that the support face P_i^v is a vertex.

Remark 3.12. Obviously, if a tuple P is ζ–prickly, then it is 2–developed with respect to ζ.

Theorem 3.13. *Consider a point $\zeta \in \mathbb{Z}^n$ and finite lattice sets $P_1, \ldots, P_n$. Suppose that $P_1, \ldots, P_n$ are 2-developed with respect to ζ. Then for any two tuples $(H_1, \ldots, H_n)$ and $(H'_1, \ldots, H'_n)$ of tropical hypersurfaces, whose equations are supported at $P_1, \ldots, P_n$, the 2-intersection numbers $\iota_2(H_1, \ldots, H_n; \zeta)$ and $\iota_2(H'_1, \ldots, H'_n; \zeta)$ coincide.*

Definition 3.14. For a tuple of polytopes $P_1, \ldots, P_n$, 2-developed with respect to $\zeta \in \mathbb{Z}^n$, consider generic tropical hypersurfaces $H_1, \ldots, H_n$, such that the equation of H_i is supported at the set of vertices of P_i. Then the function $\mathrm{MV}_2 \colon (P_1, \ldots, P_n; \zeta) \mapsto \iota_2(H_1, \ldots, H_n; \zeta)$ is well-defined. We call it the *2-mixed volume*.

Proposition 3.15. *The function MV_2 is symmetric and multiplinear with respect to the Minkowski summation of the arguments.*

3.2 Multivariate Vieta's formula

Take an arbitrary point $0 \neq a \in Z^n$ and consider an a–prickly tuple $P = (P_1, \ldots, P_n)$ of convex lattice polytopes in $\mathbb{R}^n$ (see Definition 3.11). By $\mathbb{C}_1^{P_i}$ we denote the set of all polynomials $f = \sum_{p \in P_i} c_p x^p$ such that $\mathcal{N}(f) = P_i$ and if $p \in P_i$ is a vertex, then $c_p \neq 0$. Consider the set $\mathbb{C}_1^P = \mathbb{C}_1^{P_1} \times \ldots \times \mathbb{C}_1^{P_n}$. The multivariate Vieta's formula expresses the product of the monomials x^a over all the roots x for a system of polynomial equations $f_1(x) = \ldots = f_n(x)$, where $F = (f_1, \ldots, f_n) \in \mathbb{C}_1^P$ and the coefficients of f_i at the vertices of its Newton polytope are equal to 1, in terms of the 2-mixed volume (see Section 3.1) of the polytopes $P_1, \ldots, P_n$ and the point a.

Namely, we can state the following result.

Theorem 3.16. *Under the same assumptions as above, we have*

$$\prod_{f_1(x) = \ldots = f_n(x), x \in (\mathbb{C} \setminus \{0\})^n} x^a = (-1)^{\mathrm{MV}_2(P_1, \ldots, P_n; a)}. \tag{3}$$

3.3 Computing the signs of the leading coefficients of the resultant

Definition 3.17. Consider a tuple $A = (A_0, \ldots, A_n)$ of finite sets in $\mathbb{Z}^n$ such that $\mathrm{codim}(A) = -1$ and the sets A_i jointly generate the affine lattice $\mathbb{Z}^n$. Then the *sparse mixed resultant* $\mathscr{R}_A$ is a unique (up to scaling) irreducible polynomial in $|A| = \sum_0^n |A_i|$ variables $c_{i,a}$ which vanishes whenever the Laurent polynomials $f_i(x) = \sum_{a \in A_i} c_{i,a} x^a$ have a common zero in $(\mathbb{C} \setminus \{0\})^n$.

By $|A_i|$ we denote the cardinality of the set $A_i \subset \mathbb{Z}^n$, and $|A|$ stands for the sum $\sum_0^n |A_i|$. For simplicity of notation, by $\mathscr{R}$ we denote the sparse mixed resultant $\mathscr{R}(A)$.

Consider the Newton polytope $\mathcal{N}(\mathscr{R})$ of the resultant $\mathscr{R}(A)$. Suppose that we are given a pair of gradings $\gamma = (\alpha_{i,a} \mid i \in \{0, \ldots, n\}, a \in A_i)$ and $\sigma = (\beta_{j,b} \mid j \in \{0, \ldots, n\}, b \in A_i) \in (\mathbb{Z}^*)^{|A|}$ with strictly positive coordinates such that the support faces $\mathcal{N}(\mathscr{R})^\gamma$ and $\mathcal{N}(\mathscr{R})^\sigma$ are 0-dimensional. We will now compute the quotient of the coefficients r_γ and r_σ of $\mathscr{R}$ which are leading with respect to the gradings γ and σ respectively, by reducing this problem to the multivariate Vieta's formula (see Theorem 3.16).

To the covectors γ, σ one can associate the tuple $P^{\gamma,\sigma} = (P_0^{\gamma,\sigma}, \ldots, P_n^{\gamma,\sigma})$ of polytopes in $\mathbb{R}^{n+1}$ such that

$$P_i^{\gamma,\sigma} = \mathrm{conv}(\{(a, \alpha_{i,a}) \mid a \in A_i\} \cup \{(a, -\beta_{i,a}) \mid a \in A_i\}).$$

Example 3.18. Let $A = (A_0, A_1)$, where $A_0 = \{0, 1\}, A_1 = \{0, 1, 2\} \subset \mathbb{Z}$. The Newton polytope $\mathcal{N}(\mathscr{R}(A))$ is a triangle with vertices $\bar{\gamma} = (2, 0, 0, 0, 1)$, $\bar{\sigma} = (0, 2, 1, 0, 0)$ and $\bar{\delta} = (1, 1, 0, 1, 0)$. Consider the covectors $\gamma = (2, 1, 1, 1, 2)$, $\sigma = (1, 2, 2, 1, 1)$, and $\delta = (2, 2, 1, 2, 1)$, whose support faces are the vertices $\bar{\gamma}, \bar{\sigma}, \bar{\delta}$. Thus, we obtain the the polygons $P_0^{\gamma,\sigma}$ and $P_1^{\gamma,\sigma}$ (see Figure 1a) and the polygons $P_0^{\gamma,\delta}$ and $P_1^{\gamma,\delta}$ (see Figure 1b).

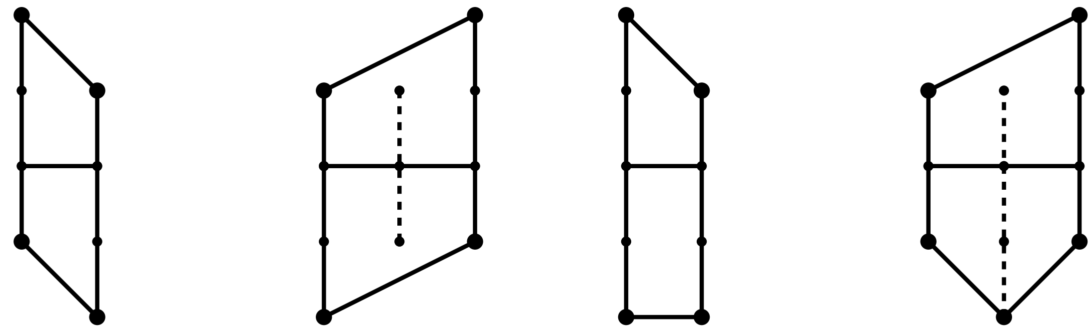

Figure 1a. Figure 1b.

Figure 1: The polygons $P_0^{\gamma,\sigma}, P_1^{\gamma,\sigma}$ and the polygons $P_0^{\gamma,\delta}, P_1^{\gamma,\delta}$

Theorem 3.19. *Let $A = (A_0, \ldots, A_n)$ be a tuple of finite sets in $\mathbb{Z}^n$ satisfying the properties given in Definition 3.17 and $\gamma, \sigma \in (\mathbb{Z}^*)^{|A|}$ be a pair of gradings with strictly positive coordinates and 0-dimensional support faces $\mathcal{N}(\mathscr{R})_\gamma$ and $\mathcal{N}(\mathscr{R})_\sigma$. Then the quotient of the coefficients r_γ and r_σ of $\mathscr{R}(A)$ that are leading with respect to the gradings γ and σ respectively can be computed as follows:*

$$\frac{r_\gamma}{r_\sigma} = (-1)^{\mathrm{MV}(P_0^{\gamma,\sigma}, \ldots, P_n^{\gamma,\sigma})}(-1)^{\mathrm{MV}_2(P_0^{\gamma,\sigma}, \ldots, P_n^{\gamma,\sigma}, \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right))}. \tag{4}$$

Example 3.20. Using Theorem 3.19, let us compute the quotient of the coefficients r_γ and r_σ corresponding to the vertices $\bar{\gamma}$ and $\bar{\sigma}$ which were considered in 3.18:

$$\frac{r_\gamma}{r_\sigma} = 1 \cdot (-1)^{\det_2\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right) + \det_2\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right) + \det_2\left(\begin{smallmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \end{smallmatrix}\right)} = (-1)^0 = 1.$$

For the coefficients r_γ and r_δ corresponding to the vertices $\bar{\gamma}$ and $\bar{\delta}$, we obtain

$$\frac{r_\gamma}{r_\delta} = 1 \cdot (-1)^{\det_2\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right) + \det_2\left(\begin{smallmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{smallmatrix}\right) + \det_2\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) + \det_2\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)} = (-1)^1 = -1.$$

Thus, we obtain the well-known formula for the resultant $\mathscr{R} = \mathscr{R}(f, g)$ of the polynomials $f = a_0 + a_1 x, g = b_0 + b_1 x + b_2 x^2$: we have $\mathscr{R} = \pm(a_0^2 b_2 + a_1^2 b_0 - a_0 a_1 b_1)$, just as expected.

The rest of this Subsection is devoted to the proof of Theorem 3.19.

Definition 3.21. To the gradings γ, σ, we can associate the *Khovanskii curve* $\mathscr{C}^{\gamma,\sigma} \subset \mathbb{C}^{|A|}$ parametrized by the complex parameter $t \neq 0$ and defined by the following equations: $z_{i,a} = t^{\alpha_{i,a}} + t^{-\beta_{i,a}}$, where $i \in \{0, \ldots, n\}$ and $a \in A_i$.

Restricting the resultant $\mathscr{R}$ to the Khovanskii curve $\mathscr{C}^{\gamma,\sigma}$, we obtain a Laurent polynomial in the variable t, which we denote by $\phi(t)$. The following statements are obvious.

Proposition 3.22. *The coefficient of the leading (lowest) term of $\phi(t)$ equals r_γ (r_σ, respectively).*

Proposition 3.23. *The equality $\phi(t_0) = 0$ holds if and only if the point with coordinates $(t_0^{\alpha_{i,a}} + t_0^{-\beta_{i,a}} \mid i \in \{0, \ldots, n\}, a \in A_i)$ belongs to the set $\{\mathscr{R} = 0\} \cap \mathscr{C}^{\gamma,\sigma}$.*

Remark 3.24. Note that the polytopes $P_0^{\gamma,\sigma}, \ldots, P_n^{\gamma,\sigma}$ are exactly the Newton polytopes of the Laurent polynomials $g_0(x,t), \ldots, g_n(x,t)$, where

$$g_i(x,t) = \sum_{a \in A_i} (t^{\alpha_{i,a}} + t^{-\beta_{i,a}}) x^a.$$

Proof of Theorem 3.19. Remark 3.24 implies that the equality (4) can be rewritten as follows:

$$\frac{r_\gamma}{r_\sigma} = (-1)^{\mathrm{MV}(\mathcal{N}(g_0),\ldots,\mathcal{N}(g_n))}(-1)^{\mathrm{MV}_2(\mathcal{N}(g_0),\ldots,\mathcal{N}(g_n),\binom{0}{1})}.$$

At the same time, using the classical Vieta's formula, we obtain

$$\frac{r_\sigma}{r_\gamma} = \prod_{\phi(t)=0} t.$$

It follows from Proposition 3.23 and the Bernstein-Kouchnirenko theorem (see [2] for the details) that

$$\prod_{\phi(t)=0} t = (-1)^{|\{\mathscr{R}=0\}\cap\mathscr{C}^{\gamma,\sigma}|} \prod_{g_0(x,t)=\ldots=g_n(x,t)=0} t = (-1)^{\mathrm{MV}(\mathcal{N}(g_0),\ldots,\mathcal{N}(g_n))} \prod_{g_0(x,t)=\ldots=g_n(x,t)=0} t.$$

Then, applying the multivariate Vieta's formula (see Theorem 3.16), we have

$$(-1)^{\mathrm{MV}(\mathcal{N}(g_0),\ldots,\mathcal{N}(g_n))} \prod_{g_0(x,t)=\ldots=g_n(x,t)=0} t = (-1)^{\mathrm{MV}(\mathcal{N}(g_0),\ldots,\mathcal{N}(g_n))}(-1)^{\mathrm{MV}_2(\mathcal{N}(g_0),\ldots,\mathcal{N}(g_n),\binom{0}{1})},$$

which finishes the proof of the theorem. □

4 Singularities of a projection of a complete intersection

In Chapter 4 we compute the sum of δ-invariants of a plane projection of a complete intersection curve given by a generic system of polynomial equations with a given support. First we introduce the necessary notation, discuss the important assumptions and then state the main result of the chapter – Theorem 4.9. For more details see [19].

4.1 Dramatis Personæ

- $(x_1,\ldots,x_n,y,t)$, coordinates in $(\mathbb{C}\setminus\{0\})^{n+2}$, $n \geqslant 1$;
- $\{e_1,\ldots,e_{n+2}\}$, the standard basis of the character lattice $\mathbb{Z}^{n+2}$;
- $A \subset \mathbb{Z}^{n+2}$, a finite subset of maximal dimension;
- $\Delta = \mathrm{conv}(A) \subset \mathbb{R}^{n+2}$, the convex hull of the set A;
- $\mathcal{F}(\Delta)$, the set of all facets of the polytope Δ;
- X_Δ, the toric variety associated to the polytope Δ;
- $f_1,\ldots,f_{n+1} \in \mathbb{C}^A$, a tuple of polynomials supported at A;
- $\widetilde{\mathcal{C}} = \{f_1 = \ldots = f_{n+1} = 0\} \subset (\mathbb{C}\setminus\{0\})^{n+2}$, the complete intersection given by the polynomials $f_1,\ldots,f_{n+1}$;
- $\pi\colon (\mathbb{C}\setminus\{0\})^{n+2} \to (\mathbb{C}\setminus\{0\})^2$, the projection forgetting the first n coordinates;
- $\mathcal{C} \subset (\mathbb{C}\setminus\{0\})^2$, the closure of the image $\pi(\widetilde{\mathcal{C}}) \subset (\mathbb{C}\setminus\{0\})^2$;
- $P \subset \mathbb{R}^2$ the Newton polygon of the curve $\mathcal{C}$;
- $\mathcal{S}$, the singular locus of the curve $\mathcal{C}$.

Remark 4.1. In general, the image $\pi(\widetilde{\mathcal{C}})$ is a plane curve with punctured points coming from the intersection of the closure of the curve $\widetilde{\mathcal{C}}$ in the toric variety X_Δ with the orbits corresponding to horizontal facets of the polytope Δ (see Definition 4.8). Therefore, we define the curve $\mathcal{C}$ to be the Zariski closure of $\pi(\widetilde{\mathcal{C}})$.

4.2 Statement of the problem

In this subsection we give a precise formulation of the question that we address in this chapter and discuss all the assumptions we make. For generic $f_1, \dots, f_{n+1} \in \mathbb{C}^A$, the complete intersection $\widetilde{\mathcal{C}} = \{f_1 = \dots = f_{n+1} = 0\} \subset (\mathbb{C} \setminus \{0\})^{n+2}$ is a smooth curve and the closure $\mathcal{C}$ of its image under the projection π is a plane curve in $(\mathbb{C} \setminus \{0\})^2$ whose singular locus consists of finitely many isolated singular points.

It is quite natural to expect that under certain genericity conditions, all the singular points of the curve $\mathcal{C}$ are nodes. However, this is the case not for all support sets. Moreover, the following example shows that when the support sets $\operatorname{supp}(f_j) = A_j$ do not coincide, one can no longer expect the singular points of the projection of the corresponding complete intersection to be nodes even for generic polynomials $f_j \in \mathbb{C}^{A_j}$. If we take a point $(y_0, t_0) \in \pi(\mathcal{C})$ and substitute it into the polynomials defining $\widetilde{\mathcal{C}}$, then we obtain a system of $n+1$ polynomial equations in n variables. The number of solutions for this system is equal to the number of preimages of the point (y_0, t_0). Now, take $n = 1$ and $\widetilde{\mathcal{C}} = \{f_1 = f_2 = 0\}$ with

$$f_1(x_1, y, t) = g_0(y, t) + x_1 g_1(y, t) + x_1^3 g_3(y, t) \text{ and } f_2(x_1, y, t) = h_0(y, t) + x_1^3 h_3(y, t),$$

where g_i, h_j are some Laurent polynomials in the variables y, t. The example below shows that in this case, if the point (y_0, t_0) has more than one preimage in $\widetilde{\mathcal{C}}$, then it has 3 preimages, so it cannot be a node. If the support sets $\operatorname{supp}(f_1)$ and $\operatorname{supp}(f_2)$ are big enough, then the curve $\mathcal{C}$ will actually have points with 3 preimages.

Example 4.2. Consider $A_1 = \{0, 1, 3\} \subset \mathbb{Z}^1$ and $A_2 = \{0, 3\} \subset \mathbb{Z}^1$. Let $f_1(x)$ and $f_2(x)$ be polynomials supported at A_1 and A_2 respectively. Suppose that the univariate system $\{f_1(x) = f_2(x) = 0\}$ has 2 distinct roots $r_1, r_2 \in (\mathbb{C} \setminus \{0\})$. Let us show that this system also has another root $r_3 \in (\mathbb{C} \setminus \{0\})$. Indeed, the assumption we made implies that $r_2 = \alpha \cdot r_1$, where α is a root of unity. Substituting these roots into the first equation, we obtain that the linear term of f_1 has to be 0. But then it is clear that the third root $r_3 = \alpha \cdot r_2$ is also a root for the first equation.

Instead of computing the number of nodes of the curve $\mathcal{C}$, we address a slightly more general problem: namely, we compute the sum of the δ-invariants of the singular points of the curve $\mathcal{C}$. On one hand, this problem makes sense for any collection of support sets. On the other hand, if the curve $\mathcal{C}$ only has nodes as singularities, then the answer is exactly the number of those nodes.

Problem **4.3.** In the same notation as above, express the sum $Đ$ of the δ-invariants of the singular points of the curve $\mathcal{C}$ in terms of the set A.

Definition 4.4. Let B be any finite set in $\mathbb{Z}^{n+2}$ and let $\widetilde{\Lambda}_B \subset \mathbb{Z}^{n+2}$ be the sublattice affinely generated by B. Denote by Λ_B the image of $\widetilde{\Lambda}_B$ under the projection $\rho\colon \mathbb{Z}^{n+2} \twoheadrightarrow \mathbb{Z}^{n+2} \big/ \langle e_{n+1}, e_{n+2} \rangle$. We define $\operatorname{ind}_v(B)$ to be the index of Λ_B in $\mathbb{Z}^{n+2} \big/ \langle e_{n+1}, e_{n+2} \rangle$.

Assumption **4.5.** The set A contains $0 \in \mathbb{Z}^{n+2}$.

This assumption can be made without loss of generality since multiplication by monomial does not change the zero set of the polynomial inside the algebraic torus. At the same time, the resulting support set is a shift of the initial one.

Assumption **4.6.** The set A satisfies the following property: $ind_v(A) = 1$.

Remark 4.7. We can make this assumption without loss of generality due to the following reason. The lattices Λ_A and $\Lambda = \mathbb{Z}^{n+2} \big/ \langle e_{n+1}, e_{n+2} \rangle$ admit a pair of aligned bases such that $\Lambda = \bigoplus \mathbb{Z} w_i$ and $\Lambda_A = \bigoplus \mathbb{Z} a_i w_i$ for some $a_i \in \mathbb{Z}$. Performing a monomial change of variables to pass from the basis $\{e_1, \dots, e_{n+2}\}$ to $(w_1, \dots, w_n, e_{n+1}, e_{n+2})$ and then another change of variables of the form $\check{x}_i = x_i^{a_i}$, we will reduce our problem to the case $ind_v(A) = 1$.

Let $Q = p(\Delta)$ be the image of the polytope Δ under the projection $\rho\colon \mathbb{R}^{n+2} \twoheadrightarrow \mathbb{R}^{n+2} \big/ \langle e_{n+1}, e_{n+2} \rangle$.

Definition 4.8. We call a face $\widetilde{\Gamma} \subset \Delta$ *horizontal*, if its projection is contained in the boundary of Q. We denote the set of all horizontal facets of the polytope Δ by $\mathcal{H}(\Delta)$.

4.3 Statement of the main result

Denote by $(\epsilon_1, \ldots, \epsilon_{n+2})$ the coordinate system induced by the basis $\{e_1, \ldots, e_{n+2}\}$ of $\mathbb{R}^{n+2}$. Let $\Gamma \subset \Delta$ be a non-horizontal facet contained in a hyperplane given by a linear equation of the form $\ell(\epsilon_1, \ldots, \epsilon_{n+2}) = c$. The function ℓ is unique up to a scalar multiple, therefore, one can assume that the coefficients of ℓ are coprime integers and that for any $\alpha \in A \setminus \Gamma$, $\ell(\alpha) < c$.

We now construct a sequence of integers $i^\Gamma = (i_1^\Gamma, i_2^\Gamma, \ldots)$ as follows. We set $B_1^\Gamma = A \cap \Gamma$. For every $r > 1$, we define

$$B_r^\Gamma = B_{r-1}^\Gamma \cup (A \cap \{\ell(\epsilon_1, \ldots, \epsilon_{n+2}) = c - (r-1)\}).$$

Finally, for every $r \geqslant 1$, we set

$$i_r^\Gamma = \mathrm{ind}_v(B_r^\Gamma).$$

It is clear that for every r, the element i_r^Γ divides i_{r-1}^Γ. Moreover, since for the set A we have $\mathrm{ind}_v(A) = 1$, any such sequence stabilizes to 1.

Theorem 4.9. *Let $A \subset \mathbb{Z}^{n+2}$ be a finite set of full dimension, satisfying Assumption 4.6, and let $\Delta \subset \mathbb{R}^{n+2}$ be its convex hull. In the same notation as above, for generic $f_1, \ldots, f_{n+1} \in \mathbb{C}^A$, the closure $\mathcal{C}$ of the image of the curve $\widetilde{\mathcal{C}} = \{f_1 = \ldots = f_{n+1} = 0\}$ under the projection $\pi\colon (\mathbb{C}\backslash\{0\})^{n+2} \to (\mathbb{C}\backslash\{0\})^2$ forgetting the first n coordinates is an algebraic plane curve, whose singular locus $\mathcal{S}$ consists of isolated singular points. Then the number $Đ = \sum_{s \in \mathcal{S}} \delta(s)$ can be computed via the following formula:*

$$Đ = \frac{1}{2}\left(\mathrm{Area}(P) - (n+1)\,\mathrm{Vol}(\Delta) + \sum_{\Gamma \in \mathcal{H}(\Delta)} \mathrm{Vol}(\Gamma) - \sum_{\Gamma \in \mathcal{F}(\Delta) \backslash \mathcal{H}(\Delta)} \mathrm{Vol}(\Gamma) \sum_{1}^{\infty} (i_r^\Gamma - 1) \right), \tag{5}$$

where $\delta(s)$ is the δ-invariant of the singular point s, $P = \int_\pi(\Delta)$ is the fiber polytope of Δ with respect to π, the set $\mathcal{F}(\Delta)$ is the set of all facets of the polytope Δ and $\mathcal{H}(\Delta)$ is the set of all horizontal facets of Δ.

5 Application: Newton polytope of the Morse discriminant

In this section, we explain how combining the methods introduced in the previous chapters, allows to compute the vertices of the Newton polytope of the Morse discriminant in the space of univariate polynomials of given degree. In other words, in the notation of subsection , we fix $A = \mathbb{Z} \cap [1, n] \subset \mathbb{Z}$, and we want to compute the Newton polytope of the polynomial $h_c h_m^2$, where the polynomials h_c and h_m define the caustic and the Maxwell stratum in $\mathbb{C}^A$, respectively.

Remark 5.1. Since rescaling the variables or multiplying all the coefficients of a polynomial by the same non-zero number does not affect whether or not it is Morse, the polytope $\mathcal{M}_A$ should lie in the intersection of two affine hyperplanes in $\mathbb{R}^{|A|}$: namely, the hyperplanes $\{e_1 + \ldots + e_n = d_1\}$ and $\{1 \cdot e_1 + 2e_1 + \ldots + n \cdot e_n = d_2\}$ for some d_1, d_2.

Remark 5.2. Since adding a constant to a polynomial $f(x)$ does not change its property of being Morse or non-Morse, one can always consider sets A without 0.

Example 5.3. For the set $A = \{1, 2, 3, 4\}$, the space $\mathbb{C}^A$ consists of polynomials of the form $\alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4$ with complex coefficients α_i, and we have

$$h_c = \alpha_2^2\alpha_3^2 - 4\alpha_1\alpha_3^3 - 4\alpha_2^3\alpha_4 + 18\alpha_1\alpha_2\alpha_3\alpha_4 - 27\alpha_1^2\alpha_4^2,$$

$$h_m = \alpha_3^3 + 8\alpha_1\alpha_4^2 - 4\alpha_2\alpha_3\alpha_4.$$

Example 5.4. For the set $A = \{1, 2, 3, 4\}$, we have

$$\mathcal{M}_A = \operatorname{conv}\big(\{(4,0,0,6),(0,2,8,0),(1,0,9,0),(0,5,2,3),(2,3,0,5)\}\big) \subset \mathbb{R}^4.$$

Problem **5.5.** Obtain a combinatorial description of the vertices of the Newton polytope $\mathcal{M}_A$ of the Morse discriminant in terms of the support set A.

5.1 The geometric interpretation of the problem

The vertices of the polytope $\mathcal{M}_A$ are in 1-to-1 correspondence with the full-dimensional cones of its dual fan. The support function μ_A is linear on each of these cones, and its coefficients on the given cone are the coordinates of the corresponding vertex.

As it was discussed in Remark 5.1, the sought polytope lies in the intersection of the following two hyperplanes in $\mathbb{R}^{|A|}$: the hyperplane $\{e_1+\ldots+e_n = d_1\}$ and the hyperplane $\{1\cdot e_1+\ldots+n\cdot e_n = d_2\}$ for some d_1, d_2. Therefore it suffices to compute μ_A only on the covectors $\gamma \in (\mathbb{R}^{|A|})^*$ with non-negative entries.

Moreover, to find the coefficients of the function μ_A on its domains of linearity, it is enough to compute μ_A on rational, or, equivalently, on integer covectors supported at the corresponding vertices. The latter observation allows to use the following geometric interpretation of the main problem. Namely, we can use the same idea as, for instance, in [1] (see Chapter 3) or the works [4] and [6].

Let γ be an integer covector with non-negative entries supported at a vertex of $\mathcal{M}_A$. Alternatively it can be viewed as a function $\gamma\colon A \to \mathbb{Z}_{\geqslant 0}$. Replacing the coefficients of $x^p,\ p \in A$ of a polynomial $f(x)$, $\operatorname{supp}(f) = A$, with polynomials of degrees $\gamma(p)$ in a new variable t turns the Morse discriminant into a polynomial in t. And since we can interpret the value $\mu_A(\gamma)$ as the number of roots of this univariate polynomial, the main problem of this chapter can be reduced to the following one.

Problem **5.6.** Let $\gamma\colon A \to \mathbb{Z}_{\geqslant 0}$ be an arbitrary function and $q_p, v_p,\ p \in A$ be generic tuples of complex numbers. For how many values of the parameter $t \in \mathbb{C}$ is the polynomial $f_t(x) = \sum\limits_{p\in A}(q_p + v_p t^{\gamma(p)})x^p$ not Morse?

This question was discussed in Example 1.1 of [6], and the answer was obtained for a special case of a concave function $\gamma\colon \{1,\ldots,n\} \to \mathbb{Z}_{\geqslant 0}$. Using a similar approach and the results of section 4, we will obtain the answer for any function $\gamma\colon A \to \mathbb{Z}_{\geqslant 0}$.

Given a function $\gamma\colon A \to \mathbb{Z}_{\geqslant 0}$, we consider the hypersurface $\mathcal{H} = \{f_t(x) - y\} \subset (\mathbb{C}\setminus\{0\})^3$. Let π be the projection $\pi\colon (\mathbb{C}\setminus\{0\})^3 \to (\mathbb{C}\setminus\{0\})^2,\quad (x,y,t) \mapsto (y,t)$, and let $\mathcal{A}_1$, $\mathcal{A}_2$ and $2\mathcal{A}_1$ be the open multisingularity strata of its restriction $\pi\,|_{\mathcal{H}}$ to the hypersurface $\mathcal{H}$. The sets $\mathcal{A}_1, \mathcal{A}_2$ and $2\mathcal{A}_1$ consist of all the points $(y,t) \in (\mathbb{C}\setminus\{0\})^2$ such that the equation $f_t(x) = y$ has exactly one root of multiplicity 2, exactly one root of multiplicity 3 and exactly two roots of multiplicity 2, respectively. Due to the choice of the set A(we recall that in our case, $A = \mathbb{Z}\cap[1,n]$), the only strata of codimension 2 are $\mathcal{A}_2$ and $2\mathcal{A}_1$, while the strata of other singularities are of strictly higher codimension.

Denote by Δ the Newton polytope of the polynomial $f_t(x) - y$. Equivalently, Δ is the convex hull of the set

$$\tilde{A} = \{(0,1,0)\} \cup \{(a_0,0,0)\} \cup \{(a_{|A|-1},0,0)\} \cup \{\big(p,0,\gamma(p)\big) \mid p \in A\} \subset \mathbb{Z}^3.$$

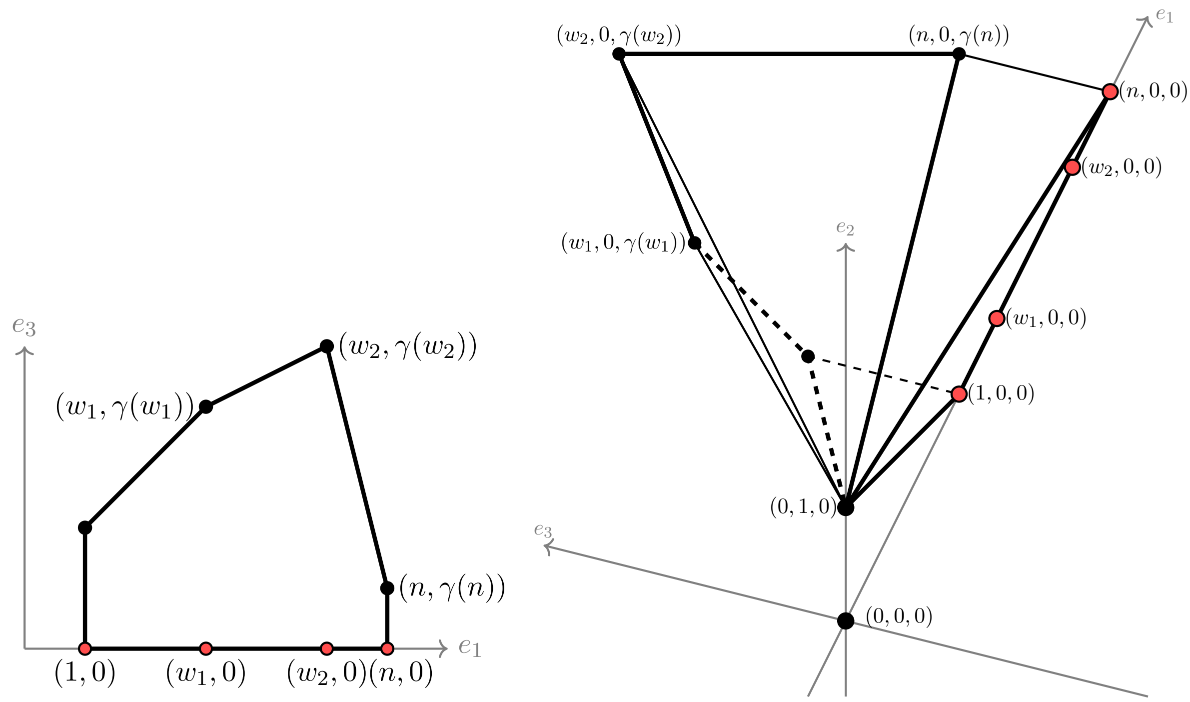

Figure 2: The polytopes N and Δ.

Figure 2 below shows what the polytopes N and Δ may look like.

All the convex subdivisions of the interval $\mathrm{conv}(A)$ are in 1-to-1 correspondence with the subsets of the form $W = \{w_0 < w_1 < \ldots < w_{k-1} < w_k\} \subset A$ with $w_0 = 1$ and $w_k = n$. Moreover, every function $\gamma\colon A \to \mathbb{Z}_{\geqslant 0}$ defines a convex subdivision of the interval $\mathrm{conv}(A)$. Indeed, let N be the Newton polygon of the polynomial $f_t(x) = \sum\limits_{p \in A} (q_p + v_p t^{\gamma(p)}) x^p$. Then the corresponding subset W consists of all the points $p \in A$ such that the point $(p, \gamma(p))$ is a vertex of N.

To obtain the desired formula, we view the set $\mathcal{D} \subset (\mathbb{C} \setminus \{0\})^2$ of the critical values of the projection π as the plane projection of the curve $\mathcal{C} = \{f(x,t) - y = x\frac{\partial f(x,t)}{\partial x} = 0\} \subset (\mathbb{C} \setminus \{0\})^3$, which is the set of the critical values of the projection π. The Newton polytope of the curve $\mathcal{D}$ is precisely the fiber polygon $P = \int_\pi \Delta$. A direct computation yields the following formula for the area of the polygon P.

Proposition 5.7. *In the same notation as above, for $A = \mathbb{Z} \cap [1, n]$ and a covector $\gamma \in (\mathbb{R}^{|A|})^*$ with the corresponding subdivision $W \subset A$, we have the following formula:*

$$\mathrm{Vol}\Big(\int_\pi \Delta\Big) = w_1(w_1 - 1)\gamma(1) + (n - w_{k-1})(2n + w_{k-1} - 2)\gamma(n) +$$
$$+ \sum_{j=1}^{j=k-1} (w_{j+1} - w_{j-1})(w_{j-1} + w_j + w_{j+1} - 2)\gamma(w_j).$$

The singular points of the curve $\mathcal{D}$ inside the torus are nodes and cusps (namely, $|2\mathcal{A}_1|$ nodes and $|\mathcal{A}_2|$ cusps). Theorem 4.9 allows to compute the total sum of the δ-invariants of the curve $\mathcal{D}$ inside the torus, taking the singularities of $\mathcal{D}$ at infinity into account. Just as in the setting of Theorem 4.9, the answer depends not only on the convex hull of the support set $\tilde{A} = \mathrm{supp}(f(t,x) - y)$, but on the sequences i^Γ for all non-horizontal facets $\Gamma \subset \Delta$.

The latter can be encoded as follows: for every number $j = 0, \ldots, k-1$, we can write down the $M^j = (m_1^j, m_2^j, \ldots)$ composed of the elements $A \setminus \{w_j, w_{j+1}\}$ ordered as follows. Let ℓ be a line passing through $(w_j, \gamma(w_j))$ and $(w_{j+1}, \gamma(w_{j+1}))$. Now, let us trace the copies of ℓ shifted by vectors $(0, -r)$, where $r \in \mathbb{Z}{>}\, 0$. If the covector γ is generic, then the points $(m, \gamma(m)) \in \mathrm{supp}(f(t,x))$ are encountered one by one by the copies of the line ℓ as r increases. The order, in which they are encountered, determines the order of the elements in the sequence M^j.

Together with every sequence M^j, we define an auxillary sequence $B^j = (b_0^j, b_1^j, \ldots)$: we set $b_0^j = \gcd(w_j, w_{j+1})$ and for every $l \geqslant 1$, we set $b_l^j = \gcd(b_{l-1}, m_l)$.

Finally, for every $j = 0, \ldots, k-1$, we can define the expressions C^j as follows.

$$C_\gamma^j = \sum_{l \geqslant 1} \Big((w_{j+1} - w_j)\gamma(m_l^j)) + (m_l^j - w_{j+1})\gamma(w_j) + (w_j - m_l^j)\gamma(w_{j+1}) \Big)(b_{l-1}^j - b_l^j). \tag{6}$$

Using Theorem 4.9, we can compute the total number of nodes and cusps of the curve $\mathcal{D}$. Together with the well-known formula for the support function of the classical discriminant, Theorem 4.9 allows to compute $2|2\mathcal{A}_1| + |\mathcal{A}_2|$. The latter coincides with the value of the sought support function μ_A at a generic integer covector $\gamma \in (\mathbb{R}^{|A|})^*$, which yields the desired formula for the support function μ_A.

5.2 The vertices of the Newton polytope of the Morse discriminant

A generic covector $\gamma \colon A \to \mathbb{R}_{\geqslant 0}$ determines a subdivision $W = \{1 < w_1 \ldots < w_{k-1} < n\} \subset [1, n]$ and for every $j = 0, \ldots, k-1$, the covector γ the sequences M^j, b^j and the linear expression C^j, as explained in the previous section.

Theorem 5.8. *In the same notation as above, for $A = \mathbb{Z} \cap [1, n]$, the value of the support function μ_A of the Morse polytope $\mathcal{M}_A$ at the covector γ can be computed via the following formula:*

$$\mu_A(\gamma) = (w_1 - 2)^2\gamma(w_0) + \sum_{j=1}^{k-1}(w_{j+1} - w_{j-1})(w_{j-1} + w_j + w_{j+1} - 5)\gamma(w_j) + \\ + (n - 1 - w_{k-1})(2n + w_{k-1} - 6)\gamma(w_k) + \sum_{j=0}^{j=k-1} C_\gamma^j. \tag{7}$$

Remark 5.9. One can easily observe that the covectors γ defining the same combinatorial data, that is, the subdivision $W = \{w_0, \ldots, w_k\}$ and the sequences M^j, $j = 0, \ldots, k-1$, form a convex full-dimensional cone in $(\mathbb{R}^A)^*$. The result above establishes a surjection between the cones of covectors defining the same combinatorial data and the linearity domains of the support function μ_A of the polytope $\mathcal{M}_A$. Thus, it allows to compute all the vertices of the polytope $\mathcal{M}_A$ by computing the tuples of coefficients of the support function μ_A in its linearity domains.

Remark 5.10. Formula (7) allows to compute the polytope $\mathcal{M}_A$ up to a shift. In the special case $A = \mathbb{Z} \cap [1, n]$, we have chosen the shift which moves the polytope as close as possible to $0 \in \mathbb{R}^A$, while keeping it in the positive octant.

5.2.1 Example: degree 4 polynomials

We will now compute the vertices of the Newton polytope of the Morse discriminant in the space of polynomials with support $A = \{1, 2, 3, 4\}$ using Theorem 5.8.

To do this, we need to enumerate all possible tuples of combinatorial data arising from generic covectors in $(\mathbb{R}^4)^*$ and evaluate the support function on the corresponding cones.

There are 5 cases, we will treat each of them separately. The graphs of typical representatives $\gamma \in (\mathbb{R}^4)^*$ (viewed as functions $\gamma \colon A \to \mathbb{R}$) of the corresponding cones are shown in Figure 3. The elements of the subsets W are marked red. By comparing the slope of the interval $[(1, \gamma(1)), (3, \gamma(3))]$ with the slope of $[(2, \gamma(2)), (4, \gamma(4))]$ we distinguish the cases 4 and 5.

- $W = \{1, 4\}$. In this case, we have $\gcd(w_0, w_1) = 1$, therefore the summand C^0 is equal to 0 and does not depend on the sequence M^0. The corresponding vertex is $(4, 0, 0, 6)$.

- $W = \{1,2,3,4\}$. Similarly to the previous case, the summands C^j do not depend on the sequences M^j and are equal to 0. The corresponding vertex is $(0,2,8,0)$.
- $W = \{1,3,4\}$. In this case, $\gcd(w_0, w_1) = 1$, $\gcd(w_1, w_2) = 1$, so the summands C^0 and C^1 do not depend on the sequences M^0 and M^1 and are equal to 0. The corresponding vertex is $(1,0,9,0)$.
- $W = \{1,2,4\}$, $M^1 = (3,1)$. In this case, $\gcd(1,2) = 1$, so $C^0 = 0$. We have $b^1 = (2,1,1)$, thus $C^1 = 2\gamma(3) - \gamma(2) - \gamma(4)$, therefore the corresponding vertex of $\mathcal{M}_A$ is $(0,6,0,4) + (0,-1,2,-1) = (0,5,2,3)$.
- $W = \{1,2,4\}$, $M^1 = (1,3)$. So, we have $C^0 = 0$, and $C^1 = 2\gamma(1) - 3\gamma(2) + \gamma(4)$, therefore, the corresponding vertex is $(0,6,0,4) + (2,-3,0,1) = (2,3,0,5)$.

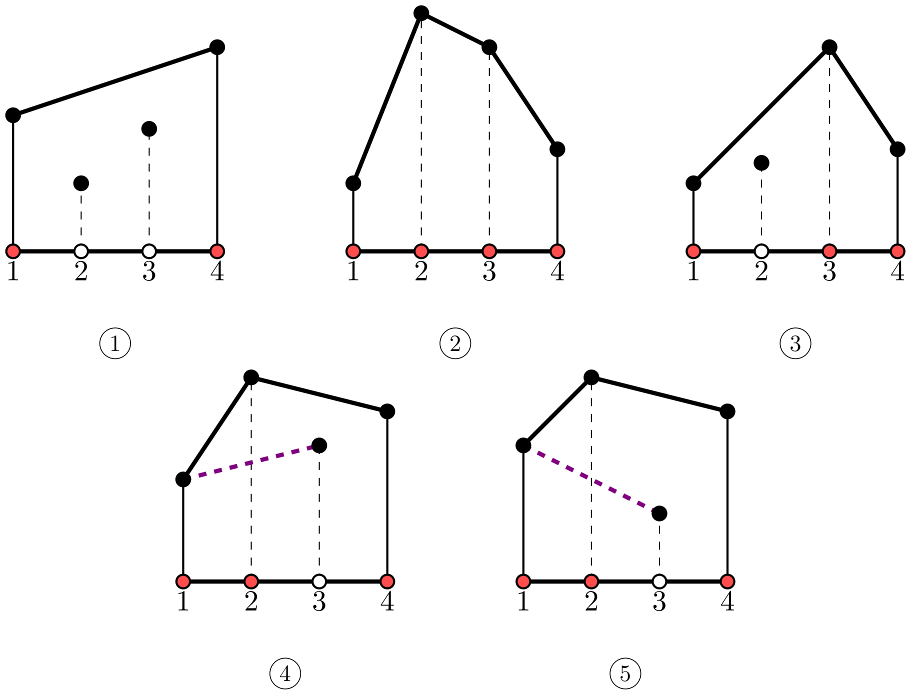

Figure 3: The 5 vertices of the polytope $\mathcal{M}_A$.

The polytope $\mathcal{M}_A$ is a polygon in $\mathbb{R}^4$, and its image under the projection forgetting the first and the last coordinates is shown in Figure 4 below.

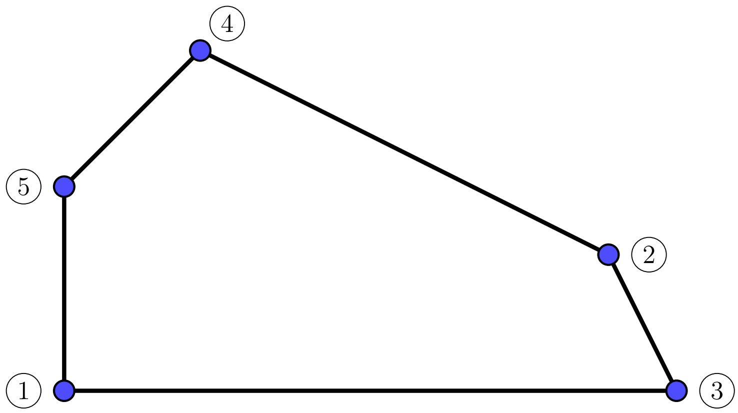

Figure 4: A projection of the polytope $\mathcal{M}_A$.

The main results of the thesis are presented in two papers:

– A. Arkhipova and A. Esterov, *"Signs of the Leading Coefficients of the Resultant"*, Geom.Funct. Anal., vol. 27, no. 1, pp. 33–66, Feb. 2017, issn: 1420-8970. doi: 10.1007/s00039-017-0393-z.

– A. Voorhaar, *"On the singular locus of a plane projection of a complete intersection"*, Math.Z., Apr. 2022, issn: 1432-1823. doi: 10.1007/s00209-022-03014-7.

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